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Stochastic Optimal Control Problem of Constrained Switching System with Delay

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Abstract. This paper concerns the stochastic optimal control problem of switching systems with delay. The evolution of the system is governed by the collection of stochastic delay differential equations with initial conditions that depend on its previous state. The restriction on the system is defined by the functional constraint that contains state and time parameters. First, maximum principle for stochastic control problem of delay switching system without constraint is established. Finally, using Ekeland's variational principle, the necessary condition of optimality for control system with constraint is obtained.

1. Introduction

Uncertainty and time delay are associated with many real phenomena, and often they are sources of complicated dynamics. Systems with uncertainties have provided a lot of interest for problems of nuclear fission, communication systems, self-oscillating systems and etc. [13, 15, 43].Stochastic differential equations have the benefit in description of the natural systems, which in one or another degree are subjected to the influence of the random noises [28, 39]. The differential equations with time delay can be used in modeling of processes with a memory; that is, the behaviour of the system is dependent of the past [33, 38]. Many problems in physics,engineering, biological and economical sciences are expressed in terms of optimality principles, which often provide the most compact description of the laws governing dynamics and design of a systems [9, 40, 47]. Optimization problems for delay control systems have attracted a lot of interest [21, 23, 25, 27]. Stochastic models and stochastic control problems have many practical applications [24, 29, 33, 35]. The modern stochastic optimal control theory has been developed along the lines of Pontryagin's maximum principle and Bellman's dynamic programming [26, 48]. The stochastic maximum principle has been first considered by Kushner [34] Earliest results on the extension of Pontryagin's maximum principle to stochastic control problems are obtained in [10, 16, 17, 30]. Modern presentations of stochastic maximum principle with backward stochastic differential equations are considered in [18, 36, 37, 42]. Switching systems consist the several subsystems and a switching law indicating the active subsystem at each time instantly. For general theory of stochastic switching systems, we refer to [19]. A manufacturing systems,power systems, communication systems, aerospace space and a lot of problems of mathematical finance are some

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applications of stochastic switching systems. Recently, optimization problems for switching systems have attracted a lot of theoretical and practical interest [4, 7, 12, 14, 20, 44, 46]. Deterministic and stochastic optimal control problems of switching systems, described by differential equations with delay, are actual at present [3, 11, 31, 45]. In this paper, backward stochastic differential equations have been used to establish a maximum principle for stochastic optimal control problems of delay switching systems with constraint. Such kind of problems without delay have been considered by the author in [1, 2, 5, 6]. The optimal control problem of delay switching systems without endpoint constraints is considered in [3]. The plan of the paper is as follows: The next section formulates the main problem, presents some concepts and assumptions. The necessary condition of optimality for delay stochastic switching systems without endpoint constraint is obtained in Section 3. In Section 4, using Ekeland's variational principle [22] investigated control system with restriction is convert into the sequence of unconstrained optimal control problems. A maximum principle and transversality condition are established for the transformed problem. Finally, the necessary condition of optimality in the case with endpoint constraints is achieved. The conclusion and final remarks are given at the last section.

2. Statement of the Problem. Assumptions and Notations

In this section we fix notations and definitions used throughout this paper. Let **N** be some positive constant, R^n denotes the n dimensional real vector space, $|.|$ denotes the Euclidean norm and $\langle\cdot,\cdot\rangle$ denotes scalar product in R^n. E represents the mathematical expectation; by $\overline{1,r}$ we denote the set of integer numbers *1,...,r* . Let $w^1(t), w^2(t), ..., w^r(t)$ are independent Wiener processes that generate the filtration $F_t^l = \bar{\sigma}(w^l(t), t_{l-1}, t_l), l = \overline{1,r}$. (Ω^l, F^l, P^l) be a probability space with corresponding filtration $\left\{F_t^l, t \in [t_{l-1}, t_l]\right\}$.

$L_F^2(a, b; R^n)$ denotes the space of all predictable processes $x(t, \omega) \equiv x(t)$ such that: $E\int\limits_a^b |x(t,\omega)|^2 dt < +\infty$. $R^{m\times n}$ is the space of linear transformations from R^m to R^n. Let $O_l \subset R^{n_l}$, $Q_l \subset R^{m_l}$, $l = \overline{1,r}$, be open sets; $\mathbf{T} = [0, T]$ be a finite interval and $0 = t_0 < t_1 < ... < t_r = T$. We also use following notations : $\mathbf{t} = (t_0, t_1, ..., t_r)$; $\mathbf{t} = (t_0, t_1, ..., t_r)$, $\mathbf{u} = (u^1, u^2, ..., u^r)$, $\mathbf{x} = (x^1, x^2, ..., x^r)$; $F_t = \bigcup\limits_{l=1}^{r} F_t^l$.

Dynamic of the system is described by the following differential equation with delay:

$$dx^l(t) = g^l\left(x^l(t), x^l(t-h), u^l(t), t\right)dt + f^l\left(x^l(t), x^l(t-h), t\right)dw^l(t), t \in (t_{l-1}, t_l] \quad l = \overline{1,r}; \tag{1}$$

$$x^{l+1}(t) = K^{l+1}(t) \ t \in (t_l - h, t_l), l = \overline{0, r-1}, \tag{2}$$

$$x^{l+1}(t_l) = \Phi^{l+1}\left(x^l(t_l), t_l\right) \ l = \overline{1, r-1}; x_{t_0}^1 = x_0, \tag{3}$$

$$u^l(t) \in U_\partial^l \equiv \left(u^l(\cdot,\cdot) \in \mathrm{L}_{F^l}^2 | u^l(t,\cdot) \in U^l \subset \mathrm{R}^{m_l}, a.c.\right), \tag{4}$$

where U_∂^l are non-empty bounded sets. The elements of U_∂^l are called the admissible controls. Let $\Lambda_l, l = \overline{1,r}$ be the set of piecewise continuous functions $K^l(\cdot)$ $l = \overline{1,r} : [t_{l-1} - h, t_{l-1}) \to N_l \subset O_l$ and $h \geq 0$.

The problem is concluded to find the optimal solution $(x, u) = \left(x^1, x^2, ..., x^r, u^1, u^2, ..., u^r\right)$ and switching sequence $\mathbf{t} = (t_1, t_2, ..., t_r)$ on the decisions of the system (1)-(4), which minimize the cost functional:

$$J(u) = \sum_{l=1}^{r} E\left[\varphi^l\left(x^l(t_l)\right) + \int\limits_{t_{l-1}}^{t_l} p^l\left(x^l(t), u^l(t), t\right)dt\right] \tag{5}$$

under the endpoint condition

$$Eq^r(x^r(t_r), t_r) \in G, \tag{6}$$

G is a closed convex set in R.

Introduce the sets:

$A_i = \mathrm{T}^{i+1} \times \prod_{j=1}^{i} O_j \times \prod_{j=1}^{i} \Lambda_j \times \prod_{j=1}^{i} Q_j,\ i = \overline{1, r},$

with the elements

$$\pi^i = (t_0, t_1, t_i, x^1(t), x^2(t), ..., x^i(t), u^1, u^2, ..., u^i).$$

Now for completeness of the presentation and convenience of the reader, we introduce following definitions from [3].

Definition 2.1. The set of functions $\{x^l(t) = x^l\left(t, \pi^l\right),\ t \in [t_{l-1} - h, t_l],\ l = \overline{1, r}\}$ is said to be a solution of the equation (1) with variable structure corresponding to an element $\pi^r \in \mathrm{A}_r$ if the function $x^l(t) \in O_l$ on the interval $[t_{l-1} - h, t_l]$ satisfies the conditions (2),(3), while on the interval $[t_{l-1}, t_l]$ it is absolutely continuous almost certainly (a.c.) and satisfies the equation (1) almost everywhere.

Definition 2.2. The element $\pi^r \in A_r$ is said to be admissible if the pairs $\left(x^l(t),\ u^l(t)\right),\ t \in [t_{l-1}, t_l],\quad l = \overline{1, r}$ are the solutions of system (1)-(4) and satisfy the constraints (6).

Definition 2.3. Let A_r^0 be the set of admissible elements. The element $\bar{\pi}^r \in A_r^0$, is said to be an optimal solution of problem (1)-(6) if there exist admissible controls $\bar{u}^l(t),\ t \in [t_{l-1}, t_l],\ l = \overline{1, r}$ and corresponding solutions $\bar{x}^l(t),\ t \in [t_{l-1}, t_l],\ l = \overline{1, r}$ of system (1)-(3) such that the pairs $\left(\bar{x}^l(t), \bar{u}^l(t)\right),\ l = \overline{1, r}$ minimize the functional (5).

Assume that the following requirements are satisfied:

I. Functions $g^l, f^l, p^l,\ l = \overline{1, r}$ and their derivatives are continuous in (x, y, u, t):
$g^l(x, y, u, t) : O_l \times O_l \times Q_l \times \mathbf{T} \to R^{n_l}$, $f^l(x, y, t) : O_l \times O_l \times \mathbf{T} \to R^{n_l \times n_l}$, $p^l(x, u, t) : O_l \times Q_l \times \mathbf{T} \to R^{n_l}$

II. For fixed (u, t) functions $g^l, f^l, p^l,\ l = \overline{1, r}$ hold the conditions:

$$\left(1 + |x| + |y|\right)^{-1} \left(\left|g^l(x, y, u, t)\right| + \left|g_x^l(x, y, u, t) + \left|g_y^l(x, y, u, t)\right|\right| + \right.$$
$$\left.\left|f^l(x, y, t)\right| + \left|f_x^l(x, y, t)\right| + \left|f_y^l(x, y, t)\right| + \left|p^l(x, u, t)\right| + \left|p_x^l(x, u, t)\right|\right) \le N.$$

III. Functions $\varphi^l(x) : R^{n_l} \to R$ are continuously differentiable and their derivatives are bounded by $N(1 + |x|)$.

IV. Functions $\Phi^l(x, t) : O_{l-1} \times \mathbf{T} \to O_l, l = \overline{1, r-1}$ are continuously differentiable in respect to (x, t) and their derivatives are bounded by $N(1 + |x|)$.

V. Function $q^r(x, t) : O_l \times \mathbf{T} \to R$ is continuously differentiable in respect to (x, t) and satisfies:

$$\left|q^r(x, t)\right| + \left|q_x^r(x, t) \leqslant\right| N(1 + |x|).$$

3. Controlled Switching Systems with Delay

Using similar technique from [2], the following necessary condition of optimality for problem (1)–(5) is obtained.

Theorem 3.1. *Suppose that conditions I – IV hold and*

$$\pi^r = (t_0, t_1, t_r, x^1(t), x^2(t), ..., x^r(t), u^1, u^2, ..., u^r)$$

is an optimal solution of problem (1)–(5). There exist random processes $(\psi^l(t), \beta^l(t)) \in L^2_F(t_{l-1}, t_l; R^{n_l}) \times L^2_F(t_{l-1}, t_l; R^{n_l \times n_l})$ *which are the solutions of the following adjoint equations:*

$$\begin{cases} d\psi^l(t) = -[H^l_x(\psi^l(t), x^l(t), y^l(t), u^l(t), t) + H^l_y(\psi^l(t+t), x^l(t+h), y^l(t+h), u^l(t+h), t+h)]dt \\ +\beta^l(t)dw^l(t), \;\; t_{l-1} \leqslant t < t_l - h, \\ d\psi^l(t) = -H^l_x(\psi^l(t), x^l(t), y^l(t), u^l(t), t)dt + \beta^l(t)dw^l(t), \;\; t_l - h \leqslant t < t_l, \\ \psi^l(t_l) = -\varphi^l_x(x^l(t_l)) + \psi^{l+1}(t_l)\Phi^l_x(x^l(t_l), t_l), \;\; l = \overline{1, r-1}, \\ \psi^r(t_r) = -\varphi^r_x(x^r(t_r)), \end{cases} \tag{7}$$

Then

a) $\forall\ \bar{u}^l \in U^l$, $l = \overline{1, r}$, *a.c. fulfills the maximum principle:*

$$H^l(\psi^l(\theta), x^l(\theta), y^l(\theta), \bar{u}^l, \theta) - H^l(\psi^l(\theta), x^l(\theta), u^l(\theta), \theta) \leq 0, \;\; a.e. \; \theta \in [t_{l-1}, t_l]; \tag{8}$$

b) following transversality conditions hold:

$$a_l\psi^{l+1}(t_l)\Phi^l\left(x^l(t_l), t_l\right) - b_l\psi^{l+1}(t_l)g^{l+1}\left(x^{l+1}(t_l+h), K^{l+1}(t_l), u^{l+1}(t_l+h), t_l+h\right) = 0, \;\; a.c., \; l = \overline{1, r-1}. \tag{9}$$

Here
$H^l\left(\psi(t), x(t), y(t), u(t), t\right) = \psi(t)g^l\left(x(t), y(t), u(t), t\right) + \beta(t)f^l\left(x(t), y(t), t\right) - p^l\left(x(t), u(t), t\right);$
$y^l(t) = x^l(t-h); a_1 = ... = a^{r-1} = 1; a^r = b_1 = 0; b_2 = ... = b^r = 1$

Proof. Let $\bar{u}^l(t), u^l(t)$ $l = \overline{1, r}$ be some admissible controls; call the vectors $\Delta\bar{u}^l(t) = \bar{u}^l(t) - u^l(t)$, $l = \overline{1, r}$ be an admissible increments of the controls $u^l(t)$. By (1)-(3), the trajectories $\bar{x}^l(t), x^l(t)$ $l = \overline{1, r}$ corresponds to the controls $\bar{u}^l(t), u^l(t)$ $l = \overline{1, r}$. The vectors $\Delta\bar{x}^l(t) = \bar{x}^l(t) - x^l(t)$, $l = \overline{1, r}$ are called increments of the solutions $x^l(t)$ $l = \overline{1, r}$ that correspond to the increments $\Delta\bar{u}^l(t), l = \overline{1, r}$. Let $0 = t_0 < t_1 < ... < t_r = T$ be switching sequence corresponds to the optimal solution. Then following identities are obtained for some sequence $0 = \bar{t}_0 < \bar{t}_1 < ... < \bar{t}_r = T$:

$$\begin{cases} d\Delta\bar{x}^l(t) = \left[\Delta_{\bar{u}^l}g^l(x^l(t), y^l(t), u^l(t), t) + g^l_x(x^l(t), y^l(t), u^l(t), t)\Delta\bar{x}^l(t) + g^l_y(x^l(t), y^l(t), u^l(t), t)\Delta\bar{y}^l(t)\right]dt \\ + \left[f^l_x(x^l(t), y^l(t), t)\Delta\bar{x}^l(t) + f^l_y(x^l(t), y^l(t), t)\Delta\bar{y}^l(t)\right]dw^l(t) \; + \eta^1_t, t \in (t_{l-1}, t_l], \\ \Delta\bar{x}^l(t) = 0, t \in [t_{l-1} - h, t_{l-1}), \Delta\bar{x}^1(t_0) = 0, \\ \Delta\bar{x}^l(t_{l-1}) = \Phi^{l-1}\left(\bar{x}^{l-1}(t_{l-1}), \bar{t}_{l-1}\right) - \Phi^{l-1}\left(x^{l-1}(t_{l-1}), t_{l-1}\right) \; l = \overline{2, r} \end{cases} \tag{10}$$

where

$$\Delta_{\bar{u}}g(x(t), y(t), u(t), t) = g(x(t), y(t), \bar{u}(t), t) - g(x(t), y(t), u(t), t),$$

$$\eta^1_t = \int_0^1 \left[g^{l*}_x(x^l(t) + \mu\Delta\bar{x}^l(t), \bar{y}^l_t, \bar{u}^l_t, t) - g^{l*}_x(x^l(t), y^l_t, u^l(t), t)\right]\Delta\bar{x}^l(t)d\mu\; dt+$$

$$\int_0^1 \left[g^{l*}_y(x^l(t), y^l_t + \mu\Delta\bar{y}^l(t), \bar{u}^l_t, t) - g^{l*}_y(x^l(t), y^l_t, u^l(t), t)\right]\Delta\bar{y}^l(t)d\mu\; dt+$$

$$\int_0^1 \left[f^{l*}_x(x^l(t) + \mu\Delta\bar{x}^l(t), \bar{y}^l(t), t) - f^{l*}_x(x^l(t), y^l(t), t)\right]\Delta\bar{x}^l(t)d\mu dw^l(t)+$$

$$\int_0^1 \left[f^{l*}_y(x^l(t), y^l(t) + \mu\Delta\bar{y}^l(t), t) - f^{l*}_y(x^l(t), y^l(t), t)\right]\Delta\bar{y}^l(t)d\mu dw^l(t).$$

According to Ito's formula [28] the following has been yielded:

$$
\begin{aligned}
&d(\psi^{l*}(t)\Delta\bar{x}^l(t)\Delta\bar{t}_l) = d\psi^{l*}(t)\Delta\bar{x}^l(t)\Delta\bar{t}_l + \psi^{l*}(t)d\Delta\bar{x}^l(t)\Delta\bar{t}_l + \psi^{l*}(t)\Delta\bar{x}^l(t)d\Delta\bar{t}_l +\\
&\Big\{\beta^{l*}(t)[f_x^l(x^l(t),y^l(t),t)\Delta\bar{x}^l(t) + f_y^l(x^l(t),y^l(t),t)\Delta\bar{y}^l(t)]\Delta\bar{t}_l\\
&+\beta^{l*}(t)\int_0^1[f_x^l(x^l(t)+\mu\Delta\bar{x}^l(t),\bar{y}^l(t),t) - f_x^l(x^l(t),y^l(t),t)]\Delta\bar{x}^l(t)\Delta\bar{t}_l d\mu +\\
&+\beta^{l*}(t)\int_0^1[f_y^l(x^l(t),y^l(t)+ +\mu\Delta\bar{y}^l(t),t) - f_y^l(x^l(t),y^l(t),t)]\Delta\bar{y}^l(t)\Delta\bar{t}_l d\mu\ \Big\}\,dt.
\end{aligned}
$$

The stochastic processes $\psi^l(t)$, $l=\overline{1,r}$, at the points $t_1, t_2, ..., t_r$ are defined as follows:

$$
\psi^l(t_l) = -\varphi_x^l\left(x^l(t_l)\right) + \psi^{l+1}(t_l)\Phi_x^l(x^l(t_l),t_l\,),\ \psi^r(t_r) = -\varphi_x^r(x(t_r)) \tag{11}
$$

Taking into consideration (9)-(11) expression of increment of the cost functional (5) along the admissible control looks like:

$$
\begin{aligned}
\Delta J(u) = &\sum_{l=1}^{r} E\left\{\varphi^l\left(\bar{x}^l(t_l)\right) - \varphi^l\left(x^l(t_l)\right) + \int_{t_{l-1}}^{t_l}\left[p^l\left(\bar{x}^l(t),\bar{u}_t^l,t\right) - p^l\left(x^l(t),u^l(t),t\right)\right]dt\right\}\\
&= -\sum_{l=1}^{r} E\int_{t_{-1}}^{t_l}\left[\psi^{l*}(t)\Delta_{\bar{u}^l}g^l\left(x^l(t),y^l(t),u^l(t),t\right) + \psi^{l*}(t)g_x^l\left(x^l(t),y^l(t),u^l(t),t\right)\Delta\bar{x}^l(t)\right.\\
&+\psi^{l*}(t)g_y^l\left(x^l(t),y^l(t),u^l(t),t\right)\Delta\bar{y}^l(t) + \beta^{l*}(t)f_x^l\left(x^l(t),y^l(t),t\right)\Delta\bar{x}^l(t) + \beta^{l*}(t)f_y^l\left(x^l(t),y^l(t),t\right)\Delta\bar{y}^l(t)\\
&-\Delta_{\bar{u}^l}p^l\left(x^l(t),u^l(t),t\right) - p_x^l\left(x^l(t),u^l(t),t\right)\Delta\bar{x}^l(t)\ ]\,\Delta\bar{t}_l dt + \sum_{l=1}^{r-1}\psi^{l+1}(t_l)\Phi_t\left(x^l(t_l),t_l\right) + \sum_{l=1}^{r}\eta_{t_{l-1}}^{t_l},
\end{aligned} \tag{12}
$$

where

$$
\begin{aligned}
\eta_{t_{l-1}}^{t_l} &= -E\int_0^1(1-\mu)\left[\varphi_x^{l*}\left(\bar{x}^l(t_l)\right) - \varphi_x^*\left(x^l(t_l)\right)\right]\Delta\bar{x}^l(t_l)d\mu -\\
&E\int_{t_{l-1}}^{t_l}\int_0^1(1-\mu)\left[H_x^l\left(\psi^l(t),x^l(t)+\mu\Delta\bar{x}^l(t_l),y^l(t),u^l(t),t\right) - H_x^l\left(\psi^l(t),x^l(t)u^l(t),t\right)\right]\Delta\bar{x}^l(t)\Delta\bar{t}_l d\mu dt\\
&E\int_{t_{l-1}}^{t_l}\int_0^1(1-\mu)\left[H_y^l\left(\psi^l(t),x^l(t_l),y^l(t)+\mu\Delta\bar{y}^l(t_l),u^l(t),t\right) - H_y^l\left(\psi^l(t),x^l(t),y^l(t),u^l(t),t\right)\right]\Delta\bar{y}^l(t)\Delta\bar{t}_l d\mu dt\\
&-E\int_0^1(1-\mu)\,\psi_{t_l}^{l+1}\left[\Phi_x^l\left(x^l(t_l)+\mu\Delta\bar{x}^l(t_l),(t_l)\right) - \Phi_x^l\left(x^l(t_l),t_l\right)\right]\Delta x^l(t_l)\Delta\bar{t}_l d\mu
\end{aligned} \tag{13}
$$

According to a necessary condition for an optimal solution, we obtain that the coefficients of the independent increments $\Delta x^l(t), \Delta y^l(t), \Delta\bar{t}_l$ equal zero. Using assumption IV and the expression (10), from the identity (12), we obtain that (9) is true.

By (9) and (11) , through the simple transformations, expression (12) can be rewritten under the following form:

$$
\begin{aligned}
\Delta J(u) = \sum_{l=1}^{r}\Delta J^l\left(u^l\right) = -\sum_{l=1}^{r}E\int_{t_{l-1}}^{t_l}\left[\Delta_{\overline{u}^l}H^l\left(\psi^l(t),x^l(t),y^l(t),u^l(t),t\right)+\right.\\
\Delta_{\overline{u}^l}H_x^l\left(\psi^l(t),x^l(t),y^l(t),u^l(t),t\right)\Delta\bar{x}^l(t) + \Delta_{\overline{u}^l}H_y^l\left(\psi^l(t),x^l(t),y^l(t),u^l(t),t\right)\Delta\bar{y}^l(t)\ ]\,\Delta\bar{t}_l dt + \sum_{l=1}^{r}\eta_{t_{l-1}}^{t_l}
\end{aligned} \tag{14}
$$

Due to the fact the space of admissible controls is not assumed to be convex , onwards we will use following spike variations:

$$\Delta u^l(t) = \Delta u^{\theta_l}_{t,\varepsilon^l} = \begin{cases} 0,\ t \notin [\theta_l, \theta_l + \varepsilon_l)\,, \varepsilon_l > 0, \quad \theta_l \in [t_{l-1}, t_l) \\ \bar{u}^l - u^l_t, t \in [\theta_l, \theta_l + \varepsilon_l)\,, \bar{u}^l \in L^2\left(\Omega, F^{\theta_l}, P; R^m\right), \end{cases}$$

where ε_l are small enough. In terms of the presented variations the expression (14) takes the form of:

$$\begin{aligned} &\Delta_\theta J(u) = -\sum_{l=1}^{r} E \int_{\theta_l}^{\theta_l+\varepsilon_l} \Big[\Delta_{\bar{u}^l} H^l\left(\psi^l(t), x^l(t), y^l(t), u^l(t), t\right) + \\ &\Delta_{\bar{u}^l} H^l_x\left(\psi^l(t), x^l(t), y^l(t), u^l(t), t\right)\Delta\bar{x}^l(t) + \Delta_{\bar{u}^l} H^l_y\left(\psi^l(t), x^l(t), y^l(t), u^l(t), t\right)\Delta\bar{y}^l(t)\ \Big]\Delta\bar{t}_l dt + \sum_{l=1}^{r}\eta^{\theta_l+\varepsilon_l}_{\theta_l} \end{aligned} \tag{15}$$

The following lemma will be used in estimation of increment (15).

Lemma 3.2. ([6]) *Assume that conditions $I - IV$ are fulfilled. Then* $\lim_{\varepsilon_l \to 0} E\left|x^{\theta_l}_{\varepsilon_l}(t) - x^l(t)\right|^2 \leq N\varepsilon_l$, *a.e. in* $[t_{l-1}, t_l)\,, l = \overline{1, r}$.

Here $x^{\theta_l}_{\varepsilon_l}(t)$ are the trajectories of system (1)-(3), corresponding to the controls $u^{\theta_l}_{\varepsilon_l}(t) = u^l(t) + \Delta u^{\theta_l}_{\varepsilon_l}(t)$.

By invoking the expression (13),using 3.2 following estimation is implied:

$$\eta^{\theta_l+\varepsilon_l}_{\theta_l} = o\,(\varepsilon_l)\,, l = \overline{1, r}.$$

According to optimality of controls $\bar{u}^l(t)$, $l = \overline{1, r}$ from (15) for each l it follows that:

$$\Delta_{\theta^l} J(u) = -\varepsilon_l E\left[\Delta_{\bar{u}^l} H(\psi^l(\theta_l), x^l(\theta_l), y^l(\theta_l), u^l(\theta_l), \theta_l)\right]\Delta\bar{t}_l + o(\varepsilon_l) \geq 0$$

According to sufficient smallness ε_l it follows that (8) is fulfilled. □

4. Necessary Condition of Optimality for Stochastic Switching Systems with Constraint

The main result of the paper, presented in this section, is proved via an approximation of the initial control problem by the sequence of unconstraint systems. Based on 3.1 the necessary condition of optimality for stochastic control systems with delay (1)-(6) with endpoint constraint is obtained.

Theorem 4.1. *Suppose that, conditions I-V hold. Let $\pi^r = (t_0, t_1, ..., t_r, x^1(t), x^2(t), , ..., x^r(t), u^1, K_1, ..., K_r, u^2, ..., u^r)$ is an optimal solution of problem (1)-(6), and random processes $(\psi^l(t), \beta^l(t)) \in L^2_{F^l}(t_{l-1}, t_l; R^{n_l}) \times L^2_{F^l}(t_{l-1}, t_l; R^{n_l \times n_l})$ are the solution of the following adjoint equations:*

$$\begin{cases} d\psi^l(t) = -[H^l_x(\psi^l(t), x^l(t), y^l(t), u^l(t), t) + H^l_y(\psi^l(t+t), x^l(t+h), y^l(t+h), u^l(t+h), t+h)]dt \\ +\beta^l(t)dw^l(t),\ \ t_{l-1} \leqslant t < t_l - h, \\ d\psi^l(t) = -H^l_x(\psi^l(t), x^l(t), y^l(t), u^l(t), t)dt + \beta^l(t)dw^l(t),\ \ t_l - h \leqslant t < t_l, \\ \psi^l(t_l) = -\lambda_l \varphi^l_x(x^l(t_l)) + \psi^{l+1}(t_l)\Phi^l_x(x^l(t_l), t_l),\ \ l = \overline{1, r-1}, \\ \psi^r(t_r) = -\lambda_0 \varphi^r_x(x^r(t_r)) - \lambda_r q^r_x(x^r(t_r), t_r). \end{cases} \tag{16}$$

Then

a) a.e. $\theta \in [t_{l-1}, t_l]$ and $\forall\ \bar{u}^l \in U^l$, $l = \overline{1, r}$, a.c. the maximum principle (8) fulfill;

b) a.c. following transversality conditions hold for each $l = \overline{1, r-1}$

$$(1 - a_l)\lambda_1 q^r(x^r(t_r), t_r) = a_l\psi^{l+1}(t_l)\Phi^l\left(x^l(t_l), t_l\right) - b_l\psi^{l+1}(t_l)g^{l+1}\left(x^{l+1}(t_l + h), K^{l+1}(t_l), u^{l+1}(t_l + h), t_l + h\right). \tag{17}$$

Proof. For any natural j let's introduce the following approximating functional for each $l = \overline{1,r}$:

$$I_j(\mathbf{u}) = S_j < E\sum_{l=1}^{r}[\varphi^l(t_l) + \int_{t_{l-1}}^{t_l} p^l\left(x^l(t), u^l(t), t\right)dt], Eq^r(x^r(t_r), t_r) >$$

$$= \min_{(c,y)\in\varepsilon}\sqrt{\left|c - 1/j - EM(\mathbf{x}, \mathbf{u}, \mathbf{t})\right|^2 + \left|y - Eq^r(x^r(t_r), t_r)\right|^2},$$

where $M(\mathbf{x}, \mathbf{u}, \mathbf{t}) = \sum_{i=1}^{r}\left[\varphi^l(x^l(t_l)) + \int_{t_{l-1}}^{t_l} p(x^l(t), u^l(t), t)dt\right]$;$\varepsilon = \left\{c : c \le J^0, y \in G\right\}$;$c = \sum_{i=1}^{r} c^l$ and J^0 minimal value of the functional in the problem (1)-(5).

Let $V^l \equiv (U^l_\partial, d)$ be space of controls obtained by means of the following metric:

$$d(u^l, v^l) = (l\otimes P)\ \left\{(t,\omega)\in[t_{l-1}, t_l]\times\Omega : v^l_t \ne u^l(t)\right\}.$$

For each $l = \overline{1,r}$, the V^l is a complete metric space [22].

Proof of the next lemma immediately follows from Ito's formula and assumptions I, II, IV.

Lemma 4.2. *Assume that $u^{l,n}(t)$, $l = \overline{1,r}$ be the sequence of admissible controls from V^l, and $x^{l,n}(t)$ be the sequence of corresponding trajectories of the system (1)-(4).*

Then, $\lim_{n\to\infty}\left\{\sup_{t_{l-1}\le t\le t_l} E\left|x^{l,n}(t) - x^l(t)\right|^2\right\} = 0$,*if :* $d(u^{l,n}(t), u^l(t)) \to 0.$

Here $x^l(t)$ is a trajectory corresponding to an admissible controls $u^l(t)$, $l = \overline{1,r}.$

Due to continuity of the functionals $I^l_j : V^l \to R^{n_l}$, according to Ekeland's variational principle, there are controls such as; $u^{l,j}(t) : d(u^{l,j}(t), u^l(t)) \le \sqrt{\varepsilon^l_j}$ and for $\forall u^l(t) \in V^l$ the following is achieved: $I^l_j(u^{l,j}) \le I^l_j(u^l) + \sqrt{\varepsilon^l_j}d(u^{l,j}, u^l), \varepsilon^l_j = \frac{1}{j}.$

This inequality means that $(t_1, ..., t_r, x^{1,j}(t), ..., x^{r,j}(t), K_1, ..., K_r, u^{1,j}(t), ..., u^{r,j}(t))$ for each $t \in (t_{l-1}, t_l]$ is a solution of the following problem:

$$\begin{cases} J_j(u) = \sum_{l=1}^{r}\left(I^l_j(u^l) + \sqrt{\varepsilon^l_j}E\int_{t_{l-1}}^{t_l}\delta(u^l(t), u^{l,j}(t))dt\right) \to \min \\ dx^l(t) = g^l(x^l(t), y^l(t), u^l(t), t)dt + f^l(x^l(t), y^l(t), t)dw(t),\ l = \overline{1,r} \\ x^{l+1}(t) = K^{l+1}(t),\ l = \overline{0, r-1} \\ x^{l+1}(t_l) = \Phi^{l+1}\left(x^l(t_l), t_l\right),\ l = \overline{0, r-1}\ ; \\ x^1(t_0) = x_0, u^l(t) \in U^l_\partial \end{cases} \tag{18}$$

Function $\delta(u, v)$ is determined in the following way: $\delta(u,v) = \begin{cases} 0, u = v \\ 1, u \ne v. \end{cases}$

Taking into account (18) from Theorem 3.1 there follows: if $(x^{1,j}(t), ..., x^{r,j}(t), u^{1,j}(t), ..., u^{r,j}(t))$ is an optimal solution of problem (18), and there exist the random processes $(\psi^{l,j}(t), \beta^{l,j}(t)) \in L^2_{F^l}(t_{l-1}, t_l; R^{n_l}) \times L^2_{F^l}(t_{l-1}, t_l; R^{n_l\times n_l})$ that are solutions of the following system:

$$\begin{cases} d\psi^{l,j}(t) = -H^l_x\left(\psi^{l,j}(t), x^{l,j}(t), y^{l,j}(t), u^{l,j}(t), t\right)dt - H^l_y\left(\psi^{l,j}(t+h), x^{l,j}(t+h), y^{l,j}(t+h), u^{l,j}(t+h), t+h\right)dt \\ +\beta^{l,j}(t)dw^l(t),\ t\in[t_{l-1}, t_l - h), \\ d\psi^{l,j}(t) = -H^l_x\left(\psi^{l,j}(t), x^{l,j}(t), y^{l,j}(t), u^{l,j}(t), t\right)dt + \beta^{l,j}(t)dw^l(t),\ t \in (t_l - h, t_l), \\ \psi^{l,j}(t_l) = -\lambda^j_l\varphi^l_x\left(x^{l,j}(t_l)\right) + \psi^{l+1}(t_l)\Phi^l_x(x^{l,j}(t_l), t_l), l = \overline{1, r-1} \\ \psi^r(t_r) = -\lambda^j_0\varphi^r_x\left(x^{r,j}_{t_r}\right) - \lambda^j_r q^r_x\left(x^{r,j}_{t_r}, t_r\right). \end{cases} \tag{19}$$

where non-zero $(\lambda_0^j, \lambda_1^j, ..., \lambda_r^j)$ meet the following requirement:

$$\begin{cases} \lambda_l^j = \left[-c^l + 1/j + \varphi^l(x^l(t_l)) + \int\limits_{t_{l-1}}^{t_l} p(x^l(t), u^l(t), t)dt\right]/J_j^0, l = \overline{0, r-1}; \\ \lambda_r^j = -y + Eq^r(x^{r,j}(t_r), t_r)/J_j^0. \end{cases} \tag{20}$$

Here

$$J_j^0 = \left(\left|y - Eq^r(x^{r,j}(t_r), t_r)\right|^2 + \left|c - 1/j - E\sum_{l=1}^{r}\left[\varphi^l(x^l(t_l)) + \int_{t_{l-1}}^{t_l} p(x^l(t), u^l(t), t)dt\right]\right|^2\right)^{1/2}$$

2) a.e. $\theta \in [t_{l-1}, t_l]$ and $\forall\ \tilde{u}^l \in V^l$, $l = \overline{1, r}$, a.c. is satisfied:

$$H^l(\psi^{l,j}(\theta), x^{l,j}(\theta), y^{l,j}(\theta), \bar{u}^{l,j}, \theta) - H^l(\psi^{l,j}(\theta), x^{l,j}(\theta), y^{l,j}(\theta), u^{l,j}, \theta) \leq 0 \tag{21}$$

3) for each $l = \overline{1, r-1}$ the following transversality conditions hold:

$$\begin{aligned} &(1 - a_l)\lambda_1^j q^r(x^{r,j}(t_r), t_r) = a_l\psi^{l+1,j}(t_l + h)\Phi^{l+1,j}\left(x^{l,j}(t_l), t_l\right) \\ &-b_l\psi^{l+1,j}(t_l + h)g^{l+1,j}\left(x^{l+1,j}(t_l + h), K^{l+1}(t_l), u^{l+1}(t_l + h), t_l + h\right), \ a.c. \end{aligned} \tag{22}$$

According to conditions I-IV it is achieve that: $(\lambda_0^j, ..., \lambda_r^j) \to (\lambda_0, ..., \lambda_r)$ if $j \to \infty$.
To complete the proof of Theorem 4.1 we need the following fact.

Lemma 4.3. *Let $\psi^l(t_l)$ be a solution of system (16), $\psi^{l,j}(t_l)$ be a solution of system (19). If $d(u^{l,j}(t), u^l(t)) \to 0,\ j \to \infty$, then*

$$E\int_{t_{l-1}}^{t_l} |\psi^{l,j}(t) - \psi^l(t)|^2 dt + E\int_{t_{l-1}}^{t_l} |\beta^{l,j}(t) - \beta^l(t)|^2 dt \to 0\,, l = \overline{1, r}.$$

Proof. It is clear that $\forall t \in [t_l - h, t_l)$:

$$\begin{aligned} &d\left(\psi^{l,j}(t) - \psi^l(t)\right) = -\left[H_x^l\left(\psi^{l,j}(t), x^{l,j}(t), y^{l,j}(t), u^{l,j}(t), t\right) - H_x^l\left(\psi^l(t), x^l(t), y^l(t), u^l(t), t\right)\right]dt + \\ &\left(\beta^{l,j}(t) - \beta^l(t)\right)dw(t) = -\left[\psi^{l,j}(t)\ g_x^l\left(x^{l,j}(t), y^{l,j}(t), u^{l,j}(t), t\right) + \beta^{l,j}(t)f_x^l\left(x^{l,j}(t), y^{l,j}(t), t\right)\right. \\ &-p_x^l\left(x^{l,j}(t), u^{l,j}(t), t\right) - \psi^l(t)g_x^l\left(x^l(t), y^l(t), u^l(t), t\right) \\ &\left.-\beta^l(t)f_x^l\left(x^l(t), y^l(t), t\right) + p_x^l\left(x^l(t), u^l(t), t\right)\right]dt + \left(\beta^{l,j}(t) - \beta^l(t)\right)dw(t) \end{aligned}$$

Let us square both sides of the last equation. According to Ito's formula $\forall s \in [t_l - h, t_l)$:

$$\begin{aligned} &E(\psi^{l,j}(t) - \psi^l(t))^2 - E(\psi^{l,j}(s) - \psi^l(s))^2 = \\ &2E\int_s^{t_l}[\psi^{l,j}(t) - \psi^l(t)][(g_x^{l*}(x^{l,j}(t), y^{l,j}(t), u^{l,j}(t), t) - g_x^{l*}(x^l(t), y^l(t), u^l(t), t))\psi^{l,j}(t) + \\ &+g_x^{l*}(x^l(t), y^l(t), u^l(t), t)(\psi^{l,j}(t) - \psi^l(t)) + (f_x^{l*}(x^{l,j}(t), y^{l,j}(t), t) - f_x^{l*}(x^l(t), y^l(t), t))\beta^{l,j}(t) + f_x^{l*}(x^l(t), y^l(t), t)\times \\ &\times(\beta^{l,j}(t) - \beta^l(t)) - p^l(x^{l,j}(t), u^{l,j}(t), t) + p_x^l(x^l(t), u^l(t), t)]dt + E\int_s^{t_l}(\beta^{l,j}(t) - \beta^l(t))^2 dt \end{aligned}$$

Now, due to assumptions I-IV we get:

$$E\int_s^{t_l}|\beta^{l,j}(t) - \beta^l(t)|^2 dt + E|\psi^{l,j}(s) - \psi^l(s)|^2 \leq EN\int_s^{t_l}|\psi^{l,j}(t) - \psi^l(t)|^2 dt + EN\varepsilon\int_s^{t_l}|\beta^{l,j}(t) - \beta^l(t)|^2 dt + E\left|\psi^{l,j}(t_l) - \psi^l(_{t_l})\right|^2.$$

Hence, by the Gronwall inequality [26] we obtain

$$E|\psi^{l,j}(s)-\psi^{l}(s)|^2 \leq De^{N(t_l-s)} \text{ a.e. in } [t_l-h,t_l], \tag{23}$$

where $D=E|\psi^{l,j}(t_l)-\psi^{l}(t_l)|^2$. Hence, it follows from (16) and (19) that: $\psi^{l,j}(t_l)\to\psi^{l}(t_l)$ which leads to $D\to 0$. Consequently, it follows that $\psi^{l,j}(s)\to\psi^{l}(s)$ in $L^2_F(t_{l-1}-h,t_l;R^{n_l})$, and thus $\beta^{l,j}(s)\to\beta^{l}(s)$ in $L^2_F(t_{l-1}-h,t_l;R^{n_l\times n_l})$.

Then $\forall t\in[t_{l-1},t_l-h)$ from expression we get:

$$d\left(\psi^{l,j}(t)-\psi^{l}(t)\right)=-\left[H^l_x\left(\psi^{l,j}_t,x^{l,j}(t),y^{l,j}(t),u^{l,j}(t),t\right)-H^l_x\left(\psi^{l}(t),x^{l}(t),y^{l}(t),u^{l}(t),t\right)\right]dt-$$
$$\left[H^l_y\left(\psi^{l,j}(t+h),x^{l,j}(t+h),y^{l,j}(t+h),u^{l,j}(t+h),t+h\right)-H^l_y\left(\psi^{l}(t+h),x^{l}(t+h),y^{l}(t+h),u^{l}(t+h),t+h\right)\right]dt$$
$$+\left(\beta^{l,j}(t)-\beta^{l}(t)\right)dw(t);$$

using simple transformations, in view of assumptions I-IV, it is achieved:

$$E\int_s^{t_l-h}|\beta^{l,j}(t)-\beta^{l}(t)|^2dt+E|\psi^{l,j}(s)-\psi^{l}_s|^2\leq EN\int_s^{t_l-h}\left|\psi^{l,j}(t)-\psi^{l}(t)\right|^2dt+$$
$$+EN\varepsilon\int_s^{t_l-h}|\beta^{l,j}(t)-\beta^{l}(t)|^2dt+E|\psi^{l,j}(t_l-h)-\psi^{l}(t_l-h)|^2.$$

Hence, according to Gronwall inequality we have:

$E|\psi^{l,j}(s)-\psi^{l}(s)|^2\leq De^{N(t_l-h-s)}$a.e. in $[t_{l-1},t_l-h)$, $l=\overline{1,r-1}$;, where constant D is determined in the following way: $D=E|\psi^{l,j}(t_l-h)-\psi^{l}(t_l-h)|^2$,which $D\to 0$. Then from (23) implies that $\psi^{l,j}(s)\to\psi^{l}(s)$ in $L^2_{F^l}(t_{l-1},t_l;R^n)$ and $\beta^{l,j}(s)\to\beta^{l}(s)$ in $L^2_{F^l}(t_{l-1},t_l;R^{n\times n})$. □

Based on Lemma 4.3, passing to the limit in system (19), we derive the fulfilment of (16). Finally,fulfilment of maximum principle and transversality conditions can be obtain to take the limits in (21) and (22). □

In order to establish the existence and uniqueness of solution of adjoint stochastic differential equations, it is enough to follow the method described in the article [16], to make use of the independence of Wiener processes $w^1(t),...,w^r(t)$.

5. Conclusion

Investigated stochastic delay systems are widely used in various optimization problems of nuclear fission, communication , self-oscillating, biology, technology, engineering and economy [9, 13, 21, 23, 32, 45, 47]. A classical approach for optimization, and particularly for control problems are to derive necessary conditions satisfied by an optimal solution. In this paper a necessary condition of optimality in form of maximum principle for stochastic control problem of constrained switching systems with delay on state is obtained. The necessary conditions developed in this study can be viewed as a stochastic analogues of the problems formulated in [8, 14, 20, 44] and extension of results [36, 42]. Theorem 3.1 and Theorem 4.1 are a improving of the results confirmed in [1, 2, 5, 6].

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