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Local K-Convoluted C-Cosine Functions and Abstract Cauchy Problems

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Abstract. Let $K : [0, T_0) \to \mathbb{F}$ be a locally integrable function, and $C : X \to X$ a bounded linear operator on a Banach space X over the field $\mathbb{F}$(=$\mathbb{R}$ or $\mathbb{C}$). In this paper, we will deduce some basic properties of a nondegenerate local K-convoluted C-cosine function on X and some generation theorems of local K-convoluted C-cosine functions on X with or without the nondegeneracy, which can be applied to obtain some equivalence relations between the generation of a nondegenerate local K-convoluted C-cosine function on X with subgenerator A and the unique existence of solutions of the abstract Cauchy problem: $u''(t) = Au(t) + f(t)$ for a.e. $t \in (0, T_0)$, $u(0) = x$, $u'(0) = y$ when K is a kernel on $[0, T_0)$, $C : X \to X$ an injection, and $A : D(A) \subset X \to X$ a closed linear operator in X such that $CA \subset AC$. Here $0 < T_0 \leq \infty$, $x, y \in X$, and $f \in L^1_{loc}([0, T_0), X)$.

1. Introduction

Let X be a Banach space over the field $\mathbb{F}$(=$\mathbb{R}$ or $\mathbb{C}$) with norm $\|\cdot\|$, and let $L(X)$ denote the family of all bounded linear operators from X into itself. For each $0 < T_0 \leq \infty$, we consider the following abstract Cauchy problem:

$$\text{ACP}(A, f, x, y) \qquad \begin{cases} u''(t) = Au(t) + f(t) & \text{for a.e. } t \in (0, T_0), \\ u(0) = x, u'(0) = y, \end{cases}$$

where $x, y \in X$, $A : D(A) \subset X \to X$ is a closed linear operator, and $f \in L^1_{loc}([0, T_0), X)$. A function u is called a (strong) solution of ACP(A, f, x, y) if $u \in C^1([0, T_0), X)$ satisfies ACP(A, f, x, y) (that is $u(0) = x$, $u'(0) = y$ and for a.e. $t \in (0, T_0)$, $u'(t)$ is differentiable and $u(t) \in D(A)$, and $u''(t)=Au(t)+f(t)$ for a.e. $t \in (0, T_0)$). For each $C \in L(X)$ and $K \in L^1_{loc}([0, T_0), \mathbb{F})$, a subfamily $C(\cdot)(= \{C(t) \,|\, 0 \leq t < T_0\})$ of $L(X)$ is called a local K-convoluted C-cosine function on X if $C(\cdot)$ is strongly continuous, $C(\cdot)C = CC(\cdot)$, and satisfies

$$2C(t)C(s)x = \left(\int_0^{t+s} - \int_0^t - \int_0^s\right) K(t+s-r)C(r)Cxdr + \int_{|t-s|}^t K(s-t+r)C(r)Cxdr + \int_{|t-s|}^s K(t-s+r)C(r)Cxdr + \int_0^{|t-s|} K(|t-s|+r)C(r)Cxdr$$

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for all $0 \leq t, s, t+s < T_0$ and $x \in X$ (see [8]). In particular, $C(\cdot)$ is called a local (0-times integrated) C-cosine function on X if $K = j_{-1}$(the Dirac measure at 0) or equivalently, it is strongly continuous, $C(\cdot)C = CC(\cdot)$, and satisfies

$$2C(t)C(s)x = C(t+s)Cx + C(|t-s|)Cx \quad \text{for all } 0 \leq t, s, t+s < T_0 \text{ and } x \in X$$

(see [4,6,19,21]). Moreover, we say that $C(\cdot)$ is nondegenerate, if $x = 0$ whenever $C(t)x = 0$ for all $0 \leq t < T_0$. The nondegeneracy of a local K-convoluted C-cosine function $C(\cdot)$ on X implies that

$$C(0) = C \text{ if } K = j_{-1}, \text{ and } C(0) = 0 \text{ (the zero operator on X) otherwise,}$$

and the (integral) generator $A : D(A) \subset X \to X$ of $C(\cdot)$ is a closed linear operator in X defined by

$$D(A) = \{x \in X \mid \text{ there exists a } y_x \in X \text{ such that } C(\cdot)x - K_0(\cdot)Cx = \widetilde{S}(\cdot)y_x \text{ on } [0, T_0)\}$$

and $Ax = y_x$ for all $x \in D(A)$. Here $K_\beta(t) = K * j_\beta(t) = \int_0^t K(t-s)j_\beta(s)ds$ for $\beta > -1$ with $j_\beta(t) = \frac{t^\beta}{\Gamma(\beta+1)}$ and the Gamma function $\Gamma(\cdot)$, $S(s)z = \int_0^s C(r)zdr$, and $\widetilde{S}(t)z = \int_0^t S(s)zds$. In general, a local K-convoluted C-cosine function on X is called a K-convoluted C-cosine function on X if $T_0 = \infty$; a (local) K-convoluted C-cosine function on X is called a (local) K-convoluted cosine function on X if $C = I$(the identity operator on X) or a (local) α-times integrated C-cosine function on X if $K = j_{\alpha-1}$ for some $\alpha \geq 0$ (see [12-14,16]); a (local) α-times integrated C-cosine function on X is called a (local) α-times integrated cosine function on X if $C = I$ (see [15]); and a (local) C-cosine function on X is called a cosine function on X if $C = I$ (see [1,5]). Moreover, a local α-times integrated cosine function on X is not necessarily extendable to an α-times integrated cosine function on X except for $\alpha = 0$ (see [5]), the nondegeneracy of a local α-times integrated C-cosine function on X does not imply the injectivity of C except for $T_0 = \infty$ (see [12]), and the injectivity of C does not imply the nondegeneracy of a local α-times integrated C-cosine function on X except for $\alpha = 0$ (see [19]). Some basic properites of a nondegenerate (local) α-times integrated C-cosine function on X have been established by many authors in [11,22] when $\alpha = 0$, in [7,17-18,23-24] when $\alpha \in \mathbb{N}$, in [12] when $\alpha > 0$ is arbitrary with $T_0 = \infty$ and in [16] for the general case $0 < T_0 \leq \infty$, which can be applied to deduce some equivalence relations between the generation of a nondegenerate (local) α-times integrated C-cosine function on X with subgenerator A (see Definition 2.4 below) and the unique existence of strong or weak solutions of the abstract Cauchy problem ACP(A, f, x, y) (see the results in [7,12] for the case $T_0 = \infty$ and in [13-14,16] for the general case $0 < T_0 \leq \infty$). The purpose of this paper is to investigate the following basic properties of a nondegenerate local K-convoluted C-cosine function $C(\cdot)$ on X when C is injective and some additional conditions are taken into consideration.

$$C^{-1}AC = A; \tag{1}$$

$$\widetilde{S}(t)x \in D(A) \quad \text{and } A\widetilde{S}(t)x = C(t)x - K_0(t)Cx \quad \text{for all } x \in X \quad \text{and } 0 \leq t < T_0; \tag{2}$$

$$C(t)x \in D(A) \quad \text{and } AC(t)x = C(t)Ax \quad \text{for all } x \in D(A) \quad \text{and } 0 \leq t < T_0; \tag{3}$$

and

$$C(t)C(s) = C(s)C(t) \quad \text{for all } 0 \leq t, s, t+s < T_0 \tag{4}$$

(see Theorems 2.7 and 2.11, and Corollary 2.12 below). We then deduce some equivalence relations between the generation of a nondegenerate local K-convoluted C-cosine function on X with subgenerator A and the unique existence of strong solutions of ACP(A, f, x, y) in section 3 just as results in [16] concerning some equivalence relations between the generation of a nondegenerate local α-times C-cosine function on X with subgenerator A and the unique existence of strong solutions of ACP(A, f, x, y). To do these, we will prove an important lemma which shows that a strongly continuous subfamily $C(\cdot)$ of $L(X)$ is a local K-convoluted C-cosine function on X is equivalent to say that $\widetilde{S}(\cdot)$ is a local K_1-convoluted C-cosine function on X (see Lemma 2.1 below), and then show that a strongly continuous subfamily $C(\cdot)$ of $L(X)$ which commutes with C on X is a local K-convoluted C-cosine function on X is equivalent to say that $\widetilde{S}(t)[C(s)-K_0(s)C]=[C(t)-K_0(t)C]\widetilde{S}(s)$ for all $0 \le t, s, t+s < T_0$ (see Theorem 2.2 below). In order, we show that $a * C(\cdot)$ is a local $a * K$-convoluted C-cosine function on X if $C(\cdot)$ is a local K-convoluted C-cosine function on X and $a \in L^1_{loc}([0,T_0),\mathbb{F})$. In particular, $j_\beta * C(\cdot)$ is a local K_β-convoluted C-cosine function on X if $C(\cdot)$ is a local K-convoluted C-cosine function on X and $\beta > -1$ (see Proposition 2.3 below), where $f * C(t)x = \int_0^t f(t-s)C(s)xds$ for all $x \in X$ and $f \in L^1_{loc}([0,T_0),\mathbb{F})$. We also show that a strongly continuous subfamily $C(\cdot)$ of $L(X)$ which commutes with C on X is a local K-convoluted C-cosine function on X when $C(\cdot)$ has a subgenerator (see Theorem 2.5 below), which had been proven in [8] by another method similar to that already employed in [12] in the case that $C(\cdot)$ has a closed subgenerator and C is injective; and the generator of a nondegenerate local K-convoluted C-cosine function $C(\cdot)$ on X is the unique subgenerator of $C(\cdot)$ which contains all subgenerators of $C(\cdot)$ and each subgenerator of $C(\cdot)$ is closable and its closure is also a subgenerator of $C(\cdot)$ when $C(\cdot)$ has a subgenerator (see Theorems 2.7 and 2.11, and Corollary 2.12 below). This can be applied to show that $CA \subset AC$ and $C(\cdot)$ is a nondegenerate local K-convoluted C-cosine function on X with generator $C^{-1}AC$ when C is injective, K_0 a kernel on $[0,T_0)$ (that is, $f = 0$ on $[0,T_0)$ whenever $f \in C([0,T_0),\mathbb{F})$ with $\int_0^t K_0(t-s)f(s)ds = 0$ for all $0 \le t < T_0$) and $C(\cdot)$ a strongly continuous subfamily of $L(X)$ with closed subgenerator A. In this case, $C^{-1}\overline{A_0}C$ is the generator of $C(\cdot)$ for each subgenerator A_0 of $C(\cdot)$ (see Theorem 2.13 below). Some illustrative examples concerning these theorems are also presented in the final part of paper.

2. Basic Properties of Local *K*-Convoluted *C*-Cosine Functions

We will deduce an important lemma which can be applied to obtain an equivalence relation between the generation of a local K-convoluted C-cosine function $C(\cdot)$ on X and the equation

$$\widetilde{S}(t)[C(s)-K_0(s)C] = [C(t)-K_0(t)C]\widetilde{S}(s) \quad \text{for all } 0 \le t, s, t+s < T_0, \tag{5}$$

(see a result in [16] for the case of local α-times integrated C-cosine function and a corresponding statement in [9] for the case of (a,k)-regularized (C_1,C_2)-existence and uniqueness family). **Lemma 2.1** *Let $C(\cdot)$ be a strongly continuous subfamily of $L(X)$. Then $C(\cdot)$ is a local K-convoluted C-cosine function on X if and only if $\widetilde{S}(\cdot)$ is a local K_1-convoluted C-cosine function on X.*

Proof. We will show that

$$\begin{aligned}
&\frac{d}{dt}\left[\left(\int_0^{t+s} - \int_0^t - \int_0^s\right) K_1(t+s-r)\widetilde{S}(r)Cxdr + \int_{|t-s|}^t K_1(s-t+r)\widetilde{S}(r)Cxdr\right.\\
&\left.+ \int_{|t-s|}^s K_1(t-s+r)\widetilde{S}(r)Cxdr + \int_0^{|t-s|} K_1(|t-s|+r)\widetilde{S}(r)Cxdr\right]\\
&=\left(\int_0^{t+s} - \int_0^t - \int_0^s\right) K_0(t+s-r)\widetilde{S}(r)Cxdr + sgn(s-t)\int_{|t-s|}^t K_0(s-t+r)\widetilde{S}(r)Cxdr\\
&+ sgn(t-s)\int_{|t-s|}^s K_0(t-s+r)\widetilde{S}(r)Cxdr + \int_0^{|t-s|} K_0(|t-s|+r)\widetilde{S}(r)Cxdr
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
&\frac{d^2}{dt^2}\left[\left(\int_0^{t+s} - \int_0^t - \int_0^s\right) K_1(t+s-r)\widetilde{S}(r)Cxdr + \int_{|t-s|}^t K_1(s-t+r)\widetilde{S}(r)Cxdr + \int_{|t-s|}^s K_1(t-s+r)\widetilde{S}(r)Cxdr\right.\\
&\left.+ \int_0^{|t-s|} K_1(|t-s|+r)\widetilde{S}(r)Cxdr\right] + 2K_0(s)\widetilde{S}(t)Cx
\end{aligned}$$

$$= \left(\int_0^{t+s} - \int_0^t - \int_0^s \right) K(t+s-r)\widetilde{S}(r)Cxdr + \int_{|t-s|}^t K(s-t+r)\widetilde{S}(r)Cxdr + \int_{|t-s|}^s K(t-s+r)\widetilde{S}(r)Cxdr$$
$$+ \int_0^{|t-s|} K(|t-s|+r)\widetilde{S}(r)Cxdr \tag{7}$$

for all $x \in X$ and $0 \leq t, s, t+s < T_0$, where $sgn(t) = 1$ if $0 < t$, $sgn(0) = 0$, and $sgn(t) = -1$ if $t < 0$. Indeed, for $0 \leq s \leq t < T_0$ with $t+s < T_0$, we have

$$\frac{d}{dt}\left[\left(\int_0^{t+s} - \int_0^t - \int_0^s \right) K_1(t+s-r)\widetilde{S}(r)Cxdr + \int_{t-s}^t K_1(s-t+r)\widetilde{S}(r)Cxdr + \int_0^s K_1(t-s+r)\widetilde{S}(r)Cxdr \right]$$
$$= \left[\left(\int_0^{t+s} - \int_0^t - \int_0^s \right) K_0(t+s-r)\widetilde{S}(r)Cxdr - K_1(s)\tilde{S}(t)Cx \right] + \left[K_1(s)\widetilde{S}(t)Cx - \int_{t-s}^t K_0(s-t+r)\tilde{S}(r)Cxdr \right]$$
$$+ \int_0^s K_0(t-s+r)\tilde{S}(r)Cxdr$$
$$= \left(\int_0^{t+s} - \int_0^t - \int_0^s \right) K_0(t+s-r)\widetilde{S}(r)Cxdr$$
$$+ sgn(s-t) \int_{|t-s|}^t K_0(s-t+r)\widetilde{S}(r)Cxdr + sgn(t-s) \int_{|t-s|}^s K_0(t-s+r)\widetilde{S}(r)Cxdr$$
$$+ \int_0^{|t-s|} K_0(|t-s|+r)\widetilde{S}(r)Cxdr$$

and

$$\frac{d}{dt}\left[\left(\int_0^{t+s} - \int_0^t - \int_0^s \right) K_0(t+s-r)\widetilde{S}(r)Cxdr - \int_{t-s}^t K_0(s-t+r)\tilde{S}(r)Cxdr + \int_0^s K_0(t-s+r)\tilde{S}(r)Cxdr \right]$$
$$+ 2K_0(s)\widetilde{S}(t)Cx$$
$$= \left(\int_0^{t+s} - \int_0^t - \int_0^s \right) K(t+s-r)\widetilde{S}(r)Cxdr - 2K_0(s)\widetilde{S}(t)Cx + \int_{t-s}^t K(s-t+r)\widetilde{S}(r)Cxdr$$
$$+ \int_0^s K(t-s+r)\widetilde{S}(r)Cxdr + 2K_0(s)\widetilde{S}(t)Cx$$
$$= \left(\int_0^{t+s} - \int_0^t - \int_0^s \right) K(t+s-r)\widetilde{S}(r)Cxdr + \int_{t-s}^t K(s-t+r)\widetilde{S}(r)Cxdr + \int_0^s K(t-s+r)\widetilde{S}(r)Cxdr$$
$$= \left(\int_0^{t+s} - \int_0^t - \int_0^s \right) K(t+s-r)\widetilde{S}(r)Cxdr + \int_{|t-s|}^t K(s-t+r)\widetilde{S}(r)Cxdr$$
$$+ \int_{|t-s|}^s K(t-s+r)\widetilde{S}(r)Cxdr + \int_0^{|t-s|} K(|t-s|+r)\widetilde{S}(r)Cxdr.$$

That is, (6) and (7) both hold for all $0 \leq s \leq t < T_0$ with $t+s < T_0$. Similarly, we can show that (6) and (7) both also hold when $0 \leq t \leq s < T_0$ with $t+s < T_0$. Clearly, the right-hand side of (7) is symmetric in t, s with $0 \leq t, s, t+s < T_0$. It follows that

$$\frac{d^2}{ds^2}\left[\left(\int_0^{t+s} - \int_0^t - \int_0^s \right) K_1(t+s-r)\widetilde{S}(r)Cxdr + \int_{|t-s|}^t K_1(s-t+r)\widetilde{S}(r)Cxdr \right.$$
$$\left. + \int_{|t-s|}^s K_1(t-s+r)\widetilde{S}(r)Cxdr + \int_0^{|t-s|} K_1(|t-s|+r)\widetilde{S}(r)Cxdr \right] + 2K_0(t)\widetilde{S}(s)Cx$$

$$
\begin{aligned}
=&\left(\int_0^{t+s}-\int_0^t-\int_0^s\right)K(t+s-r)\widetilde{S}(r)Cxdr+\int_{|t-s|}^t K(s-t+r)\widetilde{S}(r)Cxdr\\
&+\int_{|t-s|}^s K(t-s+r)\widetilde{S}(r)Cxdr+\int_0^{|t-s|}K(|t-s|+r)\widetilde{S}(r)Cxdr
\end{aligned}\tag{8}
$$

for all $x \in$ X and $0 \leq t, s, t+s < T_0$. Using integration by parts twice, we obtain

$$
\begin{aligned}
&\left(\int_0^{t+s}-\int_0^t-\int_0^s\right)K(t+s-r)\widetilde{S}(r)Cxdr+\int_{|t-s|}^t K(s-t+r)\widetilde{S}(r)Cxdr\\
&+\int_{|t-s|}^s K(t-s+r)\widetilde{S}(r)Cxdr+\int_0^{|t-s|}K(|t-s|+r)\widetilde{S}(r)Cxdr\\
=&\left(\int_0^{t+s}-\int_0^t-\int_0^s\right)K_1(t+s-r)C(r)Cxdr+\int_{|t-s|}^t K_1(s-t+r)C(r)Cxdr\\
&+\int_{|t-s|}^s K_1(t-s+r)C(r)Cxdr+\int_0^{|t-s|}K_1(|t-s|+r)C(r)Cxdr\\
&+2K_0(t)\widetilde{S}(s)Cx+2K_0(s)\widetilde{S}(t)Cx
\end{aligned}\tag{9}
$$

for all $x \in X$ and $0 \leq t, s, t+s < T_0$. Suppose that $\widetilde{S}(\cdot)$ is a local K_1-convoluted C-cosine function on X. Then we have by (8)-(9) that

$$
\begin{aligned}
&2\widetilde{S}(t)C(s)x = 2\frac{d^2}{ds^2}\widetilde{S}(t)\widetilde{S}(s)x\\
=&(\int_0^{t+s}-\int_0^t-\int_0^s)K_1(t+s-r)C(r)Cxdr\\
&+\int_{|t-s|}^t K_1(s-t+r)C(r)Cxdr+\int_{|t-s|}^s K_1(t-s+r)C(r)Cxdr\\
&+\int_0^{|t-s|}K_1(|t-s|+r)C(r)Cxdr+2K_0(s)\widetilde{S}(t)Cx
\end{aligned}
$$

for all $x \in X$ and $0 \leq t, s, t+s < T_0$, so that

$$
\begin{aligned}
2C(t)C(s)x =&2\frac{d^2}{dt^2}\widetilde{S}(t)C(s)x\\
=&\left(\int_0^{t+s}-\int_0^t-\int_0^s\right)K(t+s-r)C(r)Cxdr+\int_{|t-s|}^t K(s-t+r)C(r)Cxdr\\
&+\int_{|t-s|}^s K(t-s+r)C(r)Cxdr+\int_0^{|t-s|}K(|t-s|+r)C(r)Cxdr
\end{aligned}\tag{10}
$$

for all $x \in X$ and $0 \leq t, s, t+s < T_0$. Hence, $C(\cdot)$ is a local K-convoluted C-cosine function on X. Conversely, let $C(\cdot)$ be a local K-convoluted C-cosine function on X. We will apply Fubini's theorem for double integrals twice to obtain

$$
\begin{aligned}
2C(t)\widetilde{S}(s)x =&\Bigg[\left(\int_0^{t+s}-\int_0^t-\int_0^s\right)K_1(t+s-r)C(r)Cxdr+\int_{|t-s|}^t K_1(s-t+r)C(r)Cxdr\\
&+\int_{|t-s|}^s K_1(t-s+r)C(r)Cxdr+\int_0^{|t-s|}K_1(|t-s|+r)C(r)Cxdr\Bigg]+2K_0(t)\widetilde{S}(s)Cx
\end{aligned}\tag{11}
$$

for all $x \in X$ and $0 \le t, s, t+s < T_0$. Let $x \in X$ be given, then for $0 \le t, s, t+s < T_0$ with $t \ge s$, we have

$$\begin{aligned}&\int_0^\tau \int_t^{t+\lambda} K(t+\lambda-r)C(r)Cxdrd\lambda \\ =&\int_t^{t+\tau} \int_{r-t}^{\tau} K(t+\lambda-r)C(r)Cxd\lambda dr \\ =&\int_t^{t+\tau} K_0(t+\tau-r)C(r)Cxdr,\end{aligned} \tag{12}$$

$$\begin{aligned}&\int_0^\tau \int_0^{\lambda} K(t+\lambda-r)C(r)Cxdrd\lambda \\ =&\int_0^{\tau} \int_{r}^{\tau} K(t+\lambda-r)C(r)Cxd\lambda dr \\ =&\int_0^{\tau} K_0(t+\tau-r)C(r)Cxdr - K_0(t)S(\tau)Cx,\end{aligned} \tag{13}$$

$$\begin{aligned}&\int_0^\tau \int_{t-\lambda}^{t} K(\lambda-t+r)C(r)Cxdrd\lambda \\ =&\int_{t-\tau}^{t} \int_{t-r}^{\tau} K(\lambda-t+r)C(r)Cxd\lambda dr \\ =&\int_{t-\tau}^{t} K_0(\tau-t+r)C(r)Cxdr,\end{aligned} \tag{14}$$

and

$$\begin{aligned}&\int_0^\tau \int_0^{\lambda} K(t-\lambda+r)C(r)Cxdrd\lambda \\ =&\int_0^{\tau} \int_{r}^{\tau} K(t-\lambda+r)C(r)Cxd\lambda dr \\ =&K_0(t)S(\tau)Cx - \int_0^{\tau} K_0(t-\tau+r)C(r)Cxdr\end{aligned} \tag{15}$$

for all $0 \le \tau \le s$. Observe that (12)-(15) also imply

$$\int_0^s \int_t^{t+\tau} K_0(t+\tau-r)C(r)Cxdrd\tau = \int_t^{t+s} K_1(t+s-r)C(r)Cxdr, \tag{16}$$

$$\begin{aligned}&\int_0^s \left[\int_0^{\tau} K_0(t+\tau-r)C(r)Cxdr - K_0(t)S(\tau)Cx\right] d\tau \\ =&\left[\int_0^{s} K_1(t+s-r)C(r)Cxdr - K_1(t)S(s)Cx\right] - K_0(t)\widetilde{S}(s)Cx,\end{aligned} \tag{17}$$

$$\int_0^s \int_{t-\tau}^{t} K_0(\tau-t+r)C(r)Cxdrd\tau = \int_{t-s}^{t} K_1(s-t+r)C(r)Cxdr, \tag{18}$$

and

$$\begin{aligned}&\int_0^s [K_0(t)S(\tau)Cx - \int_0^\tau K_0(t-\tau+r)C(r)Cxdr]d\tau\\ =&K_0(t)\widetilde{S}(s)Cx + [\int_0^s K_1(t-s+r)C(r)Cxdr - K_1(t)S(s)Cx].\end{aligned} \tag{19}$$

Combining (16)-(17), we obtain (11) for all $0 \le t, s, t+s < T_0$ with $t \ge s$. Similarly, we can show that (11) also holds when $0 \le t, s, t+s < T_0$ with $s \ge t$. By (7), (9) and (11), we have

$$\begin{aligned}&2C(t)\widetilde{S}(s)x\\ =&\frac{d^2}{dt^2}[(\int_0^{t+s} - \int_0^t - \int_0^s)K_1(t+s-r)\widetilde{S}(r)Cxdr\\ &+ \int_{|t-s|}^t K_1(s-t+r)\widetilde{S}(r)Cxdr + \int_{|t-s|}^s K_1(t-s+r)\widetilde{S}(r)Cxdr\\ &+ \int_0^{|t-s|} K_1(|t-s|+r)\widetilde{S}(r)Cxdr]\end{aligned}$$

for all $x \in X$ and $0 \le t, s, t+s < T_0$. Combining this and (6) with $t = 0$, we conclude that $\widetilde{S}(\cdot)$ is a local K_1-convoluted C-cosine function on X. □

Theorem 2.2 *Let $C(\cdot)$ be a strongly continuous subfamily of $L(X)$ which commutes with C on X. Then $C(\cdot)$ is a local K-convoluted C-cosine function on X if and only if $\widetilde{S}(t)[\,C(s) - K_0(s)C\,] = [\,C(t) - K_0(t)C\,]\widetilde{S}(s)$ for all $0 \le t, s, t+s < T_0$.*

Proof. Let $C(\cdot)$ be a local K-convoluted C-cosine function on X. By (7) and (8), we have $2C(t)\widetilde{S}(s)x + 2K_0(s)\widetilde{S}(t)Cx = 2\widetilde{S}(t)C(s)x + 2K_0(t)\widetilde{S}(s)Cx$ for all $x \in X$ and $0 \le t, s, t+s < T_0$ or equivalently, $\widetilde{S}(t)[C(s) - K_0(s)C] = [C(t) - K_0(t)C]\widetilde{S}(s)$ for all $0 \le t, s, t+s < T_0$. Conversely, suppose that (5) holds for all $0 \le t, s, t+s < T_0$. Then $\widetilde{S}(t)C(s)x - C(t)\widetilde{S}(s)x = K_0(s)\widetilde{S}(t)Cx - K_0(t)\widetilde{S}(s)Cx$ for all $x \in X$ and $0 \le t, s, t+s < T_0$. Fix $x \in X$ and $0 \le t, s, t+s < T_0$ with $t \ge s$. Then we have

$$\begin{aligned}&\widetilde{S}(t+s-r)C(r)x - C(t+s-r)\widetilde{S}(r)x\\ =&K_0(r)\widetilde{S}(t+s-r)Cx - K_0(t+s-r)\widetilde{S}(r)Cx\end{aligned} \tag{20}$$

for all $0 \le r \le t$, and

$$\begin{aligned}&\widetilde{S}(s-t+r)C(r)x - C(s-t+r)\widetilde{S}(r)x\\ =&K_0(r)\widetilde{S}(s-t+r)Cx - K_0(s-t+r)\widetilde{S}(r)Cx\end{aligned} \tag{21}$$

for all $t - s \le r \le t$. Using integration by parts to left-hand sides of the integrations of (20)-(21) and change of variables to right-hand sides of the integrations of (20)-(21), we obtain

$$\widetilde{S}(s)S(t)x + S(s)\widetilde{S}(t)x = \left(\int_0^{t+s} - \int_0^t - \int_0^s\right)K_0(t+s-r)\widetilde{S}(r)Cxdr$$

and

$$\widetilde{S}(s)S(t)x - S(s)\widetilde{S}(t)x = \int_0^s K_0(t-s+r)\widetilde{S}(r)Cxdr - \int_{t-s}^t K_0(s-t+r)\widetilde{S}(r)Cxdr,$$

so that

$$2\widetilde{S}(s)S(t)x = \left(\int_0^{t+s} - \int_0^t - \int_0^s\right) K_0(t+s-r)\widetilde{S}(r)Cxdr - \int_{t-s}^t K_0(s-t+r)\widetilde{S}(r)Cxdr + \int_0^s K_0(t-s+r)\widetilde{S}(r)Cxdr.$$

Hence,

$$2\widetilde{S}(s)C(t)x = \left(\int_0^{t+s} - \int_0^t - \int_0^s\right) K(t+s-r)\widetilde{S}(r)Cxdr + \int_{t-s}^t K(s-t+r)\widetilde{S}(r)Cxdr + \int_0^s K(t-s+r)\widetilde{S}(r)Cxdr - 2K_0(s)\widetilde{S}(t)Cx,$$

which implies that

$$\begin{aligned} 2\widetilde{S}(s)C(t)x + 2K_0(s)\widetilde{S}(t)Cx = & \left(\int_0^{t+s} - \int_0^t - \int_0^s\right) K(t+s-r)\widetilde{S}(r)Cxdr \\ & + \int_{|t-s|}^t K(s-t+r)\widetilde{S}(r)Cxdr + \int_{|t-s|}^s K(t-s+r)\widetilde{S}(r)Cxdr \\ & + \int_0^{|t-s|} K(|t-s|+r)\widetilde{S}(r)Cxdr. \end{aligned} \tag{22}$$

Similarly, we can show that (22) also holds when $x \in X$ and $0 \leq t, s, t+s < T_0$ with $s \geq t$. Combining this with (7), we have

$$\begin{aligned} 2\widetilde{S}(s)C(t)x = & \frac{d^2}{dt^2}\Bigg[\left(\int_0^{t+s} - \int_0^t - \int_0^s\right) K_1(t+s-r)\widetilde{S}(r)Cxdr + \int_{|t-s|}^t K_1(s-t+r)\widetilde{S}(r)Cxdr \\ & + \int_{|t-s|}^s K_1(t-s+r)\widetilde{S}(r)Cxdr + \int_0^{|t-s|} K_1(|t-s|+r)\widetilde{S}(r)Cxdr\Bigg]. \end{aligned}$$

for all $x \in X$ and $0 \leq t, s, t+s < T_0$. Consequently, $\widetilde{S}(\cdot)$ is a local K_1-convoluted C-cosine function on X. Combining this with Lemma 2.1, we get that $C(\cdot)$ is a local K-convoluted C-cosine function on X. □

By slightly modifying the proof of [16, Proposition 2.3], the next result concerning local K-convoluted C-cosine functions on X is also attained.

Proposition 2.3 *Let $C(\cdot)$ be a local K-convoluted C-cosine function on X and $a \in L^1_{loc}([0, T_0), \mathbb{F})$. Then $a * C(\cdot)$ is a local $a * K$-convoluted C-cosine function on X. In particular, for each $\beta > -1$ $j_\beta * C(\cdot)$ is a local K_β-convoluted C-cosine function on X. Moreover, $C(\cdot)$ is a local K-convoluted C-cosine function on X if it is a strongly continuous subfamily of $L(X)$ such that $S(\cdot)$ is a local K_0-convoluted C-cosine function on X.*

Definition 2.4 *Let $C(\cdot)$ be a strongly continuous subfamily of $L(X)$. A linear operator A in X is called a subgenerator of $C(\cdot)$ if*

$$C(t)x - K_0(t)Cx = \int_0^t \int_0^s C(r)Axdrds \tag{23}$$

for all $x \in D(A)$ and $0 \leq t < T_0$, and

$$\int_0^t \int_0^s C(r)xdrds \in D(A) \quad \text{and } A\int_0^t \int_0^s C(r)xdrds = C(t)x - K_0(t)Cx \tag{24}$$

for all $x \in X$ and $0 \leq t < T_0$. A subgenerator A of $C(\cdot)$ is called the maximal subgenerator of $C(\cdot)$ if it is an extension of each subgenerator of $C(\cdot)$ to $D(A)$.

Applying Theorem 2.2, we can obtain the next result concerning the generation of a local K-convoluted C-cosine function $C(\cdot)$ on X, which had been proven in [8] by another method similar to that already employed in [12] in the case that $C(\cdot)$ has a closed subgenerator and C is injective.

Theorem 2.5 *Let $C(\cdot)$ be a strongly continuous subfamily of $L(X)$ which commutes with C on X. Assume that $C(\cdot)$ has a subgenerator. Then $C(\cdot)$ is a local K-convoluted C-cosine function on X. Moreover, $C(\cdot)$ is nondegenerate if the injectivity of C is added and K_0 is a non-zero function on $[0, T_0)$.*

Proof. Let A be a subgenerator of $C(\cdot)$. By (24), we have

$$[C(t) - K_0(t)C]\widetilde{S}(\cdot)x = \widetilde{S}(t)A\widetilde{S}(\cdot)x = \widetilde{S}(t)[C(\cdot) - K_0(\cdot)C]x$$

on $[0, T_0 - t)$ for all $x \in X$ and $0 \leq t < T_0$. Applying Theorem 2.2, we get that $C(\cdot)$ is a local K-convoluted C-cosine function on X. Suppose that C is injective, K_0 is a non-zero function, $x \in X$ and $C(t)x = 0$, $t \in [0, T_0)$. By (24), we have $K_0(\cdot)Cx = 0$ on $[0, T_0)$, and so $Cx = 0$. Hence, $x = 0$, which implies that $C(\cdot)$ is nondegenerate. □

Lemma 2.6 *Let A be a closed subgenerator of a strongly continuous subfamily $C(\cdot)$ of $L(X)$, and K_0 a kernel on $[0, t_0)$ (or equivalently, K is a kernel on $[0, t_0)$) for some $0 < t_0 \leq T_0$. Assume that C is injective and $u \in C([0, t_0), X)$ satisfies $u(\cdot) = Aj_1 * u(\cdot)$ on $[0, t_0)$. Then $u = 0$ on $[0, t_0)$.*

Proof. We observe from (23) and (24) that $A\int_0^t\int_0^s C(r)xdrds = \int_0^t\int_0^s C(r)Axdrds$ for all $x \in D(A)$ and $0 \leq t < T_0$. Combining this with the closedness of A, we have $C(t)Ax = AC(t)x$ for all $x \in D(A)$ and $0 \leq t < T_0$, and so $\int_0^t C(t-s)u(s)ds = \int_0^t C(t-s)Aj_1 * u(s)ds = \int_0^t AC(t-s)j_1 * u(s)ds = A\int_0^t C(t-s)j_1 * u(s)ds = A\widetilde{S} * u(t) = \int_0^t C(t-s)u(s)ds - C\int_0^t K_0(t-s)u(s)ds$ for all $0 \leq t < t_0$. Hence, $\int_0^t K_0(t-s)u(s)ds = 0$ for all $0 \leq t < t_0$, which implies that $u(t) = 0$ for all $0 \leq t < t_0$. □

Theorem 2.7 *Let $C(\cdot)$ be a nondegenerate local K-convoluted C-cosine function on X with generator A. Assume that $C(\cdot)$ has a subgenerator. Then A is the maximal subgenerator of $C(\cdot)$, and each subgenerator of $C(\cdot)$ is closable and its closure is also a subgenerator of $C(\cdot)$. Moreover, if C is injective. Then (1)-(3) hold, and (4) also holds when K_0 is a kernel on $[0, T_0)$ or $T_0 = \infty$.*

Proof. Let B be a subgenerator of $C(\cdot)$. Clearly, $B \subset A$. It follows that $C(t)z - K_0(t)Cz = B\int_0^t\int_0^s C(r)zdrds = A\int_0^t\int_0^s C(r)zdrds$ for all $z \in X$ and $0 \leq t < T_0$, which together with the definition of A implies that A is also a subgenerator of $C(\cdot)$. To show that each subgenerator of $C(\cdot)$ is closable and its closure is also a subgenerator of $C(\cdot)$. We will show that B is closable. Let $x_k \in D(B)$, $x_k \to 0$, and $Bx_k \to y$ in X. Then $x_k \in D(A)$ and $Ax_k = Bx_k \to y$. By the closedness of A, we have $y = 0$. In order to show that $\overline{B}$ is a subgenerator of $C(\cdot)$. Let $x \in D(\overline{B})$ be given, then $x_k \to x$ and $Bx_k \to \overline{B}x$ in X for sequence $\{x_k\}_{k=1}^{\infty}$ in $D(B)$. By (23), we have $C(t)x_k - K_0(t)Cx_k = \int_0^t\int_0^s C(r)Bx_kdrds$ for all $k \in \mathbb{N}$ and $0 \leq t < T_0$. Letting $k \to \infty$, we get that $C(t)x - K_0(t)Cx = \int_0^t\int_0^s C(r)\overline{B}xdrds$ for all $0 \leq t < T_0$. Since $B \subset \overline{B}$, we also have $C(t)z - K_0(t)Cz = B\int_0^t\int_0^s C(r)zdrds = \overline{B}\int_0^t\int_0^s C(r)zdrds$ for all $z \in X$ and $0 \leq t < T_0$. Consequently, the closure of B is a subgenerator of $C(\cdot)$. To show that A is the maximal subgenerator of $C(\cdot)$. We will apply Zorn's lemma to show that $C(\cdot)$ has a subgenerator which does not have a proper extension that is still a subgenerator of $C(\cdot)$. To do this. Let $\mathcal{F}$ be the family of all subgenerators of $C(\cdot)$. We define a partial order "$\subset$" on $\mathcal{F}$ by $f \subset g$ if g is an extension of f to $D(g)$. Suppose that $\mathcal{A}$ is a chain of $\mathcal{F}$. Define $A_0 : D(A_0) \subset X \to X$ by $D(A_0) = \cup_{f \in \mathcal{A}} D(f)$ and $A_0x = fx$ whenever $x \in D(A_0)$ with $x \in D(f)$ for some $f \in \mathcal{A}$, then A_0 is well-defined and a subgenerator of $C(\cdot)$, and so A_0 is an upper bound of $\mathcal{A}$ in $(\mathcal{F}, \subset)$. By Zorn's lemma, $(\mathcal{F}, \subset)$ has a maximal element B which is a subgenerator of $C(\cdot)$, and does not have a proper extension that is still a subgenerator of $C(\cdot)$. In particular, $B \subset A$. Similarly, we can show that B is the maximal subgenerator of $C(\cdot)$, which implies that $A \subset B$. Clearly, (2) and (3) both hold because A is the maximal subgenerator of $C(\cdot)$. To show that (1) holds when C is injective, we will show that $A \subset C^{-1}AC$ or equivalently, $CA \subset AC$. Let $x \in D(A)$ be given, then $K_2(t)Cx = \widetilde{S}(t)x - j_1 * \widetilde{S}(t)Ax \in D(A)$ and

$$\begin{aligned} AK_2(t)Cx =& A\widetilde{S}(t)x - Aj_1 * \widetilde{S}(t)Ax \\ =& A\widetilde{S}(t)x - [\widetilde{S}(t)Ax - K_2(t)CAx] \\ =& K_2(t)CAx \end{aligned}$$

for all $0 \le t < T_0$, so that $CAx = ACx$. Hence, $CA \subset AC$. In order to show that $C^{-1}AC \subset A$. Let $x \in D(C^{-1}AC)$ be given, then $Cx \in D(A)$ and $ACx \in R(C)$. By the definition of generator and the commutativity of C with $C(\cdot)$, we have $C[C(t)x - K_0(t)Cx] = C(t)Cx - K_0(t)C^2x = \int_0^t S(r)ACxdr = \int_0^t S(r)CC^{-1}ACxdr = C\int_0^t S(r)C^{-1}ACxdr$. Since C is injective, we have $x \in D(A)$ and $Ax = C^{-1}ACx$. Consequently, $A \subset C^{-1}AC$. Finally, we will show that (4) holds when K_0 is a kernel on $[0, T_0)$. Clearly, it suffices to show that $\widetilde{S}(t)\widetilde{S}(s)x = \widetilde{S}(s)\widetilde{S}(t)x$ for all $x \in X$ and $0 \le t, s < T_0$. Let $x \in X$ and $0 \le s < T_0$ be given. By (3) and the closedness of A, we have

$$\begin{aligned} \widetilde{S}(\cdot)\widetilde{S}(s)x - Aj_1 * \widetilde{S}(\cdot)\widetilde{S}(s)x =& K_2(\cdot)C\widetilde{S}(s)x \\ =& \widetilde{S}(s)K_2(\cdot)Cx \\ =& \widetilde{S}(s)[\widetilde{S}(\cdot)x - Aj_1 * \widetilde{S}(\cdot)x] \\ =& \widetilde{S}(s)\widetilde{S}(\cdot)x - \widetilde{S}(s)Aj_1 * \widetilde{S}(\cdot)x \\ =& \widetilde{S}(s)\widetilde{S}(\cdot)x - Aj_1 * \widetilde{S}(s)\widetilde{S}(\cdot)x \end{aligned}$$

on $[0, T_0)$, and so $[\widetilde{S}(\cdot)\widetilde{S}(s)x - \widetilde{S}(s)\widetilde{S}(\cdot)x] = Aj_1 * [\widetilde{S}(\cdot)\widetilde{S}(s)x - \widetilde{S}(s)\widetilde{S}(\cdot)x]$ on $[0, T_0)$. Hence, $\widetilde{S}(\cdot)\widetilde{S}(s)x = \widetilde{S}(s)\widetilde{S}(\cdot)x$ on $[0, T_0)$, which implies that $\widetilde{S}(t)\widetilde{S}(s)x = \widetilde{S}(s)\widetilde{S}(t)x$ for all $0 \le t, s < T_0$. □

Lemma 2.8 *Let $C(\cdot)$ be a local K-convoluted C-cosine function on X, and $0 \in suppK_0$ (the support of K_0). Assume that $C(\cdot)x = 0$ on $[0, t_0)$ for some $x \in X$ and $0 < t_0 < T_0$. Then $CC(\cdot)x = 0$ on $[0, T_0)$. In particular, $C(t)x = 0$ for all $0 \le t < T_0$ if the injectivity of C is added.*

Proof. Let $0 \le t < T_0$ be given, then $t + s < T_0$ and $K_0(s)$ is nonzero for some $0 < s < t_0$, so that $\widetilde{S}(s)C(t)x = C(t)\widetilde{S}(s)x = 0$, $C(s)\widetilde{S}(t)x = \widetilde{S}(t)C(s)x = 0$ and $\widetilde{S}(s)K_0(t)Cx = K_0(t)C\widetilde{S}(s)x = 0$. By Theorem 2.2, we have $K_0(s)\widetilde{S}(t)Cx = K_0(s)C\widetilde{S}(t)x = 0$. Hence, $\widetilde{S}(t)Cx = 0$. Since $0 \le t < T_0$ is arbitrary, we have $CC(t)x = C(t)Cx = 0$ for all $0 \le t < T_0$. In particular, $C(t)x = 0$ for all $0 \le t < T_0$ if the injectivity of C is added. □

Theorem 2.9 *Let $C(\cdot)$ be a local K-convoluted C-cosine function on X, and $0 \in suppK_0$. Assume that C is injective. Then $C(\cdot)$ is nondegenerate if and only if it has a subgenerator.*

Proof. By Theorem 2.5, we need only to show that A is a subgenerator of $C(\cdot)$ when $C(\cdot)$ is a nondegenerate local K-convoluted C-cosine function on X with generator A and $0 \in \mathrm{supp}K_0$. Observe (23)-(24) and the definition of A, we need only to show that (23) holds. Let $0 \le t_0 < T_0$ be fixed. Then for each $x \in X$ and $0 \le s < T_0$, we set $y = \widetilde{S}(t_0)x$. By Theorem 2.2, we have

$$\begin{aligned} \widetilde{S}(r)[C(s) - K_0(s)C]y =& [C(r) - K_0(r)C]\widetilde{S}(s)y \\ =& \widetilde{S}(s)[C(r) - K_0(r)C]y \\ =& \widetilde{S}(s)([C(r) - K_0(r)C]\widetilde{S}(t_0)x) \\ =& \widetilde{S}(s)(\widetilde{S}(r)[C(t_0) - K_0(t_0)C]x) \\ =& [\widetilde{S}(s)\widetilde{S}(r)][C(t_0) - K_0(t_0)C]x \\ =& \widetilde{S}(r)\widetilde{S}(s)[C(t_0) - K_0(t_0)C]x \end{aligned}$$

for all $0 \le r < T_0$ with $r+s, r+t_0 < T_0$ or equivalently, $C(r)[C(s) - K_0(s)C]y = C(r)\widetilde{S}(s)[C(t_0) - K_0(t_0)C]x$ for all $0 \le r < T_0$ with $r+s, r+t_0 < T_0$. It follows from Lemma 2.8 and the nondegeneracy of $C(\cdot)$ that we have $[C(s) - K_0(s)C]y = \widetilde{S}(s)[C(t_0) - K_0(t_0)C]x$. Since $0 \le s < T_0$ is arbitrary, we have $y \in D(A)$ and $Ay = [C(t_0) - K_0(t_0)C]x$. Since $0 \le t_0 < T_0$ is arbitrary, we conclude that (23) holds. □

By slightly modifying the proof of Theorem 2.9, we can obtain the next result concerning nondegenerate K-convoluted C-cosine functions.

Theorem 2.10 *Let $C(\cdot)$ be a nondegenerate K-convoluted C-cosine function on X. Then C is injective, and $C(\cdot)$ has a subgenerator.*

Combining Theorem 2.10 with Theorem 2.7, the next result concerning nondegenerate K-convoluted C-cosine functions is also obtained.

Theorem 2.11 *Let $C(\cdot)$ be a nondegenerate K-convoluted C-cosine function on X with generator A. Then A is the maximal subgenerator of $C(\cdot)$, and each subgenerator of $C(\cdot)$ is closable and its closure is also a subgenerator of $C(\cdot)$. Moreover, (1)-(4) hold.*

Since $0 \in \operatorname{supp} K_0$ implies that K_0 is a kernel on $[0, T_0)$, we can apply Theorems 2.7 and 2.9 to obtain the next corollary.

Corollary 2.12 *Let $C(\cdot)$ be a nondegenerate local K-convoluted C-cosine function on X with generator A, and $0 \in \operatorname{supp} K_0$. Assume that C is injective. Then A is the maximal subgenerator of $C(\cdot)$, and each subgenerator of $C(\cdot)$ is closable and its closure is also a subgenerator of $C(\cdot)$. Moreover, (1)-(4) hold.*

Theorem 2.13 *Let A be a closed subgenerator of a strongly continuous sufamily $C(\cdot)$ of $L(X)$, and K_0 a kernel on $[0, T_0)$. Assume that C is injective. Then $CA \subset AC$, and $C(\cdot)$ is a nondegenerate local K-convoluted C-cosine function on X with generator $C^{-1}AC$. In particular, $C^{-1}\overline{A_0}C$ is the generator of $C(\cdot)$ for each subgenerator A_0 of $C(\cdot)$.*

Proof. To show that $C(\cdot)$ is a nondegenerate local K-convoluted C-cosine function on X. By Theorem 2.5, we need only to show that $CC(\cdot) = C(\cdot)C$ or equivalently, $C\widetilde{S}(\cdot) = \widetilde{S}(\cdot)C$. Just as in the proof of Theorem 2.7, we have $CA \subset AC$ and $[\widetilde{S}(\cdot)Cx - C\widetilde{S}(\cdot)x] = Aj_1 * [\widetilde{S}(\cdot)Cx - C\widetilde{S}(\cdot)x]$ on $[0, T_0)$. By Lemma 2.6, we also have $\widetilde{S}(\cdot)Cx = C\widetilde{S}(\cdot)x$ on $[0, T_0)$. We will prove that $C^{-1}AC$ is the generator of $C(\cdot)$. Let B denote the generator of $C(\cdot)$. By Theorem 2.7, we have $A \subset B$. By (1), we also have $C^{-1}AC \subset C^{-1}BC = B$. Conversely, let $x \in D(B)$ be given, then $K_2(t)Cx = \widetilde{S}(t)x - j_1 * \widetilde{S}(t)Bx \in D(A)$ for all $0 \le t < T_0$, so that $Cx \in D(A)$ and

$$
\begin{aligned}
AK_2(\cdot)Cx &= A\widetilde{S}(\cdot)x - Aj_1 * \widetilde{S}(\cdot)Bx \\
&= A\widetilde{S}(\cdot)x - [\widetilde{S}(\cdot)Bx - K_2(\cdot)CBx] \\
&= A\widetilde{S}(\cdot)x - [B\widetilde{S}(\cdot)x - K_2(\cdot)CBx] \\
&= K_2(\cdot)CBx
\end{aligned}
$$

on $[0, T_0)$. Hence, $ACx = CBx \in R(C)$, which implies that $x \in D(C^{-1}AC)$ and $C^{-1}ACx = Bx$. Consequently, $B \subset C^{-1}AC$. □

Corollary 2.14 *Let $C(\cdot)$ be a nondegenerate local K-convoluted C-cosine function on X, and $0 \in \operatorname{supp} K_0$. Assume that C is injective. Then $C^{-1}\overline{A_0}C$ is the generator of $C(\cdot)$ for each subgenerator A_0 of $C(\cdot)$.*

Remark 2.15 *Let $C(\cdot)$ be a local K-convoluted C-cosine function on X. Then*

(i) *$C(\cdot)$ is nondegenerate if and only if $S(\cdot)$ is;*
(ii) *A is the generator of $C(\cdot)$ if and only if it is the generator of $S(\cdot)$;*
(iii) *A is a closed subgenerator of $C(\cdot)$ if and only if it is a closed subgenerator of $S(\cdot)$.*

Remark 2.16 *A strongly continuous subfamily of $L(X)$ may not have a subgenerator; a local K-convoluted C-cosine function on X is degenerate when it has a subgenerator but does not have a maximal subgenerator; and a closed linear operator in X generates at most one nondegenerate local K-convoluted C-cosine function on X when C is injective and K_0 a kernel on $[0, T_0)$.*

3. Abstract Cauchy Problems

In the following, we always assume that $C \in L(X)$ is injective, K_0 a kernel on $[0, T_0)$, and A a closed linear operator in X such that $CA \subset AC$. We also note some basic properties concerning the strong solutions of ACP(A, f, x, y) just results in [12] when A is the generator of a nondegenerate (local) α-times integrated C-cosine function on X.

Proposition 3.1. *Let A be a subgenerator of a nondegenerate local K_0-convoluted C-cosine function $C(\cdot)$ on X. Then for each $x \in D(A)$ $C(\cdot)x$ is the unique solution of ACP$(A, K(\cdot)Cx, 0, 0)$ in $C([0, T_0), [D(A)])$. Here $[D(A)]$ denotes the Banach space $D(A)$ equipped with the graph norm $|x|_A = \|x\| + \|Ax\|$ for $x \in D(A)$.*

Proposition 3.2. *Let A be a subgenerator of a nondegenerate local K-convoluted C-cosine function $C(\cdot)$ on X and $C^1 = \{x \in X \,|\, C(\cdot)x$ is continuously differentiable on $(0, T_0)\}$. Then*

(i) *for each $x \in C^1$ $S(t)x \in D(A)$ for a.e. $t \in (0, T_0)$;*
(ii) *for each $x \in C^1$ $S(\cdot)x$ is the unique solution of ACP$(A, K(\cdot)Cx, 0, 0)$;*
(iii) *for each $x \in D(A)$ $S(\cdot)x$ is the unique solution of ACP$(A, K(\cdot)Cx, 0, 0)$ in $C^1([0, T_0), [D(A)])$.*

Proposition 3.3. *Let A be the generator of a nondegenerate local K-convoluted C-cosine function $C(\cdot)$ on X and $x \in X$. Assume that $C(t)x \in R(C)$ for all $0 \le t < T_0$, and $C^{-1}C(\cdot)x \in C([0, T_0), X)$ is differentiable a.e. on $(0, T_0)$. Then $C^{-1}S(t)x \in D(A)$ for a.e. $t \in (0, T_0)$, and $C^{-1}S(\cdot)x$ is the unique solution of ACP$(A, K(\cdot)x, 0, 0)$.*

Proof. Clearly, $S(\cdot)x \in C^1([0, T_0), X)$, and $C(\cdot)x = CC^{-1}C(\cdot)x$ is differentiable a.e. on $(0, T_0)$. By Theorem 2.11, we have $C\frac{d^2}{dt^2}C^{-1}S(t)x = \frac{d^2}{dt^2}S(t)x = AS(t)x + K(t)Cx = ACC^{-1}S(t)x + K(t)Cx$ for a.e. $t \in (0, T_0)$, so that for a.e. $t \in (0, T_0)$, $C^{-1}S(t)x \in D(C^{-1}AC) = D(A)$ and $\frac{d^2}{dt^2}C^{-1}S(t)x = (C^{-1}AC)C^{-1}S(t)x + K(t)x = AC^{-1}S(t)x + K(t)x$. Hence, $C^{-1}S(\cdot)x$ is a solution of ACP$(A, K(\cdot)x, 0, 0)$. □

Applying Theorem 2.13, we can prove an important result concerning the relation between the generation of a nondegenerate local K-convoluted C-cosine function on X with subgenerator A and the unique existence of strong solutions of ACP(A, f, x, y), which has been established by the author in [15] when $K = j_{\alpha-1}$, in [12] when $K = j_{\alpha-1}$ with $T_0 = \infty$, and in [11] when $K = j_{-1}$ and $T_0 = \infty$.

Theorem 3.4. *The following statements are equivalent :*

(i) A is a subgenerator of a nondegenerate local K-convoluted C-cosine function $C(\cdot)$ on X;
(ii) for each $x \in X$ and $g \in L^1_{loc}([0, T_0), X)$ the problem ACP$(A, K_0(\cdot)Cx + K_0 * Cg(\cdot), 0, 0)$ has a unique solution in $C^2([0, T_0), X) \cap C([0, T_0), [D(A)])$;
(iii) for each $x \in X$ the problem ACP$(A, K_0(\cdot)Cx, 0, 0)$ has a unique solution in $C^2([0, T_0), X) \cap C([0, T_0), [D(A)])$;
(vi) for each $x \in X$ the integral equation $v(\cdot) = Aj_1 * v(\cdot) + K_0(\cdot)Cx$ has a unique solution $v(\cdot; x)$ in $C([0, T_0), X)$.

In this case, $\widetilde{S}(\cdot)x + \widetilde{S} * g(\cdot)$ is the unique solution of ACP$(A, K_0(\cdot)Cx + K_0 * Cg(\cdot), 0, 0)$ and $v(\cdot; x) = C(\cdot)x$.

Proof. We will prove that (*i*) implies (*ii*). Let $x \in X$ and $g \in L^1_{loc}([0, T_0), X)$ be given. We set $u(\cdot) = \widetilde{S}(\cdot)x + \widetilde{S} * g(\cdot)$, then $u \in C^2([0, T_0), X) \cap C([0, T_0), [D(A)])$, $u(0) = u'(0) = 0$, and

$$\begin{aligned} Au(t) =& A\widetilde{S}(t)x + A\int_0^t \widetilde{S}(t-s)g(s)ds \\ =& C(t)x - K_0(t)Cx + \int_0^t [C(t-s) - K_0(t-s)C]g(s)ds \\ =& C(t)x + \int_0^t C(t-s)g(s)ds - [K_0(t)Cx + K_0 * Cg(t)] \\ =& u''(t) - [K_0(t)Cx + K_0 * Cg(t)] \end{aligned}$$

for all $0 \le t < T_0$. Hence, u is a solution of $ACP(A, K_0(\cdot)Cx + K_0 * Cg(\cdot), 0, 0)$ in $C^2([0, T_0), X) \cap C([0, T_0), [D(A)])$. The uniqueness of solutions for $ACP(A, K_0(\cdot)Cx + K_0 * Cg(\cdot), 0, 0)$ follows directly from the uniqueness of solutions for $ACP(A, 0, 0, 0)$. Clearly, "$(ii) \Rightarrow (iii)$" holds, and (iii) and (iv) both are equivalent. We remain only to show that "$(iv) \Rightarrow (i)$" holds. Let $C(t) : X \to X$ be defined by $C(t)x = v(t; x)$ for all $x \in X$ and $0 \le t < T_0$. Clearly, $C(\cdot)$ is strongly continuous, and satisfies (24). Combining the uniqueness of solutions for the integral equation $v(\cdot) = Aj_1 * v(\cdot) + K_0(\cdot)Cx$ with the assumption $CA \subset AC$, we have $v(\cdot; Cx) = Cv(\cdot; x)$ for each $x \in X$, which implies that $C(t)$ for $0 \le t < T_0$ are linear, and commute with C. Let $\{t_k\}_{k=1}^{\infty}$ be an increasing sequence in $(0, T_0)$ such that $t_k \to T_0$, and $C([0, T_0), X)$ a Frechet space with the quasi-norm $|\cdot|$ defined by $|v| = \sum_{k=1}^{\infty} \frac{\|v\|_k}{2^k(1+\|v\|_k)}$ for $v \in C([0, T_0), X)$. Here $\|v\|_k = \max_{t \in [0, t_k]} \|v(t)\|$ for all $k \in \mathbb{N}$. To show that $C(\cdot)$ is a subfamily of $L(X)$, we need only to show that the linear map $\eta : X \to C([0, T_0), X)$ defined by $\eta(x) = v(\cdot; x)$ for $x \in X$, is continuous or equivalently, $\eta : X \to C([0, T_0), X)$ is a closed linear operator. Let $\{x_k\}_{k=1}^{\infty}$ be a sequence in X such that $x_k \to x$ in X and $\eta(x_k) \to v$ in $C([0, T_0), X)$, then $v(\cdot; x_k) = Aj_1 * v(\cdot; x_k) + K_0(\cdot)Cx_k$ on $[0, T_0)$. Combining the closedness of A with the uniform convergence of $\{\eta(x_k)\}_{k=1}^{\infty}$ on $[0, t_k]$, we have $v(\cdot) = Aj_1 * v(\cdot) + K_0(\cdot)Cx$ on $[0, T_0)$. By the uniqueness of solutions for integral equations, we have $v(\cdot) = v(\cdot; x) = \eta(x)$. Consequently, $\eta : X \to C([0, T_0), X)$ is a closed linear operator. To show that A is a subgenerator of $C(\cdot)$, we remain only to show that $\widetilde{S}(t)A \subset A\widetilde{S}(t)$ for all $0 \le t < T_0$. Let $x \in D(A)$ be given, then $\widetilde{S}(t)x - K_2(t)Cx = Aj_1 * \widetilde{S}(t)x = j_1 * A\widetilde{S}(t)x$ for all $0 \le t < T_0$, and so

$$\begin{aligned}\widetilde{S}(t)Ax - Aj_1 * \widetilde{S}(t)Ax &= K_2(t)CAx \\ &= AK_2(t)Cx \\ &= A\widetilde{S}(t)x - Aj_1 * \widetilde{S}(t)Ax\end{aligned}$$

for all $0 \le t < T_0$. Hence, $Aj_1 * [\widetilde{S}(\cdot)Ax - A\widetilde{S}(\cdot)x] = \widetilde{S}(\cdot)Ax - A\widetilde{S}(\cdot)x$ on $[0, T_0)$. By the uniqueness of solutions for $ACP(A, 0, 0, 0)$, we have $\widetilde{S}(\cdot)Ax = A\widetilde{S}(\cdot)x$ on $[0, T_0)$. Applying Theorem 2.5, we get that $C(\cdot)$ is a nondegenerate local K-convoluted C-cosine function on X with subgenerator A. □

By slightly modifying the proof of [15, Theorem 3.5], we can apply Theorem 3.4 to obtain the next result.
Theorem 3.5. *Assume that $R(C) \subset R(\lambda - A)$ for some $\lambda \in \mathbb{F}$, and $ACP(A, K(\cdot)x, 0, 0)$ has a unique solution in $C([0, T_0), [D(A)])$ for each $x \in D(A)$ with $(\lambda - A)x \in R(C)$. Then A is a subgenerator of a nondegenerate local K_0-convoluted C-cosine function on X.*

Proof. Clearly, it suffices to show that for each $x \in X$ the integral equation

$$v(\cdot) = A \int_0^{\cdot} \int_0^s v(r)drds + K_1(\cdot)Cx \tag{25}$$

has a (unique) solution $v(\cdot; x)$ in $C([0, T_0), X)$. Let $x \in X$ be given, then there exists a $y_x \in D(A)$ such that $(\lambda - A)y_x = Cx$. By hypothesis, $ACP(A, K(\cdot)y_x, 0, 0)$ has a unique solution $u(\cdot; y_x)$ in $C([0, T_0), [D(A)])$. In particular, $u''(\cdot; y_x) = Au(\cdot; y_x) + K(\cdot)y_x \in L^1_{loc}([0, T_0), X)$. By the closedness of A and the continuity of $Au(\cdot; y_x)$, we have $\int_0^t \int_0^s u(r; y_x)drds \in D(A)$ and

$$A \int_0^t \int_0^s u(r; y_x)drds = \int_0^t \int_0^s Au(r; y_x)drds = u(t; y_x) - K_1(t)y_x \in D(A)$$

for all $0 \le t < T_0$, so that

$$\begin{aligned}(\lambda - A)u(t; y_x) &= (\lambda - A)[A \int_0^t \int_0^s u(r; y_x)drds + K_1(t)y_x] \\ &= A \int_0^t \int_0^s (\lambda - A)u(r; y_x)drds + K_1(t)Cx\end{aligned} \tag{26}$$

for all $0 \le t < T_0$. Hence, $v(\cdot; x) = (\lambda - A)u(\cdot; y_x)$ is a solution of (25) in $C([0, T_0), X)$. □

Combining Theorem 3.4 with Theorem 3.5, the next theorem is also attained.

Theorem 3.6. *Assume that $R(C) \subset R(\lambda - A)$ for some $\lambda \in \mathbb{F}$, and $ACP(A, K(\cdot)x, 0, 0)$ has a unique solution in $C^1([0, T_0), [D(A)])$ for each $x \in D(A)$ with $(\lambda - A)x \in R(C)$. Then A is a subgenerator of a nondegenerate local K-convoluted C-cosine function on X.*

Proof. Let $x \in X$ be given, and let $u(\cdot; y_x)$ and $v(\cdot; x)$ be given as in the proof of Theorem 3.5. By hypothesis, $v(\cdot; x)$ is continuously differentiable on $[0, T_0)$ and $v'(t; x) = (\lambda - A)u'(t; y_x)$ for all $0 \le t < T_0$. By (26), we also have $v'(t; x) = A \int_0^t v(r; x)dr + K_0(t)Cx$ for all $0 \le t < T_0$. In particular, $v'(0; x) = 0$, and so $v'(\cdot; x)=Aj_1 * v'(\cdot; x) + K_0(\cdot)Cx$ on $[0, T_0)$. Hence, $v'(\cdot; x)$ is a (unique) solution of the integral equation $v(\cdot)=Aj_1 * v(\cdot) + K_0(\cdot)Cx$ in $C([0, T_0), X)$. □

Since $C^{-1}AC = A$ and $R((\lambda - A)^{-1}C) = C(D(A))$ if $\rho(A) \neq \emptyset$, we can apply Proposition 3.1, Theorem 3.5 and Theorem 3.6 to obtain the next two corollaries.

Corollary 3.7. *Assume that the resolvent set of A is nonempty. Then A is the generator of a nondegenerate local K_0-convoluted C-cosine function on X if and only if for each $x \in D(A)$ $ACP(A, K(\cdot)Cx, 0, 0)$ has a unique solution in $C([0, T_0), [D(A)])$.*

Corollary 3.8. *Assume that the resolvent set of A is nonempty. Then A is the generator of a nondegenerate local K-convoluted C-cosine function on X if and only if for each $x \in D(A)$ $ACP(A, K(\cdot)Cx, 0, 0)$ has a unique solution in $C^1([0, T_0), [D(A)])$.*

Just as in [15, Theorems 3.9 and 3.10], we can apply Theorem 3.4 to obtain the next two wellposed theorems. The wellposedness of abstract fractional Cauchy problems and abstract Cauchy problems associated with various classes of Volterra integro-differential equations in locally convex spaces have been recently considered in [10].

Theorem 3.9. *Assume that A is densely defined. Then the following are equivalent :*

(i) *A is a subgenerator of a nondegenerate local K_0-convoluted C-cosine function $S(\cdot)$ on X;*
(ii) *for each $x \in D(A)$ $ACP(A, K(\cdot)Cx, 0, 0)$ has a unique solution $u(\cdot; Cx)$ in $C([0, T_0), [D(A)])$ which depends continuously on x. That is, if $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(D(A), \|\cdot\|)$, then $\{u(\cdot; Cx_n)\}_{n=1}^{\infty}$ converges uniformly on compact subsets of $[0, T_0)$.*

Proof. $(i)\Rightarrow(ii)$. It is easy to see from the definition of a subgenerator of $S(\cdot)$ that $S(\cdot)x$ is the unique solution of $ACP(A, K(\cdot)Cx, 0, 0)$ in $C([0, T_0), [D(A)])$ which depends continuously on $x \in D(A)$. $(ii)\Rightarrow(i)$. In view of Theorem 3.4, we need only to show that for each $x \in X$ (25) has a unique solution $v(\cdot; x)$ in $C([0, T_0), X)$. Let $x \in X$ be given. By the denseness of $D(A)$, we have $x_m \to x$ in X for some sequence $\{x_m\}_{m=1}^{\infty}$ in $D(A)$. We set $u(\cdot; Cx_m)$ to denote the unique solution of $ACP(A, K(\cdot)Cx_m, 0, 0)$ in $C([0, T_0), [D(A)])$. Then $u(\cdot; Cx_m) \to u(\cdot)$ uniformly on compact subsets of $[0, T_0)$ for some $u \in C([0, T_0), X)$, and so $\int_0^{\cdot} \int_0^s u(r; Cx_m)drds \to \int_0^{\cdot} \int_0^s u(r)drds$ uniformly on compact subsets of $[0, T_0)$. Since $u''(\cdot; Cx_m) = Au(\cdot; Cx_m) + K(\cdot)Cx_m$ a.e. on $(0, T_0)$, we have

$$A \int_0^{\cdot} \int_0^s u(r; Cx_m)drds = \int_0^{\cdot} \int_0^s Au(r; Cx_m)drds = u(\cdot; Cx_m) - K_1(\cdot)Cx_m \tag{27}$$

on $[0, T_0)$ for all $m \in \mathbb{N}$. Clearly, the right-hand side of the last equality of (27) converges uniformly to $u(\cdot) - K_1(\cdot)Cx$ on compact subsets of $[0, T_0)$. It follows from the closedness of A that $\int_0^t \int_0^s u(r)drds \in D(A)$ for all $0 \le t < T_0$ and $A \int_0^{\cdot} \int_0^s u(r)drds=u(\cdot) - K_1(\cdot)Cx$ on $[0, T_0)$, which implies that $u(\cdot)$ is a (unique) solution of (25) in $C([0, T_0), X)$. □

Theorem 3.10. *Assume that A is densely defined. Then the following are equivalent :*

(i) *A is a subgenerator of a nondegenerate local K-convoluted C-cosine function $C(\cdot)$ on X;*
(ii) *for each $x \in D(A)$ $ACP(A, K(\cdot)Cx, 0, 0)$ has a unique solution $u(\cdot; Cx)$ in $C^1([0, T_0), [D(A)])$ which depends continuously differentiable on x. That is, if $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(D(A), \|\cdot\|)$, then $\{u(\cdot; Cx_n)\}_{n=1}^{\infty}$ and $\{u'(\cdot; Cx_n)\}_{n=1}^{\infty}$ both converge uniformly on compact subsets of $[0, T_0)$.*

Proof. (*i*)⇒(*ii*). For each $0 \le t < T_0$ and $x \in D(A)$, we set $S(t)x = \int_0^t C(r)xdr$. Then $S(\cdot)x$ is the unique solution of ACP$(A, K(\cdot)Cx, 0, 0)$ in $C^1([0,T_0),[D(A)])$. Let $\{x_n\}_{n=1}^\infty$ be a Cauchy sequence in $(D(A), \|\cdot\|)$. We set $u(\cdot; Cx_n) = S(\cdot)x_n$ for $n \in \mathbb{N}$, then $\{u(\cdot; Cx_n)\}_{n=1}^\infty$ and $\{u'(\cdot; Cx_n)\}_{n=1}^\infty$ both converge uniformly on compact subsets of $[0,T_0)$. (*ii*)⇒(*i*). For each $x \in X$ and $0 \le t < T_0$, we define $u(t) = \lim_{n\to\infty} u(t; Cx_n)$ whenever $\{x_n\}_{n=1}^\infty$ is a sequence in $D(A)$ which converges to x in X. By hypothesis, $u(\cdot; Cx_m) \to u(\cdot)$ and $u'(\cdot; Cx_m) \to u'(\cdot)$ uniformly on compact subsets of $[0,T_0)$ for some $u \in C^1([0,T_0), X)$. Just as in the proof of Theorem 3.9, we have

$$A\int_0^t\int_0^s u'(r; Cx_m)drds = A\int_0^t u(s; Cx_m)ds = u'(\cdot; Cx_m) - K_0(\cdot)Cx_m$$

on $[0,T_0)$ for all $m \in \mathbb{N}$. Similarly, we also have $A\int_0^t\int_0^s u'(r)drds = u'(\cdot) - K_0(\cdot)Cx$ on $[0,T_0)$, which implies that $u'(\cdot)$ is a solution of the integral equation $v(\cdot) = Aj_1 * v(\cdot) + K_0(\cdot)Cx$ in $C([0,T_0), X)$. The uniqueness of solutions for the integral equation $v(\cdot) = Aj_1 * v(\cdot) + K_0(\cdot)Cx$ in $C([0,T_0), X)$ follows from the uniqueness of solutions for the integral equation (3.1) in $C([0,T_0), X)$. □

We end this paper with several illustrative examples.

Example 1. *Let $X = C_b(\mathbb{R})$, and $C(t)$ for $t \ge 0$ be bounded linear operators on X defined by $C(t)f(x) = \frac{1}{2}[f(x+t) + f(x-t)]$ for all $x \in \mathbb{R}$. Then for each $K \in L^1_{loc}([0,T_0),\mathbb{F})$ and $\beta > -1$, $K_\beta * C(\cdot) = \{K_\beta * C(t) | 0 \le t < T_0\}$ is local a K_β-convoluted cosine function on X which is also nondegenerate with a closed subgenerator $\frac{d^2}{dx^2}$ acting with its maximal distributional domain when K_0 is not the zero function on $[0,T_0)$ (or equivalently, K is not the zero in $L^1_{loc}([0,T_0),\mathbb{F})$), but $K * C(\cdot)$ may not be a local K-convoluted cosine function on X except for $K \in L^1_{loc}([0,T_0),\mathbb{F})$ so that $K * C(\cdot)$ is a strongly continuous family in $L(X)$ for which $\frac{d^2}{dx^2}$ is a closed subgenerator of $K * C(\cdot)$ when K_0 is not the zero function on $[0,T_0)$. Moreover, (1)-(4) hold and $\frac{d^2}{dx^2}$ is its generator and maximal subgenerator when K_0 is a kernel on $[0,T_0)$. In this case, $\frac{d^2}{dx^2} = \overline{A_0}$ for each subgenerator A_0 of $C(\cdot)$.*

Example 2. *Let k be a fixed nonnegative integer and K_0 a kernel on $[0,\infty)$, and let $C(t)$ for $t \ge 0$ and C be bounded linear operators on c_0 (the family of all convergent sequences in $\mathbb{F}$ with limit 0) defined by $C(t)x = \{x_n(n-k)e^{-n}\int_0^t K(t-s)\cosh nsds\}_{n=1}^\infty$ and $Cx = \{x_n(n-k)e^{-n}\}_{n=1}^\infty$ for all $x = \{x_n\}_{n=1}^\infty \in c_0$, then $\{C(t) | 0 \le t < 1\}$ is a local K-convoluted C-cosine function on c_0 which is degenerate except for $k = 0$ and generator A defined by $Ax = \{n^2x_n\}_{n=1}^\infty$ for all $x = \{x_n\}_{n=1}^\infty \in c_0$ with $\{n^2x_n\}_{n=1}^\infty \in c_0$, and for each $r > 1$ $\{C(t) | 0 \le t < r\}$ is not a local K-convoluted C-cosine function on c_0. Suppose that $k \in \mathbb{N}$. Then $A_a : c_0 \to c_0$ for $a \in \mathbb{F}$ defined by $A_ax = \{n^2y_n\}_{n=1}^\infty$ for all $x = \{x_n\}_{n=1}^\infty \in c_0$ with $\{n^2x_n\}_{n=1}^\infty \in c_0$, are subgenerators of $\{C(t) | 0 \le t < 1\}$ which do not have proper extensions that are still subgenerators of $\{C(t) | 0 \le t < 1\}$. Here $y_n = ak^2x_k$ if $n = k$, and $y_n = n^2x_n$ otherwise. Consequently, $\{C(t) | 0 \le t < 1\}$ does not have a maximal subgenerator when $k \in \mathbb{N}$.*

Example 3. *Let $X = C_b(\mathbb{R})$ (or $L^\infty(\mathbb{R})$), and A be the maximal differential operator in X defined by $Au = \sum_{j=0}^k a_jD^ju$ on $\mathbb{R}$ for all $u \in D(A)$, then $UC_b(\mathbb{R})$ (or $C_0(\mathbb{R})$) $= \overline{D(A)}$. Here $a_0, a_1, \cdots, a_k \in \mathbb{C}$ and $D^ju(x) = u^{(j)}(x)$ for all $x \in \mathbb{R}$. It is shown in [2, Theorem 6.7] that $\{C(t) | 0 \le t < T_0\}$ defined by $(C(t)f)(x) = \frac{1}{\sqrt{2\pi}}\int_0^t\int_{-\infty}^\infty K(t-s)\widetilde{\phi_s}(x-y)f(y)dyds$ for all $f \in X$ and $0 \le t < T_0$, is a norm continuous local K_0-convoluted cosine function on X with closed subgenerator A if the real-valued polynomial $p(x) = \sum_{j=0}^k a_j(ix)^j$ satisfies $\sup_{x\in\mathbb{R}} p(x) < \infty$, and $K \in L^1_{loc}([0,T_0),\mathbb{F})$ is not the zero function on $[0,T_0)$. Here $\widetilde{\phi_t}$ denotes the inverse Fourier transform of ϕ_t with $\phi_t(x) = \int_0^t \cosh(\sqrt{p(x)}s)ds$ for all $t \ge 0$. Suppose that K_0 is a kernel on $[0,T_0)$. Then A is its generator and maximal subgenerator. Applying Theorem 3.4, we get that for each $f \in X$ and continuous function g on $[0,T_0) \times \mathbb{R}$ with $\int_0^t \sup_{x\in\mathbb{R}} |g(s,x)|ds < \infty$ for all $0 \le t < T_0$, the function u on $[0,T_0) \times \mathbb{R}$ defined by $u(t,x) = \frac{1}{\sqrt{2\pi}}\int_0^t\int_{-\infty}^\infty K_1(t-s)\widetilde{\phi_s}(x-y)f(y)dyds + \frac{1}{\sqrt{2\pi}}\int_0^t\int_0^{t-r}\int_{-\infty}^\infty K_1(t-r-s)\widetilde{\phi_s}(x-$*

y)g(r, y)dydsdr for all $0 \le t < T_0$ *and* $x \in \mathbb{R}$, *is the unique solution of*

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} \\ = \displaystyle\sum_{j=0}^{k} a_j (\frac{\partial}{\partial x})^j u(t,x) + K_1(t) f(x) + \int_0^t K_1(t-s) g(s,x) ds \text{ for } t \in (0,T_0) \text{ and a.e. } x \in \mathbb{R}, \\ u(0,x) = 0 \text{ and } \dfrac{\partial u}{\partial t}(0,x) = 0 \quad \text{for a.e. } x \in \mathbb{R} \end{cases}$$

in $C^2([0,T_0),X) \cap C([0,T_0),[D(A)])$.

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