Filomat 28:10 (2014), 2047–2057
DOI 10.2298/FIL1410047N

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Cyclic Generalized φ-contractions in b-metric Spaces and an Application to Integral Equations

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Abstract. We introduce the notion of cyclic generalized φ-contractive mappings in b-metric spaces and discuss the existence and uniqueness of fixed points for such mappings. Our results generalize many existing fixed point theorems in the literature. Examples are given to support the usability of our results. Finally, an application to existence problem for an integral equation is presented.

1. Introduction and Preliminaries

The Banach's fixed point theorem (or the contractions mapping principle) is the most important metrical fixed point theorem in solving existence problems in many branches of mathematical analysis. There is a great number of generalizations of the Banach contraction principle. The underlying metric space can be generalized in many ways. For example, a new notion of b-metric space was introduced by Bakhtin in [4] and then extensively used by Czerwik in [8]. Since then some research works have dealt with fixed point theory for single valued and multivalued operators in b-metric (sometimes also called metric type) spaces (see [2, 9, 11, 12, 19] and the references therein).

Definition 1.1. *(Czerwik [8, 9]) Let X be a non-empty set and $s \geq 1$ a real number. A function $d : X \times X \to \mathbb{R}^+$ (nonnegative real numbers) is said to be a b-metric if, for all $x, y, z \in X$,*

(M1) $d(x, y) = 0$ *if and only if* $x = y$;
(M2) $d(x, y) = d(y, x)$;
(M3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space with parameter s.

Obviously, each metric space is a b-metric space (for $s = 1$). However, Czerwik [8, 9] has shown that a b-metric on X need not be a metric on X. The following simple examples can be used to show this.

2010 *Mathematics Subject Classification*. Primary 47H10; Secondary 54H25
Keywords. b-metric space, cyclic map, fixed point, control function
Received: 11 September 2013; Accepted: 08 March 2014
Communicated by V. Rakočević
The second author is thankful to the Ministry of Education, Science and Technological Development of Serbia.
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Example 1.2. Let $X = \{x_1, x_2, x_3\}$ and $d : X \times X \to \mathbb{R}^+$ be such that $d(x_1, x_2) = x \geq 3$, $d(x_1, x_3) = 2$, $d(x_2, x_3) = 1$, $d(x_n, x_n) = 0$, $d(x_n, x_k) = d(x_k, x_n) = 0$ for $n, k = 1, 2, 3$. Then

$$d(x_n, x_k) \leq \frac{x}{3}[d(x_n, x_i) + d(x_i, x_k)], \quad n, k, i = 1, 2, 3.$$

Hence, (X, d) is a b-metric space (with $s = x/3$), and not a metric space if $x > 3$.

Example 1.3. Let (X, ρ) be a metric space and $d(x, y) = (\rho(x, y))^p$, where $p > 1$ is a real number. Then d is a b-metric with $s = 2^{p-1}$. Condition (M3) follows easily from the convexity of function $f(x) = x^p$ $(x > 0)$.

Let (X, d) be a b-metric space. As in the metric case, the b-metric d induces a topology. For every $r > 0$ and any $x \in X$, we set $\mathcal{B}(x, r) = \{y \in X : d(x, y) < r\}$. The topology $\tau(d)$ on X associated with d is given by setting $\mathcal{U} \in \tau(d)$ if, and only if, for each $x \in \mathcal{U}$, there exists some $r > 0$ such that $\mathcal{B}(x, r) \subset \mathcal{U}$. The space X will be equipped with the topology $\tau(d)$. In particular a sequence $\{x_n\}$ converges to a point $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = 0$. Almost all the concepts and results obtained for metric spaces can be extended to the case of b-metric spaces.

Lemma 1.4. *(see, e.g., [12, 19]) Let (X, d) be a b-metric space with parameter s, and $\{y_n\}$ a sequence in X such that*

$$d(y_n, y_{n+1}) \leq q d(y_{n-1}, y_n), \quad \forall\, n \in \mathbb{N}.$$

Then $\{y_n\}$ is a Cauchy sequence in X, provided that $sq < 1$.

It is well known that in a standard metric space (X, d), the function d is continuous in both variables, in the sense that if $\{x_n\}, \{y_n\}$ are sequences in X such that $x_n \to x$, $y_n \to y$ as $n \to \infty$, then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$. A b-metric need not posses this property as the following example shows.

Example 1.5. *[11, Example 2]* Let $X = \mathbb{N} \cup \{\infty\}$ and let $d : X \times X \to \mathbb{R}$ be defined by

$$d(m, n) = \begin{cases} 0, & \text{if } m = n, \\ |\frac{1}{m} - \frac{1}{n}|, & \text{if one of } m, n \text{ is even and the other is even or } \infty, \\ 5, & \text{if one of } m, n \text{ is odd and the other is odd (and } m \neq n) \text{ or } \infty, \\ 2, & \text{otherwise.} \end{cases}$$

Then, considering all possible cases, it can be checked that for all $m, n, p \in X$, we have

$$d(m, p) \leq \frac{5}{2}(d(m, n) + d(n, p)).$$

Thus, (X, d) is a b-metric space (with $s = 5/2$). Let $x_n = 2n$ for each $n \in \mathbb{N}$. Then

$$d(2n, \infty) = \frac{1}{2n} \to 0 \quad \text{as } n \to \infty,$$

that is, $x_n \to \infty$, but $d(x_n, 1) = 2 \not\to 5 = d(\infty, 1)$ as $n \to \infty$.

One of the remarkable generalizations of the Banach contraction principle was reported by Kirk, Srinivasan and Veeramani [13] via so-called cyclic contractions. A mapping $T : A \cup B \to A \cup B$ is called cyclic if $T(A) \subseteq B$ and $T(B) \subseteq A$, where A, B are nonempty subsets of a metric space (X, d). Moreover, T is called a cyclic contraction if there exists $k \in (0, 1)$ such that $d(Tx, Ty) \leq k d(x, y)$ for all $x \in A$ and $y \in B$. Notice that although a contraction is continuous, cyclic contractions need not be. This is one of the important gains of this approach.

Definition 1.6. *([13, 20]) Let (X, d) be a metric space. Let p be a positive integer, $A_1, A_2, \ldots, A_p$ be nonempty subsets of X, $Y = \bigcup_{i=1}^p A_i$ and $T : Y \to Y$. Then $Y = \bigcup_{i=1}^p A_i$ is said to be a cyclic representation of Y with respect to T if* *(i)* A_i, $i = 1, 2, \ldots, p$ *are nonempty closed sets, and*

(ii) $T(A_1) \subseteq A_2, \ldots, T(A_{p-1}) \subseteq A_p$, $T(A_p) \subseteq A_1$.

Following [13], a number of fixed point theorems on cyclic representations of Y with respect to a self-mapping T have appeared (see, e.g., [1, 14–18, 20–22]).

In this paper, we introduce a new variant of cyclic contractive mappings, named as cyclic generalized φ-contractions in b-metric spaces and then derive the existence and uniqueness of fixed points for such mappings. Our main result generalizes and improves many existing theorems in the literature. Some examples are provided to demonstrate the validity of our results. As an application, in the last section, the existence of solution of an integral equation is proved under appropriate conditions.

2. Main Result

All the way through this paper, by $\mathbb{R}^+$, we designate the set of all nonnegative real numbers, while $\mathbb{N}$ is the set of all natural numbers.

We denote by Φ the set of functions $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(t) < \frac{t}{2s}$ for each $t > 0$, $\varphi(0) = 0$.

We introduce the notion of cyclic generalized φ-contraction in b-metric spaces as follows.

Definition 2.1. *Let (X, d) be a b-metric space with parameter s. Let p be a positive integer, $A_1, A_2, \ldots, A_p$ be nonempty subsets of X and $Y = \bigcup_{i=1}^p A_i$. An operator $T : Y \to Y$ satisfies a cyclic generalized φ-contraction for some $\varphi \in \Phi$, if*

(I) $Y = \bigcup_{i=1}^p A_i$ *is a cyclic representation of Y with respect to T;*

(II) for any $(x, y) \in A_i \times A_{i+1}$, $i = 1, 2, \ldots, p$ (with $A_{p+1} = A_1$),

$$d(Tx, Ty) \leq M(x, y) + L \min\{\varphi(d(x, Tx)), \varphi(d(y, Ty)), \varphi(d(x, Ty)), \varphi(d(y, Tx))\}, \tag{1}$$

where $L \geq 0$, and

$$M(x, y) = \max\left\{\varphi(d(x, y)), \varphi(d(x, Tx)), \varphi\left(\frac{d(x, Tx) + d(y, Ty)}{2}\right), \varphi\left(\frac{d(y, Tx) + d(x, Ty)}{2s}\right)\right\}. \tag{2}$$

The main result of this section is as follows:

Theorem 2.2. *Let (X, d) be a complete b-metric space, $p \in \mathbb{N}$, $A_1, A_2, \ldots, A_p$ nonempty closed subsets of X and $Y = \bigcup_{i=1}^p A_i$. Suppose $T : Y \to Y$ is a cyclic generalized φ-contractive mapping, for some $\varphi \in \Phi$. Then T has a unique fixed point. Moreover, the fixed point of T belongs to $\bigcap_{i=1}^p A_i$.*

Proof. Let $x_0 \in A_1$ (such a point exists since $A_1 \neq \emptyset$). Define the sequence $\{x_n\}$ in X by

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \ldots$$

We shall prove that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \tag{3}$$

If for some k, we have $x_{k+1} = x_k$, then it is easy to show (using (1)) that (3) holds. Hence, we can suppose that $d(x_n, x_{n+1}) > 0$ for all n. From condition (I), we observe that for all n, there exists $i = i(n) \in \{1, 2, \ldots, p\}$ such that $(x_n, x_{n+1}) \in A_i \times A_{i+1}$. Let $\delta_n = d(x_n, x_{n+1})$. Now we claim that for all $n \in \mathbb{N}$, we have

$$\delta_n < \frac{\delta_{n-1}}{2s}. \tag{4}$$

Indeed, from condition (1), we have

$$\begin{aligned}d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n)\\ &\le M(x_{n-1},x_n)+L\min\{\varphi(d(x_{n-1},Tx_{n-1})),\varphi(d(x_n,Tx_n)),\varphi(d(x_{n-1},Tx_n)),\varphi(d(x_n,Tx_{n-1}))\}\\ &= M(x_{n-1},x_n). \end{aligned}\tag{5}$$

By (2), we have

$$\begin{aligned}&M(x_{n-1},x_n)\\ &=\max\left\{\varphi(d(x_{n-1},x_n)),\varphi(d(x_{n-1},Tx_{n-1})),\varphi\left(\frac{d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)}{2}\right),\right.\\ &\qquad\left.\varphi\left(\frac{d(x_n,Tx_{n-1})+d(x_{n-1},Tx_n)}{2s}\right)\right\}\\ &=\max\left\{\varphi(d(x_{n-1},x_n)),\varphi(d(x_{n-1},x_n)),\varphi\left(\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}\right),\right.\\ &\qquad\left.\varphi\left(\frac{d(x_n,x_n)+d(x_{n-1},x_{n+1})}{2s}\right)\right\}\\ &=\max\left\{\varphi(d(x_{n-1},x_n)),\varphi\left(\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}\right),\varphi\left(\frac{d(x_{n-1},x_{n+1})}{2s}\right)\right\}.\end{aligned}$$

Consider the following possibilities.

- If $M(x_{n-1},x_n)=\varphi(d(x_{n-1},x_n))$, by (5) and using the fact that $\varphi(t)<\frac{t}{2s}$ for all $t>0$, we have

$$\delta_n=d(x_n,x_{n+1})\le\varphi(d(x_{n-1},x_n))<\frac{d(x_{n-1},x_n)}{2s}=\frac{\delta_{n-1}}{2s}.$$

- If $M(x_{n-1},x_n)=\varphi\left(\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}\right)$, we get

$$d(x_n,x_{n+1})\le\varphi\left(\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}\right)\le\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{4s},$$

wherefrom

$$\delta_n=d(x_n,x_{n+1})<\frac{1}{4s-1}d(x_{n-1},x_n)<\frac{\delta_{n-1}}{2s}.$$

- If $M(x_{n-1},x_n)=\varphi\left(\frac{1}{2s}d(x_{n-1},x_{n+1})\right)$, we get

$$d(x_n,x_{n+1})\le\varphi\left(\frac{1}{2s}d(x_{n-1},x_{n+1})\right)<\frac{1}{4s^2}d(x_{n-1},x_{n+1}).$$

On the other hand, by the property (M3), we have

$$d(x_{n-1},x_{n+1})\le s[d(x_{n-1},x_n)+d(x_n,x_{n+1})].$$

Thus, we have

$$d(x_n,x_{n+1})<\frac{1}{4s}d(x_{n-1},x_n)+\frac{1}{4s}d(x_n,x_{n+1}),$$

which implies that

$$\delta_n=d(x_n,x_{n+1})<\frac{1}{4s-1}d(x_{n-1},x_n)<\frac{\delta_{n-1}}{2s}.$$

Then, in all cases, we have $\delta_n < \frac{\delta_{n-1}}{2s}$ for all $n \in \mathbb{N}$. Therefore, we conclude that (4) holds.

Now, from (4) it follows that the sequence δ_n satisfies

$$0 < \delta_n < \frac{\delta_{n-1}}{2s} < \frac{\delta_{n-2}}{(2s)^2} < \frac{\delta_{n-3}}{(2s)^3} < \cdots < \frac{\delta_0}{(2s)^n}$$

and passing to the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Moreover, by Lemma 1.4, (4) implies that $\{x_n\}$ is a Cauchy sequence.

From the completeness of X, there exists $z \in X$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = z. \tag{6}$$

We shall prove that

$$z \in \bigcap_{i=1}^{p} A_i. \tag{7}$$

From the condition (I), and since $x_0 \in A_1$, we have $\{x_{np}\}_{n\geq 0} \subseteq A_1$. Since A_1 is closed, from (6), we get that $z \in A_1$. Again, from the condition (I), we have $\{x_{np+1}\}_{n\geq 0} \subseteq A_2$. Since A_2 is closed, from (6), we get that $z \in A_2$. Continuing this process, we obtain (7).

Now, we shall prove that z is a fixed point of T. Indeed, from (7), since for all n, there exists $i(n) \in \{1, 2, \ldots, p\}$ such that $x_n \in A_{i(n)}$, applying (II) with $x = x_n$ and $y = z$ and using (M3) we obtain

$$\begin{aligned}
d(z, Tz) &\leq s[d(z, Tx_n) + d(Tx_n, Tz)] \\
&\leq sd(z, x_{n+1}) + sM(x_n, z) + sL \min\{\varphi(d(x_n, x_{n+1})), \varphi(d(z, Tz)), \varphi(d(x_n, Tz)), \varphi(d(x_{n+1}, z))\} \\
&= sd(z, x_{n+1}) \\
&\quad + s \max\left\{\varphi(d(x_n, z)), \varphi(d(x_n, x_{n+1})), \varphi\left(\frac{d(x_n, x_{n+1}) + d(z, Tz)}{2}\right), \varphi\left(\frac{d(x_n, Tz) + d(x_{n+1}, z)}{2s}\right)\right\} \\
&\quad + sL \min\{\varphi(d(x_n, x_{n+1})), \varphi(d(z, Tz)), \varphi(d(x_n, Tz)), \varphi(d(x_{n+1}, z))\} \\
&\leq sd(z, x_{n+1}) + \frac{1}{2} \max\left\{d(x_n, z), d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) + d(z, Tz)}{2}, \frac{d(x_n, Tz) + d(x_{n+1}, z)}{2s}\right\} \\
&\quad + \frac{L}{2} \min\{d(x_n, x_{n+1}), d(z, Tz), d(x_n, Tz), d(x_{n+1}, z)\}.
\end{aligned}$$

Now, using that $d(x_n, Tz) \leq s[d(x_n, z) + d(z, Tz)]$, we get that

$$\begin{aligned}
d(z, Tz) \leq sd(z, x_{n+1}) + \frac{1}{2} \max\Big\{&d(x_n, z), d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) + d(z, Tz)}{2}, \\
&\frac{d(x_n, z) + d(z, Tz)}{2} + \frac{d(x_{n+1}, z)}{2s}\Big\} + \frac{L}{2} \min\{d(x_n, x_{n+1}), d(z, Tz), d(x_n, Tz), d(x_{n+1}, z)\}.
\end{aligned}$$

Passing to the limit as $n \to \infty$, we have

$$d(z, Tz) \leq \frac{1}{2} \cdot \frac{d(z, Tz)}{2},$$

which is only possible if $d(z, Tz) = 0$, i.e., $Tz = z$. (Note that continuity of d was not needed for this conclusion.)

We claim that z is the unique fixed point of T. Assume to the contrary that $Tu = u$ and $u \neq z$. Then (1) implies that

$$\begin{aligned} d(z,u) &= d(Tz,Tu) \\ &\le M(z,u) + L\min\{\varphi(d(z,Tz)), \varphi(d(u,Tu)), \varphi(d(z,Tu)), \varphi(d(u,Tz))\}, \end{aligned}$$

where

$$\begin{aligned} M(z,u) &= \max\left\{\varphi(d(z,u)), \varphi(d(z,Tz)), \varphi\left(\frac{d(z,Tz)+d(u,Tu)}{2}\right), \varphi\left(\frac{d(u,Tz)+d(z,Tu)}{2s}\right)\right\} \\ &< \max\left\{\frac{d(z,u)}{2s}, \frac{d(z,u)}{2s^2}\right\} = \frac{d(z,u)}{2s}, \end{aligned}$$

a contradiction. Hence, $z = u$. □

3. Consequences

In this section, we derive some fixed point theorems from our main result given by Theorem 2.2.

If we take $p = 1$ and $A_1 = X$ in Theorem 2.2, then we get immediately the following fixed point theorem.

Corollary 3.1. *Let (X,d) be a complete b-metric space and let $T : X \to X$ satisfy the following condition: there exists $\varphi \in \Phi$ such that*

$$d(Tx,Ty) \le M(x,y) + L\min\{\varphi(d(x,Tx)), \varphi(d(y,Ty)), \varphi(d(x,Ty)), \varphi(d(y,Tx))\},$$

where $L \ge 0$, and

$$M(x,y) = \max\left\{\varphi(d(x,y)), \varphi(d(x,Tx)), \varphi\left(\frac{d(x,Tx)+d(y,Ty)}{2}\right), \varphi\left(\frac{d(y,Tx)+d(x,Ty)}{2s}\right)\right\}.$$

for all $x, y \in X$. Then T has a unique fixed point.

Remark 3.2. *Corollary 3.1 extends many existing fixed point theorems in the literature, see e.g. [3, 5, 6], to complete b-metric space.*

Corollary 3.3. *Let (X,d) be a complete b-metric space, $p \in \mathbb{N}$, $A_1, A_2, \ldots, A_p$ nonempty closed subsets of X, $Y = \bigcup_{i=1}^{p} A_i$ and $T : Y \to Y$. Suppose that there exists a nondecreasing function $\varphi \in \Phi$ such that*

(a) $Y = \bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to T;

(b) for any $(x,y) \in A_i \times A_{i+1}$, $i = 1, 2, \ldots, p$ (with $A_{p+1} = A_1$),

$$d(Tx,Ty) \le \varphi(M_1(x,y)) + L\min\{\varphi(d(x,Tx)), \varphi(d(y,Ty)), \varphi(d(x,Ty)), \varphi(d(y,Tx))\},$$

where

$$M_1(x,y) = \max\left\{d(x,y), d(x,Tx), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(y,Tx)+d(x,Ty)}{2s}\right\},$$

Then T has a unique fixed point. Moreover, the fixed point of T belongs to $\bigcap_{i=1}^{p} A_i$

Proof. It follows from Theorem 2.2 by observing that if φ is nondecreasing, we have

$$\varphi(M_1(x,y)) = \max\left\{\varphi(d(x,y)), \varphi(d(x,Tx)), \varphi\left(\frac{d(x,Tx)+d(y,Ty)}{2}\right), \varphi\left(\frac{d(y,Tx)+d(x,Ty)}{2s}\right)\right\}.$$

□

Corollary 3.4. *Let (X, d) be a complete b-metric space, $p \in \mathbb{N}$, $A_1, A_2, \dots, A_p$ nonempty closed subsets of X, $Y = \bigcup_{i=1}^{p} A_i$ and $T : Y \to Y$. Suppose that there exists $\varphi \in \Phi$ such that*

(a) $Y = \bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to T;

(b) for any $(x, y) \in A_i \times A_{i+1}$, $i = 1, 2, \dots, p$ (with $A_{p+1} = A_1$),

$$d(Tx, Ty) \le \varphi(d(x, y)).$$

Then T has a unique fixed point. Moreover, the fixed point of T belongs to $\bigcap_{i=1}^{p} A_i$

Remark 3.5. *Corollary 3.4 is similar to Theorem 2.1 in [20] but considered in complete b-metric space.*
Taking in Corollary 3.4, $\varphi(t) = kt$ with $k \in (0, 1)$, we obtain Theorem 1.3 in [13] in complete b-metric space.

Corollary 3.6. *Let (X, d) be a complete b-metric space, $p \in \mathbb{N}$, $A_1, A_2, \dots, A_p$ nonempty closed subsets of X, $Y = \bigcup_{i=1}^{p} A_i$ and $T : Y \to Y$. Suppose that there exists $\varphi \in \Phi$ such that*

(a) $Y = \bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to T;

(b) for any $(x, y) \in A_i \times A_{i+1}$, $i = 1, 2, \dots, p$ (with $A_{p+1} = A_1$),

$$d(Tx, Ty) \le \varphi\left(\frac{d(x, Ty) + d(y, Tx)}{2s}\right).$$

Then T has a unique fixed point. Moreover, the fixed point of T belongs to $\bigcap_{i=1}^{p} A_i$

Remark 3.7. *Taking in Corollary 3.6, $\varphi(t) = kt$ with $k \in (0, 1)$, we obtain an analogue of Theorem 3 from [21] in complete b-metric space.*

Corollary 3.8. *Let (X, d) be a complete b-metric space, $p \in \mathbb{N}$, $A_1, A_2, \dots, A_p$ nonempty closed subsets of X, $Y = \bigcup_{i=1}^{p} A_i$ and $T : Y \to Y$. Suppose that there exists $\varphi \in \Phi$ such that*

(a) $Y = \bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to T;

(b) for any $(x, y) \in A_i \times A_{i+1}$, $i = 1, 2, \dots, p$ (with $A_{p+1} = A_1$),

$$d(Tx, Ty) \le \max\{\varphi(d(x, Tx)), \varphi(d(y, Ty))\}.$$

Then T has a unique fixed point. Moreover, the fixed point of T belongs to $\bigcap_{i=1}^{p} A_i$

Remark 3.9. *Taking in Corollary 3.8, $\varphi(t) = kt$ with $k \in (0, 1)$, we obtain an analogue of Theorem 5 from [21] in complete b-metric space.*

Corollary 3.10. *Let (X, d) be a complete b-metric space, $p \in \mathbb{N}$, $A_1, A_2, \dots, A_p$ nonempty closed subsets of X, $Y = \bigcup_{i=1}^{p} A_i$ and $T : Y \to Y$. Suppose that there exists $\varphi \in \Phi$ such that*

(a) $Y = \bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to T;

(b) for any $(x, y) \in A_i \times A_{i+1}$, $i = 1, 2, \dots, p$ (with $A_{p+1} = A_1$),

$$d(Tx, Ty) \le \max\{\varphi(d(x, y)), \varphi(d(x, Tx)), \varphi(d(y, Ty))\}.$$

Then T has a unique fixed point. Moreover, the fixed point of T belongs to $\bigcap_{i=1}^{p} A_i$

The following result (see Theorem 7 in [21]) extends Reich's fixed point theorem [7] to complete *b*-metric spaces.

Corollary 3.11. *Let (X, d) be a complete b-metric space, $p \in \mathbb{N}$, $A_1, A_2, \dots, A_p$ nonempty closed subsets of X, $Y = \bigcup_{i=1}^{p} A_i$ and $T : Y \to Y$. Suppose that there exist three positive constants a, b, c with $a + b + c < 1$ such that*

(a) *$Y = \bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to T;*

(b) *for any $(x, y) \in A_i \times A_{i+1}$, $i = 1, 2, \dots, p$ (with $A_{p+1} = A_1$),*

$$d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty).$$

Then T has a unique fixed point. Moreover, the fixed point of T belongs to $\bigcap_{i=1}^{p} A_i$

Proof. It follows from Corollary 3.10 by taking $\varphi(t) = (a + b + c)t$. □

Corollary 3.12. *Let (X, d) be a complete b-metric space, $p \in \mathbb{N}$, $A_1, A_2, \dots, A_p$ nonempty closed subsets of X, $Y = \bigcup_{i=1}^{p} A_i$ and $T : Y \to Y$. Suppose that there exist four positive constants a_1, a_2, a_3, a_4 with $a_1 + a_2 + a_3 + a_4 < 1$ such that*

(a) *$Y = \bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to T;*

(b) *for any $(x, y) \in A_i \times A_{i+1}$, $i = 1, 2, \dots, p$ (with $A_{p+1} = A_1$),*

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 \left(\frac{d(x, Ty) + d(y, Tx)}{2s} \right).$$

Then T has a unique fixed point. Moreover, the fixed point of T belongs to $\bigcap_{i=1}^{p} A_i$

Proof. It follows from Theorem 2.2 by taking $\varphi(t) = (a_1 + a_2 + a_3 + a_4)t$. □

Remark 3.13. *Corollary 3.12 extends and generalizes the well known fixed point theorem of Hardy and Rogers [10] to complete b-metric spaces. It also improves Theorem 3.7 of [12].*

4. Examples

We present some examples showing how our results can be used.

Example 4.1. Consider the set $X = \mathbb{R}$ equipped with the function $d : X \times X \to \mathbb{R}^+$ given as $d(x, y) = (x - y)^2$. Then, d is a b-metric with parameter $s = 2$ by Example 1.3. Let $A_1 = [0, +\infty)$ and $A_2 = (-\infty, 0]$. Then $A_1 \cup A_2 = X$ and $A_1 \cap A_2 = \{0\}$.

Consider the mapping $T : X \to X$ given by

$$Tx = \begin{cases} 0, & \text{if } x = 0, \\ -\frac{x}{6} \left| \sin \frac{1}{x} \right|, & \text{otherwise.} \end{cases}$$

Then, obviously, $X = A_1 \cup A_2$ is a cyclic representation of X with respect of T. Let $x \in A_1 \setminus \{0\}$ and $y \in A_2 \setminus \{0\}$ (the other possibility is treated symmetrically). Then

$$\begin{aligned} d(Tx, Ty) &= \left[-\frac{x}{6} \left| \sin \frac{1}{x} \right| + \frac{y}{6} \left| \sin \frac{1}{y} \right| \right]^2 = \frac{1}{36} \left[x \left| \sin \frac{1}{x} \right| + |y| \left| \sin \frac{1}{y} \right| \right]^2 \leq \frac{1}{18}(x^2 + y^2), \\ d(x, Tx) &= \left(x + \frac{x}{6} \left| \sin \frac{1}{x} \right| \right)^2 \geq x^2, \\ d(y, Ty) &= \left(y + \frac{y}{6} \left| \sin \frac{1}{y} \right| \right)^2 \geq y^2. \end{aligned}$$

Hence,

$$d(Tx,Ty) \le \frac{1}{18}(x^2+y^2) \le \frac{1}{18}(d(x,Tx)+d(y,Ty))$$
$$\le \frac{1}{9}\max\left\{d(x,y), d(x,Tx), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2s}\right\}$$
$$+ L\min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\},$$

for arbitrary $L > 0$. Take the function $\varphi \in \Phi$ given as $\varphi(t) = \frac{t}{9}$ (note that $\varphi(t) < \frac{t}{4} = \frac{t}{2s}$ for $t > 0$). Then,

$$d(Tx,Ty) \le \max\left\{\varphi(d(x,y)), \varphi(d(x,Tx)), \varphi\left(\frac{d(x,Tx)+d(y,Ty)}{2}\right), \varphi\left(\frac{d(x,Ty)+d(y,Tx)}{2s}\right)\right\}$$
$$+ L\min\{\varphi(d(x,Tx)), \varphi(d(y,Ty)), \varphi(d(x,Ty)), \varphi(d(y,Tx))\}.$$

The previous inequality is also satisfied if one of x, y (or both) is equal to 0; hence, all the conditions of Theorem 2.2 (as a matter of fact, of Corollary 3.8) are satisfied. Obviously, T has a unique fixed point 0, belonging to $A_1 \cap A_2$.

Example 4.2. Consider the b-metric space (X,d) given in Example 1.5 and the mapping $T : X \to X$ given as

$$Tn = \begin{cases} 6n, & \text{if } n \in \mathbb{N}, \\ \infty, & \text{if } n = \infty. \end{cases}$$

If $A_1 = \{n : n \in \mathbb{N}\} \cup \{\infty\}$ and $A_2 = \{6n : n \in \mathbb{N}\} \cup \{\infty\}$ then $A_1 \cup A_2$ is a cyclic representation of X with respect to T.

Take $\varphi \in \Phi$ given as $\varphi(t) = \frac{11}{60}t$ (note that $\varphi(t) < \frac{1}{5}t = \frac{t}{2s}$). In order to check the contractive condition (1), consider the following cases.

If $x, y \in \mathbb{N}$ then

$$d(Tx,Ty) = d(6x,6y) = \frac{1}{6}\left|\frac{1}{x}-\frac{1}{y}\right| \le \frac{11}{60}\left|\frac{1}{x}-\frac{1}{y}\right|$$
$$= \varphi\left(\left|\frac{1}{x}-\frac{1}{y}\right|\right) \le \varphi(d(x,y))$$
$$\le M(x,y) + L\min\{\varphi(d(x,Tx)), \varphi(d(y,Ty)), \varphi(d(x,Ty)), \varphi(d(y,Tx))\},$$

and (1) holds. If $x = \infty$ and y is an even integer then

$$d(Tx,Ty) = d(\infty,6y) = \frac{1}{6y} \le \varphi\left(\frac{1}{y}\right) = \varphi(d(\infty,y))$$
$$\le M(x,y) + L\min\{\varphi(d(x,Tx)), \varphi(d(y,Ty)), \varphi(d(x,Ty)), \varphi(d(y,Tx))\}.$$

Finally, if $x = \infty$ and y is an odd integer then $d(x,y) = 5$ and (1) trivially holds.

Hence, all the conditions of Theorem 2.2 (as a matter of fact, of Corollary 3.4) are satisfied. Obviously, T has a unique fixed point ∞, belonging to $A_1 \cap A_2$.

5. An Application to Integral Equations

Inspired by [16], we will consider the following integral equation for an unknown function u:

$$u(t) = \int_a^b G(t,s)f(s,u(s))\,ds, \qquad t \in [a,b], \tag{8}$$

where $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ and $G : [a,b]^2 \to [0,+\infty)$ are given continuous functions.

Let X be the set $C[a,b]$ of real continuous functions on $[a,b]$ and let $d : X \times X \to \mathbb{R}^+$ be given by

$$d(u,v) = \max_{a \le t \le b}[u(t) - v(t)]^2.$$

It is easy to see that d is a b-metric with parameter $s = 2$ and that (X,d) is a complete b-metric space. If $T : X \to X$ is defined by

$$Tu(t) = \int_a^b G(t,s) f(s,u(s))\, ds, \quad t \in [a,b],$$

then it is clear that a function u is a solution of the given equation (8) if and only if it is a fixed point of the mapping T. We will prove the existence and uniqueness of the fixed point of T under the following conditions.

(I) There exist functions $\alpha, \beta \in X$ and real numbers α_0, β_0 such that

$$\alpha_0 \le \alpha(t) \le \beta(t) \le \beta_0, \quad t \in [a,b]$$

and that

$$T\alpha(t) \le \beta(t) \quad \text{and} \quad T\beta(t) \ge \alpha(t), \qquad t \in [a,b].$$

(II) The function $f(s,\cdot)$ is nonincreasing, i.e., for each $s \in [a,b]$,

$$x,y \in \mathbb{R},\ x \le y \implies f(s,x) \ge f(s,y).$$

(III) $\displaystyle\max_{a \le t \le b} \int_a^b G^2(t,s)\, ds \le \frac{1}{b-a}$.

(IV) There exists a nondecreasing function $\varphi \in \Phi$ such that for each $s \in [a,b]$ and all $x,y \in \mathbb{R}$ satisfying ($x \ge \alpha_0$ and $y \le \beta_0$) or ($x \le \beta_0$ and $y \ge \alpha_0$), the following inequality holds:

$$[f(s,x) - f(s,y)]^2 \le \varphi((x-y)^2).$$

Note that (since $s = 2$), φ has to satisfy the condition $\varphi(t) < \frac{t}{4}$ for $t > 0$. Examples of such function are $\varphi(t) = \frac{t}{5}$, $\varphi(t) = \frac{t^2}{4(1+t)}$, $\varphi(t) = \frac{1}{4}\ln(1+t)$ and many others.

Theorem 5.1. *Under the conditions* (I)–(IV), *the equation* (8) *has a unique solution* $u^* \in X$ *and it belongs to* $C = \{u \in X : \alpha(t) \le u(t) \le \beta(t),\ t \in [a,b]\}$.

Proof. Consider closed subsets

$$A_1 = \{u \in X : u(t) \le \beta(t),\ t \in [a,b]\} \quad \text{and} \quad A_2 = \{u \in X : u(t) \ge \alpha(t),\ t \in [a,b]\}$$

of the space (X,d). We will prove that $T(A_1) \subseteq A_2$, and $T(A_2) \subseteq A_1$, i.e., $A_1 \cup A_2 = Y$ is a cyclic representation of Y with respect to T. Indeed, let $u \in A_1$, i.e., $u(s) \le \beta(s)$ for each $s \in [a,b]$. Using the condition (II) and that $G(t,s)$ is nonnegative, we get that

$$G(t,s) f(s,u(s)) \ge G(t,s) f(s,\beta(s)), \quad t,s \in [a,b],$$

which implies that

$$Tu(t) = \int_a^b G(t,s) f(s,u(s)) \ge \int_a^b G(t,s) f(s,\beta(s)) = T\beta(t) \ge \alpha(t), \quad t \in [a,b],$$

by (I). Hence, $Tu(t) \ge \alpha(t)$, $t \in [a,b]$, i.e., $Tu \in A_2$. The inclusion $T(A_2) \subseteq A_1$ can be proved in a similar way.

Let now $(u,v) \in A_1 \times A_2$, i.e., $u(t) \leq \beta(t) \leq \beta_0$, $t \in [a,b]$ and $v(t) \geq \alpha(t) \geq \alpha_0$, $t \in [a,b]$. Then, using the conditions (III), (IV) and the Cauchy-Schwarz inequality, we obtain for $t \in [a,b]$:

$$\begin{aligned}
[Tu(t) - Tv(t)]^2 &= \left\{\int_a^b G(t,s)[f(s,u(s)) - f(s,v(s))]\,ds\right\}^2 \\
&\leq \int_a^b G^2(t,s)\,ds \int_a^b [f(s,u(s)) - f(s,v(s))]^2\,ds \\
&\leq \frac{1}{b-a}\int_a^b \varphi([u(s)-v(s)]^2)\,ds \\
&\leq \frac{1}{b-a}\cdot(b-a)\varphi(d(u,v)).
\end{aligned}$$

Thus, $d(Tu,Tv) \leq \varphi(d(u,v))$ holds.

We conclude that all the conditions of Corollary 3.3 are fulfilled. Hence, the mapping T has a unique fixed point $u^* \in A_1 \cap A_2 = C$, i.e., the equation (8) has a unique solution belonging to this set. □

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