



Permutability Degrees of Finite Groups

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Abstract. Given a finite group G , we introduce the *permutability degree* of G , as

$$pd(G) = \frac{1}{|G| |\mathcal{L}(G)|} \sum_{X \in \mathcal{L}(G)} |P_G(X)|,$$

where $\mathcal{L}(G)$ is the subgroup lattice of G and $P_G(X)$ the permutizer of the subgroup X in G , that is, the subgroup generated by all cyclic subgroups of G that permute with $X \in \mathcal{L}(G)$. The number $pd(G)$ allows us to find some structural restrictions on G . Successively, we investigate the relations between $pd(G)$, the probability of commuting subgroups $sd(G)$ of G and the probability of commuting elements $d(G)$ of G . Proving some inequalities between $pd(G)$, $sd(G)$ and $d(G)$, we correlate these notions.

1. Introduction

All the groups of the present paper are supposed to be finite. Given a group G and its subgroup lattice $\mathcal{L}(G)$, the *subgroup commutativity degree*

$$sd(G) = \frac{|\{(H, K) \in \mathcal{L}(G) \times \mathcal{L}(G) \mid HK = KH\}|}{|\mathcal{L}(G)|^2}$$

of G and the *commutativity degree*

$$d(G) = \frac{|\{(x, y) \in G \times G \mid xy = yx\}|}{|G|^2}$$

of G have been largely studied in the last years. Fundamental properties and interesting generalizations of $sd(G)$ can be found in [11–13, 22, 27–29], and for $d(G)$ in [1, 2, 6, 8–10, 14, 17–20, 24]. To study these notions, various perspectives have been considered in literature, because both measure theory and combinatorial techniques may be applied in order to get restrictions on the structure of a group.

The present paper investigates a similar concept, the *permutability degree* of G

$$pd(G) = \frac{1}{|G| |\mathcal{L}(G)|} \sum_{X \in \mathcal{L}(G)} |P_G(X)|$$

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and its connections with $sd(G)$ and $d(G)$. In the previous formula, the *permutizer* $P_G(X)$ of a subgroup X of G is defined to be the subgroup generated by all cyclic subgroups of G that permute with X , that is, $P_G(X) = \langle g \in G \mid \langle g \rangle X = X \langle g \rangle \rangle$. This means that $X \in \mathcal{L}(P_G(X))$ and $X \neq P_G(X)$ if and only if $X \langle g \rangle = \langle g \rangle X$ for some $g \in G - X$.

We concentrate on permutizers because several classifications are available in literature on this topic. Recall that a group G such that $X \neq P_G(X)$ for every proper subgroup X of G is said to satisfy the *permutizer condition P*, or briefly **P**-group. Therefore the permutizer condition generalizes the well-known *normalizer condition* (see [26]) and gives information on how the group is near to be supersolvable. The study of permutizers is not new and it is based on a series of fundamental contributions [3, 21, 23, 30] in the last 20 years. From [3, Corollary 2], we know that for groups of odd order the permutizer condition is equivalent of being supersolvable and actually, a complete classification of **P**-groups can be found in [3].

Now we may define the subgroup

$$P(G) = \bigcap_{H \in \mathcal{L}(G)} P_G(H)$$

and correlate it with other subgroups of G . For instance, it is easy to check that the *norm* $N(G)$ of G (see [26] for the properties of $N(G)$) satisfies the following relation

$$Z(G) = \bigcap_{x \in G} C_G(x) \subseteq N(G) = \bigcap_{H \in \mathcal{L}(G)} N_G(H) \subseteq \bigcap_{H \in \mathcal{L}(G)} P_G(H) = P(G).$$

This relation emphasizes how $P(G)$ is connected with other subgroups, widely investigated in literature, such as intersections of normalizers or of centralizers. Note that for any $X \in \mathcal{L}(P(G))$ one has $P_G(X) = G$.

The subgroup $P(G)$ is also important because it allows us to “manipulate” the expression of $pd(G)$, for getting some analogies with

$$sd(G) = \frac{1}{|\mathcal{L}(G)|^2} \sum_{H \in \mathcal{L}(G)} |C_{\mathcal{L}(G)}(H)| \quad \text{and} \quad d(G) = \frac{1}{|G|^2} \sum_{x \in G} |C_G(x)|,$$

where $P_G(X)$ is the natural substitute of $C_{\mathcal{L}(G)}(X) = \{Y \in \mathcal{L}(G) \mid YX = XY\}$ in [22, 27] and of $C_G(x) = \{y \in G \mid xy = yx\}$ in [2, 8]. This manipulation of the expression of $pd(G)$ will allow us to detect whether $P(G)$ is cyclic or not by looking only at the size of $pd(G)$.

2. Basic Properties and Terminology

Some of the following observations will be useful later on.

Remark 2.1. Since $P_G(X)$ is a subgroup of G , for all $X \in \mathcal{L}(G)$, and it always contains the trivial subgroup, then $|P_G(X)| \leq |G|$ and also

$$0 < \sum_{X \in \mathcal{L}(G)} |P_G(X)| \leq |\mathcal{L}(G)| |G|$$

so that $pd(G) \in]0, 1]$.

Remark 2.2. A group G has $pd(G) = 1$ if and only if the sum of all $|P_G(X)|$ for $X \in \mathcal{L}(G)$ is equal to $|G||\mathcal{L}(G)|$. By default, a *quasihamiltonian group* G , that is, a group in which every subgroup is permutable, has $pd(G) = 1$. A classification of quasihamiltonian groups can be found in [26, Theorems 2.4.11 and 2.4.16] and, roughly speaking, these groups are direct products of abelian groups by a copy of the quaternion group of order 8. In particular, abelian groups have permutability degree equal to 1.

Another case in which the permutability degree reaches 1 is the following.

Remark 2.3. A **P**-group G in which all proper subgroups are maximal has $pd(G) = 1$. In such a case for all proper subgroups X of G one has $X \subset P_G(X) = G$ and so $pd(G) = 1$. One might be tempted to think that all **P**-groups have permutability degree equal to 1, but Example 3.2 below shows this is false, and then the additional condition “in which all proper subgroups are maximal” cannot be omitted.

Now we rewrite the original expression of permutability degree in the following more useful form. Since $\mathcal{L}(P(G))$ is a sublattice of $\mathcal{L}(G)$, it turns out that

$$pd(G) = \frac{1}{|G||\mathcal{L}(G)|} \left(\sum_{X \in \mathcal{L}(P(G))} |P_G(X)| + \sum_{X \in \mathcal{L}(G) - \mathcal{L}(P(G))} |P_G(X)| \right). \tag{2.1}$$

Note also that a cyclic group C (or better a quasihamiltonian group Q) has $sd(C) = pd(C) = d(C) = 1$ (or better $sd(Q) = pd(Q) = 1$). Therefore, relations between $sd(G)$ and $pd(G)$ are meaningful when G is noncyclic and nonquasihamiltonian (see Remark 2.2).

For the sake of completeness, we recall some results of Beidleman and Heineken in [4]. The *quasicenter* $Q(G)$ of G is the subgroup of G generated by all elements $g \in G$ such that $\langle g \rangle K = K \langle g \rangle$, where K is an arbitrary subgroup of G . The subgroup $Q(G)$ was introduced by Mukherjee and studied by several authors in the last years (see [4, 5, 25]), who investigated chains of quasicenters and relations with supersolvable groups. On the other hand, the *hyperquasicenter* of G , denoted by $Q_\infty(G)$, is the largest term of the chain $1 = Q_0(G) \leq Q_1(G) = Q(G) \leq \dots \leq Q_i(G) \leq Q_{i+1}(G) \leq \dots$ of normal subgroups of G , where, for any $i \geq 0$, $Q_{i+1}(G)/Q_i(G) = Q(G/Q_i(G))$ and $Q_\infty(G) = \bigcup_{i \geq 0} Q_i(G)$.

Recall that a normal subgroup N of G is said to be *hypercyclically embedded* in G if it contains a G -invariant series whose factors are cyclic. It is easy to see that G contains a unique largest hypercyclically embedded subgroup, which we denote $\Sigma(G)$. More precisely, [4, Theorem 1] shows that $\Sigma(G) = Q_\infty(G)$ is true for any group G . Some interesting connections hold between \mathbf{P} -groups, $P(G)$ and $Q_\infty(G)$. For instance, [3, (3.1), p. 697] shows that a group G is a \mathbf{P} -group if and only if $G/\Sigma(G)$ is a \mathbf{P} -group. As a first consequence, a group G is a \mathbf{P} -group if and only if $G/Q_\infty(G)$ is a \mathbf{P} -group. As a second consequence, $Z(G) \subseteq Q(G) \subseteq P(G)$ is true for any group G . Furthermore, $Q_\infty(G) = P(G)$ if and only if $P(G) = \Sigma(G)$.

3. Examples

Now we specify some of the previous notions for the symmetric group S_3 on 3 objects. This will help us to visualize analogies and differences between permutability degrees, subgroup commutativity degrees and commutativity degrees.

Example 3.1. The smallest nonabelian group S_3 has $\mathcal{L}(S_3) = \{\{1\}, S_3, A_3, H, K, L\}$, where $A_3 = \langle(123)\rangle = \langle a \rangle$, $H = \langle(12)\rangle = \langle h \rangle$, $K = \langle(13)\rangle = \langle k \rangle$, $L = \langle(23)\rangle = \langle l \rangle$. Noting that $HK \neq KH, HL \neq LH, KL \neq LK$, one has

$$P_{S_3}(\{1\}) = P_{S_3}(A_3) = P_{S_3}(S_3) = S_3 ; P_{S_3}(H) = P_{S_3}(K) = P_{S_3}(L) = S_3 ; P(S_3) = S_3.$$

The fact that $P_{S_3}(\{1\}) = P_{S_3}(A_3) = P_{S_3}(S_3) = S_3$ is clear, since we are dealing with permutizers of normal subgroups. On the other hand, $A_3 = \langle a \rangle \subseteq P_{S_3}(H)$ by definition. Now, $H \subseteq P_{S_3}(H)$ is obvious. Then HA_3 , which is a subgroup of S_3 , should be contained in $P_{S_3}(H)$ and $|HA_3| = \frac{|H||A_3|}{|H \cap A_3|} = 6$. This forces $P_{S_3}(H)$ to be equal to S_3 . Similarly we get $P_{S_3}(K) = P_{S_3}(L) = S_3$.

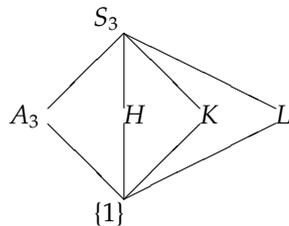


Fig.III.1. Hasse diagram of $\mathcal{L}(S_3)$.

It is interesting to note that an example as easy as this has a lot of properties in our perspective of study. The group S_3 is supersolvable by looking at the series $\{1\} \triangleleft A_3 \triangleleft S_3$, but is not quasihamiltonian, due to $HK \neq KH$. At the same time, S_3 does not satisfy the normalizer condition, since it is not nilpotent. Moreover,

S_3 has $Z(S_3) = \{1\}$, $\Sigma(S_3) = Q_2(S_3) = Q_\infty(S_3) = S_3$, $Q(S_3) = A_3$ and it is a \mathbf{P} -group, since $S_3/\Sigma(S_3) = \{1\}$ is obviously a \mathbf{P} -group.

A direct calculation shows that

$$6 \cdot 6 \text{pd}(S_3) = \sum_{X \in \mathcal{L}(S_3)} |P_{S_3}(X)| = |P_{S_3}(H)| + |P_{S_3}(K)| + |P_{S_3}(L)| + |P_{S_3}(A_3)| + |P_{S_3}(S_3)| + |P_{S_3}(\{1\})| = 36,$$

then $\text{pd}(S_3) = 1 > \text{sd}(S_3) = \frac{5}{6} > \frac{1}{2} = d(S_3)$, agreeing with the computations in [27, p.2510] and [8, 9].

Another easy (but interesting) example is the following.

Example 3.2. The dihedral group of order 8 is $D_8 = \langle a, b \mid a^2 = b^4 = 1, a^{-1}ba = b^{-1} \rangle$ and has $\mathcal{L}(D_8) = \{\{1\}, \langle b \rangle, \langle b^2 \rangle, \langle a \rangle, \langle ba \rangle, \langle b^2a \rangle, \langle b^3a \rangle, \{1, b^2, a, b^2a\}, \{1, b^2, ba, b^3a\}, D_8\}$. The normal subgroups are $D_8, \{1\}, B = \langle b \rangle$, $Z(D_8) = \langle b^2 \rangle$, $M_1 = \{1, b^2, a, b^2a\}$ and $M_2 = \{1, b^2, ba, b^3a\}$. Notice that $H = \langle b^2a \rangle$ and $K = \langle a \rangle$ are contained in M_1 , while $U = \langle ba \rangle$ and $V = \langle b^3a \rangle$ in M_2 .

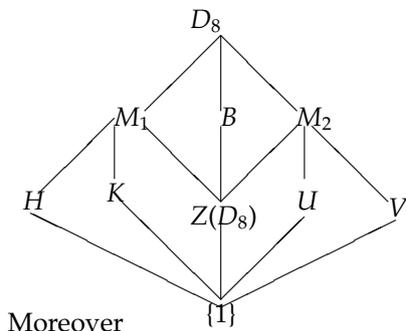


Fig.III.2. Hasse diagram of $\mathcal{L}(D_8)$.

Moreover

$$8 = |D_8| = |P_{D_8}(\{1\})| = |P_{D_8}(D_8)| = |P_{D_8}(\langle b \rangle)| = |P_{D_8}(Z(D_8))| = |P_{D_8}(M_1)| = |P_{D_8}(M_2)|,$$

$$4 = |M_1| = |P_{D_8}(H)| = |P_{D_8}(K)|, \quad 4 = |M_2| = |P_{D_8}(U)| = |P_{D_8}(V)|$$

and $Q(D_8) = P(D_8) = Z(D_8) < \Sigma(D_8) = Q_\infty(D_8) = D_8$. In fact, $D_8 = \Sigma(D_8)$ is supersolvable and it is also a \mathbf{P} -group, but nevertheless its permutability degree is different from 1, because

$$8 \cdot 10 \text{pd}(D_8) = \sum_{X \in \mathcal{L}(D_8)} |P_{D_8}(X)| = 6 \cdot |D_8| + 2 \cdot |\{1, b^2, a, b^2a\}| + 2 \cdot |\{1, b^2, ba, b^3a\}| = 64.$$

More precisely,

$$d(D_8) = \frac{5}{8} < \text{pd}(D_8) = \frac{64}{80} = \frac{4}{5} < \text{sd}(D_8) = \frac{46}{55}.$$

The value of $d(D_8)$ can be found in [8] and that of $\text{sd}(D_8)$ in [27]. This example shows that there exist \mathbf{P} -groups with permutability degree different from 1. Note that D_8 satisfies $8 = |D_8| < |\mathcal{L}(D_8)| = 10$ but

$$8 = |\{\{1\}, D_8, \langle b \rangle, \{1, b^2, ba, b^3a\}, \{1, b^2, a, b^2a\}, \langle b^2a \rangle, \langle b^2 \rangle, \langle a \rangle\}| = |\mathcal{C}_{\mathcal{L}(D_8)}(\langle a \rangle)|$$

$$\not\leq |\mathcal{Z}_{\mathcal{L}(D_8)}(\langle a \rangle)| = |\{\{1\}, \langle b^2a \rangle, \langle b^2 \rangle, \langle a \rangle\}| = 4.$$

Examples 3.1 and 3.2 illustrate a series of problems for the computation of the permutability degree, arising from the nature of the subgroup lattice of the groups under consideration. We will come back to this point later on.

4. General Properties of the Permutability Degree

We note that [8, Theorems 2.5, 3.3] shows that the commutativity degree is monotone. This is a well-known property, which is due to the fact that we are dealing with a positive monotone measure of probability. Similar situations can be found for $\text{sd}(G)$ in [27, Proposition 2.4, Corollaries 2.5, 2.6, 2.7, Theorems 3.1.1, 3.1.5] and in [9, 14, 22]. For $\text{pd}(G)$ we have something similar.

Theorem 4.1. *Let H be a subgroup of a group G . Then*

$$\frac{|\mathcal{L}(H)|}{|\mathcal{L}(G)| |G : H|} pd(H) \leq pd(G).$$

Moreover, if $|P_G(X) : P_H(X)| \leq |G : H|$ for all $X \in \mathcal{L}(G)$, $P(G) \leq H$ and $|\mathcal{L}(G) - \mathcal{L}(P(G))| \leq |\mathcal{L}(P(G))|$, then $|\mathcal{L}(G)| pd(G) \leq 2|\mathcal{L}(H)| pd(H)$. In particular,

$$\frac{|\mathcal{L}(H)|}{|\mathcal{L}(G)| |G : H|} pd(H) \leq pd(G) \leq \frac{2|\mathcal{L}(H)|}{|\mathcal{L}(G)|} pd(H).$$

Proof. We start by proving the first inequality. Since $|P_H(X)| \leq |P_G(X)|$ for all $X \in \mathcal{L}(G)$, we have

$$|G| |\mathcal{L}(G)| pd(G) = \sum_{X \in \mathcal{L}(G)} |P_G(X)| \geq \sum_{X \in \mathcal{L}(G)} |P_H(X)| = \sum_{X \in \mathcal{L}(H)} |P_H(X)| = pd(H) |H| |\mathcal{L}(H)|$$

and the result follows.

Now we prove the second inequality. Since by hypothesis, $|P_G(X) : P_H(X)| \leq |G : H|$, we have $|P_G(X)| \leq |G : H| |P_H(X)|$ and so

$$|G| |\mathcal{L}(G)| pd(G) = \sum_{X \in \mathcal{L}(G)} |P_G(X)| \leq |G : H| \sum_{X \in \mathcal{L}(G)} |P_H(X)|$$

but, from (2.1), this means

$$= |G : H| \left(\sum_{X \in \mathcal{L}(P(G))} |P_H(X)| + \sum_{X \in \mathcal{L}(G) - \mathcal{L}(P(G))} |P_H(X)| \right)$$

and the inequality $|\mathcal{L}(G) - \mathcal{L}(P(G))| \leq |\mathcal{L}(P(G))|$ provided by hypothesis, implies

$$\begin{aligned} &\leq |G : H| \left(\sum_{X \in \mathcal{L}(P(G))} |P_H(X)| + \sum_{X \in \mathcal{L}(P(G))} |P_H(X)| \right) = 2|G : H| \sum_{X \in \mathcal{L}(P(G))} |P_H(X)| \\ &\leq 2|G : H| \sum_{X \in \mathcal{L}(H)} |P_H(X)| = 2 |G : H| |H| |\mathcal{L}(H)| pd(H) \end{aligned}$$

from which we have $|\mathcal{L}(G)| pd(G) \leq 2|\mathcal{L}(H)| pd(H)$. \square

A classic splitting result for the product probability of two independent events is described by the following corollary. The proof may be generalized to finitely many factors, whose orders are pairwise coprime.

Proposition 4.2. *Let G and H be two groups such that $\gcd(|G|, |H|) = 1$. Then $pd(G \times H) = pd(G) pd(H)$.*

Proof. Given three groups A, B and C such that $A \times B \subseteq C$, we know that $N_C(A \times B) = N_C(A) \times N_C(B)$. This holds similarly for the permutizers and it is easy to see that $P_C(A \times B) = P_C(A) \times P_C(B)$. Now this fact and the assumption $\gcd(|G|, |H|) = 1$ allow us to conclude that

$$\frac{1}{|G \times H| |\mathcal{L}(G \times H)|} \sum_{X \times Y \in \mathcal{L}(G \times H)} |P_{G \times H}(X \times Y)| = \frac{1}{|G| |\mathcal{L}(G)|} \frac{1}{|H| |\mathcal{L}(H)|} \sum_{X \in \mathcal{L}(G)} |P_G(X)| \sum_{Y \in \mathcal{L}(H)} |P_H(Y)|.$$

\square

The underlying problem we deal with is the order of the subgroup lattices, which is hard to predict in general. If we concentrate on some groups arising from finite geometries, then the situation is more clear (dihedral groups, semidihedral groups and generalized quaternion groups were studied in [2, 8, 11, 22, 27–29] from a similar perspective). Having in mind Examples 3.1 and 3.2, we observe from [26, pp. 26–29] and [22, 27] that the dihedral group

$$D_{2n} = \langle x, y \mid x^2 = y^n = 1, x^{-1}yx = y^{-1} \rangle = C_2 \times C_n = \langle x \rangle \times \langle y \rangle \tag{4.1}$$

of symmetries of a regular polygon with $n \geq 1$ edges has order $2n$ and splits in the semidirect product of a cyclic group $\langle y \rangle \simeq C_n$ of order n by a cyclic group $\langle x \rangle \simeq C_2$ of order 2 acting by inversion on C_n . In particular, $S_3 \simeq D_6$ for $n = 3$ and one can note that the Hasse diagram of $\mathcal{L}(D_6) = \mathcal{L}(S_3)$ forms a *diamond* in which there are only 4 *atomic elements* (see [26] for this terminology) in between $\{1\}$ and D_6 and their number can be easily computed. The following Fig.III.3 summarizes the information of Example 3.1 in a more general situation.

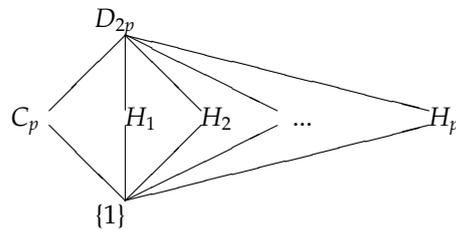


Fig.III.3. Hasse diagram of $\mathcal{L}(D_{2p})$ with odd prime $p \geq 3$;
 $H_1 \simeq H_2 \simeq \dots \simeq H_p \simeq C_2$.

From Figs.III.1, III.2 and III.3, it is clear that $\mathcal{L}(D_{2p})$ has $p + 1$ proper subgroups, and this fact comes out from the following formula

$$|\mathcal{L}(D_{2n})| = \sigma(n) + \tau(n), \tag{4.2}$$

where $\sigma(n)$ and $\tau(n)$ are the sum and the number of all divisors of n (here n is arbitrary, not necessarily an odd prime), respectively.

In particular, if $n = p^m$ is a power of a prime p (possibly $p = 2$) for some $m \geq 0$, then the set of all divisors of p^m is $\text{Div}(p^m) = \{1, p, p^2, \dots, p^m\}$ so that

$$\sigma(p^m) = \sum_{j=0}^m p^j = \frac{1 - p^{m+1}}{1 - p} \quad \text{and} \quad \tau(p^m) = |\text{Div}(p^m)| = m + 1. \tag{4.3}$$

The reader has probably noted that we used the formula for the sum of a geometric series in the previous expression for $\sigma(p^m)$. Then we may conclude that

$$|\mathcal{L}(D_{2p^m})| = 1 + m + \frac{1 - p^{m+1}}{1 - p} = m + \frac{p^{m+1} + p - 2}{p - 1}. \tag{4.4}$$

The next result shows an upper bound for $pd(G)$, when $|\mathcal{L}(G)|$ is of type (4.4).

Theorem 4.3. *Let G be a noncyclic group and p the smallest prime divisor of $|G|$. If $|P(G)| = p$ and $|\mathcal{L}(G)| = m + \frac{p^{m+1} + p - 2}{p - 1}$ for some $m \geq 0$, then*

$$pd(G) \leq \frac{p^{m+1} + 2p^2 + (m - 3)p - m}{p^{m+2} + (m + 1)p^2 - (m + 2)p}.$$

Proof. If $P_G(X) = G$ for some $X \in \mathcal{L}(G)$, then $|G| = |P_G(X)| = |P(G)|$ and G would be cyclic, contradicting our assumption. Without loss of generality we may assume $P_G(X) \neq G$ for all $X \in \mathcal{L}(G)$. The minimality of p implies that $|P_G(X)| \leq \frac{|G|}{p}$ for all $X \in \mathcal{L}(G)$. Of course, $|\mathcal{L}(P(G))| = 2$ and (2.1) becomes

$$\begin{aligned} |G| \cdot \left(\frac{mp - m + p^{m+1} + p - 2}{p - 1} \right) \cdot pd(G) &= 2|G| + \sum_{X \in \mathcal{L}(G) - \mathcal{L}(P(G))} |P_G(X)| \\ &\leq 2|G| + (|\mathcal{L}(G) - \mathcal{L}(P(G))|) \frac{|G|}{p} = 2|G| + \frac{|G|}{p} (|\mathcal{L}(G)| - 2) = 2|G| + \frac{|G|}{p} \left(\frac{mp - m + p^{m+1} + p - 2}{p - 1} - 2 \right) \\ &= \frac{|G|}{p} \left(\frac{mp - m + p^{m+1} + p - 2 - 2p + 2 + 2p^2 - 2p}{p - 1} \right) = \frac{|G|}{p} \left(\frac{p^{m+1} + 2p^2 + (m - 3)p - m}{p - 1} \right). \end{aligned}$$

This proves what we claimed. \square

5. Some Theorems of Structure

The present section is devoted to prove restrictions on $P(G)$, arising from exact bounds for $pd(G)$, when G is an arbitrary group. The evidences of Examples 3.1 and 3.2 motivated most of the following results.

Theorem 5.1. *Let G be a group (with $pd(G) \neq 1$) and p the smallest prime divisor of $|G|$. Then*

$$\left(1 - \frac{p}{|G|} \right) \frac{|\mathcal{L}(P(G))|}{|\mathcal{L}(G)|} + \frac{p}{|G|} \leq pd(G).$$

Moreover, if $P_G(X)$ is a proper subgroup of G for all $X \in \mathcal{L}(G) - \mathcal{L}(P(G))$, then

$$pd(G) \leq \frac{1}{p} + \frac{(p - 1)|\mathcal{L}(P(G))|}{p |\mathcal{L}(G)|}.$$

Proof. In order to prove the lower bound, it is enough to note from (2.1) that

$$|\mathcal{L}(G)||G|pd(G) = |\mathcal{L}(P(G))||G| + \sum_{X \in \mathcal{L}(G) - \mathcal{L}(P(G))} |P_G(X)| \geq |\mathcal{L}(P(G))||G| + (|\mathcal{L}(G)| - |\mathcal{L}(P(G))|)p, \tag{5.1}$$

where we have used in the last step that $|P_G(X)| \geq p$. Then we continue

$$= (|G| - p)|\mathcal{L}(P(G))| + p|\mathcal{L}(G)|$$

from which we get

$$pd(G) \geq \frac{(|G| - p)|\mathcal{L}(P(G))| + p|\mathcal{L}(G)|}{|\mathcal{L}(G)||G|} = \frac{(|G| - p)|\mathcal{L}(P(G))|}{|\mathcal{L}(G)||G|} + \frac{p}{|G|}.$$

Now we prove the upper bound. Formula (2.1) becomes again (5.1)

$$|G||\mathcal{L}(G)| pd(G) = |\mathcal{L}(P(G))||G| + \sum_{X \in \mathcal{L}(G) - \mathcal{L}(P(G))} |P_G(X)|$$

once one uses the fact that $P_G(X) = G$ for every $X \in \mathcal{L}(P(G))$. Now, since $|G : P_G(X)| \neq 1$ for every $X \in \mathcal{L}(G) - \mathcal{L}(P(G))$, we get $|G : P_G(X)| \geq p$, that is, $|P_G(X)| \leq \frac{|G|}{p}$. Therefore (5.1) is upper bounded by

$$\leq |\mathcal{L}(P(G))||G| + (|\mathcal{L}(G)| - |\mathcal{L}(P(G))|) \frac{|G|}{p} = \frac{(p - 1)|\mathcal{L}(P(G))||G|}{p} + \frac{|\mathcal{L}(G)||G|}{p}$$

and the result follows. \square

Of course, D_8 satisfies the lower bound, but not the upper bound, of Theorem 5.1. Details can be deduced from the information of Example 3.2. This is to justify that Theorem 5.1 originates from evidences of computational nature. On the other hand, Example 3.2 shows also that $Z(D_8) = P(D_8) \simeq C_2$. Then, when can we say that $P(D_8)$ is noncyclic? The next two results concern this question.

Theorem 5.2. *If $P(G)$ is a nontrivial proper subgroup of a group G and $pd(G) = \frac{1}{2} + \frac{|\mathcal{L}(P(G))|}{2|\mathcal{L}(G)|}$, then $P(G)$ is noncyclic.*

Proof. By assumption we exclude the cases $P(G) = G$ and $P(G) = \{1\}$, which are the extremal situations already known. Assume that $P(G)$ is cyclic of prime order $q \geq p \geq 2$, where p is the smallest prime dividing $|G|$. We may apply the arguments of the proof of Theorem 4.3 and, noting that $|\mathcal{L}(P(G))| = 2$, we find that

$$pd(G) = \frac{1}{2} + \frac{|\mathcal{L}(P(G))|}{2|\mathcal{L}(G)|} = \frac{1}{2} + \frac{1}{|\mathcal{L}(G)|} = \frac{|\mathcal{L}(G)| + 2}{2|\mathcal{L}(G)|} \leq \frac{2|G|}{|G||\mathcal{L}(G)|} + \frac{|G|(|\mathcal{L}(G)| - 2)}{p|G||\mathcal{L}(G)|} = \frac{(2p + |\mathcal{L}(G)| - 2)|G|}{p|G||\mathcal{L}(G)|}$$

and then the inequality

$$\frac{|\mathcal{L}(G)| + 2}{2} = \frac{|\mathcal{L}(G)|}{2} + 1 \leq \frac{(2p - 2)}{p} + \frac{|\mathcal{L}(G)|}{p}$$

which means

$$\frac{|\mathcal{L}(G)|}{2} - \frac{|\mathcal{L}(G)|}{p} = \frac{(p - 2)|\mathcal{L}(G)|}{2p} \leq 1 - \frac{2}{p} = \frac{p - 2}{p}.$$

From this we derive the contradiction $\frac{|\mathcal{L}(G)|}{2} \leq 1$, as at least $\{1\}$ and G are contained in $\mathcal{L}(G)$. Therefore, $P(G)$ cannot be cyclic of prime order and we may assume that $P(G)$ is cyclic of order $k \geq 2$. Now, we note that $|\mathcal{L}(P(G))| = |\text{Div}(k)|$, where $\text{Div}(k)$ is the set of all divisors of k . Here the argument we just used for q may still be applied. In fact we have

$$\begin{aligned} \frac{1}{2} + \frac{|\text{Div}(k)|}{2|\mathcal{L}(G)|} &= \frac{|\mathcal{L}(G)| + |\text{Div}(k)|}{2|\mathcal{L}(G)|} \leq \frac{|\text{Div}(k)||G|}{|G||\mathcal{L}(G)|} + \frac{|G|(|\mathcal{L}(G)| - |\text{Div}(k)|)}{p|G||\mathcal{L}(G)|} \\ &= \frac{(|\text{Div}(k)|p + |\mathcal{L}(G)| - |\text{Div}(k)|)|G|}{p|G||\mathcal{L}(G)|} \end{aligned}$$

then

$$\frac{|\mathcal{L}(G)| + |\text{Div}(k)|}{2} = \frac{|\mathcal{L}(G)|}{2} + \frac{|\text{Div}(k)|}{2} \leq \frac{(p - 1)|\text{Div}(k)|}{p} + \frac{|\mathcal{L}(G)|}{p}$$

which means

$$\frac{|\mathcal{L}(G)|}{2} - \frac{|\mathcal{L}(G)|}{p} = \frac{(p - 2)|\mathcal{L}(G)|}{2p} \leq \frac{(p - 1)|\text{Div}(k)|}{p}$$

and this would imply that $|\mathcal{L}(G)| \leq |\text{Div}(k)| = |\mathcal{L}(P(G))|$, that is, $\mathcal{L}(G) \subseteq \mathcal{L}(P(G))$ and then $\mathcal{L}(G) = \mathcal{L}(P(G))$. This condition implies $G = P(G)$, a contradiction. \square

The reader may note that Theorem 5.2 describes a very general situation, which cannot be reduced to those in Examples 3.1 and 3.2. In fact, looking at Example 3.1, $P(D_6) = D_6$ and so $P(D_6)$ is not a proper subgroup of D_6 , and this means that one of the assumptions of Theorem 5.2 is not satisfied. On the other hand, $P(D_8) = Z(D_8)$ is a nontrivial proper subgroup of D_8 (see Example 3.2), but $\frac{4}{5} = pd(D_8) \neq \frac{1}{2} + \frac{1}{10} = \frac{3}{5}$, and in fact $P(D_8)$ is cyclic. Once again, Theorem 5.2 can not be applied. These two examples show that we cannot strengthen further Theorem 5.2.

However, we may detect groups G with cyclic $P(G)$. The following result shows this circumstance.

Theorem 5.3. *Let $P(G)$ be a nontrivial proper subgroup of a group G with $pd(G) = \frac{4}{5}$ and p be the smallest prime divisor of $|G|$. If $\frac{4|G| - 5p}{5|G| - 5p} \leq \frac{2}{|\mathcal{L}(G)|}$, then $P(G)$ is cyclic of prime order.*

Proof. From the lower bound of Theorem 5.1, we have

$$pd(G) = \frac{4}{5} \geq \left(1 - \frac{p}{|G|}\right) \frac{|\mathcal{L}(P(G))|}{|\mathcal{L}(G)|} + \frac{p}{|G|} \Leftrightarrow \frac{\frac{4}{5}}{1 - \frac{p}{|G|}} - \frac{\frac{p}{|G|}}{1 - \frac{p}{|G|}} \geq \frac{|\mathcal{L}(P(G))|}{|\mathcal{L}(G)|}$$

$$\Leftrightarrow \frac{\frac{4|G|-5p}{5|G|}}{\frac{|G|-p}{|G|}} \geq \frac{|\mathcal{L}(P(G))|}{|\mathcal{L}(G)|} \Leftrightarrow \frac{4|G|-5p}{5|G|-5p} \geq \frac{|\mathcal{L}(P(G))|}{|\mathcal{L}(G)|}.$$

We conclude that $\frac{|\mathcal{L}(P(G))|}{|\mathcal{L}(G)|} \leq \frac{2}{|\mathcal{L}(G)|}$, hence $|\mathcal{L}(P(G))| \leq 2$. This forces $P(G)$ to be cyclic of prime order. \square

6. Computations for Dihedral Groups

We describe an instructive example, which correlates most of the notions which we have seen until now.

Proposition 6.1. *Let p be an odd prime. Then*

$$1 = pd(D_{2p}) > sd(D_{2p}) = \frac{7p^3 - 5p^2 - 11p + 9}{p^4 + 4p^3 - 2p^2 - 12p + 9} > \frac{p + 3}{4p} = d(D_{2p}).$$

Proof. Noting that $D_{2p} = C_2 \rtimes C_p$ (see (4.1)) and that $\mathcal{L}(D_{2p})$ forms a diamond (as in Fig.III.3), we conclude that $Z(D_{2p})$ is trivial and $C_{D_{2p}}(C_p) = C_p$ is the unique maximal normal subgroup of D_{2p} . Moreover D_{2p} is a \mathbf{P} -group, because $Q_\infty(D_{2p}) = D_{2p}$. Therefore a proper subgroup H of D_{2p} should be properly contained in $P_{D_{2p}}(H)$ and necessarily $P_{D_{2p}}(H) = D_{2p}$. Thus, we find that $pd(D_{2p}) = 1$.

On the other hand, we may specialize the formula

$$sd(D_{2p}) = \frac{\tau(p)^2 + 2\tau(p)\sigma(p) + g(p)}{(\tau(p) + \sigma(p))^2},$$

given in [27, Theorem 3.1.1], where

$$g(p) = \frac{3p^3 - 5p^2 + p + 1}{p^2 - 2p + 1}$$

is the arithmetic function in [27, Eq. 10, p.2514]. From (4.3) we deduce $\tau(p) = 2$ and $\sigma(p) = p + 1$ so that

$$sd(D_{2p}) = \frac{4 + 2 \cdot 2 \cdot (p + 1) + \frac{3p^3 - 5p^2 + p + 1}{p^2 - 2p + 1}}{p^2 + 6p + 9} = \frac{8 + 4p + \frac{3p^3 - 5p^2 + p + 1}{p^2 - 2p + 1}}{p^2 + 6p + 9} = \frac{8p^2 - 16p + 8 + 4p^3 - 8p^2 + 4p + 3p^3 - 5p^2 + p + 1}{p^2 + 6p + 9}$$

$$= \frac{7p^3 - 5p^2 - 11p + 9}{(p^2 + 6p + 9)(p^2 - 2p + 1)} = \frac{7p^3 - 5p^2 - 11p + 9}{p^4 + 4p^3 - 2p^2 - 12p + 9}.$$

Now [20, Remark 4.2] shows that $d(D_{2p}) = \frac{p+3}{4p}$ and for all odd primes we have

$$0 > p^5 - 21p^4 + 30p^3 + 26p^2 - 63p + 27$$

$$\Leftrightarrow 28p^4 - 20p^3 - 44p^2 + 36p > p^5 + 7p^4 + 10p^3 - 18p^2 - 27p + 27$$

$$\Leftrightarrow 28p^4 - 20p^3 - 44p^2 + 36p > p^5 + 4p^4 - 2p^3 - 12p^2 + 9p + 3p^4 + 12p^3 - 6p^2 - 36p + 27$$

$$\Leftrightarrow 4p(7p^3 - 5p^2 - 11p + 9) > (p + 3)(p^4 + 4p^3 - 2p^2 - 12p + 9)$$

$$\Leftrightarrow \frac{7p^3 - 5p^2 - 11p + 9}{p^4 + 4p^3 - 2p^2 - 12p + 9} > \frac{p + 3}{4p}.$$

The result follows. \square

From Proposition 6.1, $pd(D_{2p})$ has a constant value for all odd primes, while $sd(D_{2p})$ and $d(D_{2p})$ are functions of p . This is an important difference of the permutability degree with respect to the subgroup commutativity degree and the commutativity degree. This reflects the fact that we are looking at permutizers in a group, and not at centralizers.

7. Some Open Questions

We end with a series of open questions. They arise naturally from the study of the present subject. The first is motivated by the families of dihedral groups, analyzed in Section 6 (and in other parts of the present paper). Most of these groups may be described in terms of product of groups.

Open Question 7.1. Let $G = NH$ be a product of a normal subgroup N by a subgroup H . What can be said about the permutability degree of G ?

It is well known that most of dihedral groups, generalized quaternion groups and semidihedral groups has this structure. They present further analogies in terms of central quotients: one, for instance, is that $D_8/Z(D_8) \simeq Q_8$ and, roughly speaking, one can generalize this isomorphism to $Q_{2^4} = Q_{16}$, $Q_{2^5} = Q_{32}$ and so on. The reader may refer to [15, 16] for recent studies on these groups. Therefore:

Open Question 7.2. What is the permutability degree of generalized quaternion groups and semidihedral groups?

There is also another question, which is more general and may require some computational efforts. A classical result of Cayley allows us to embed a group in a suitable symmetric group. The knowledge of symmetric groups plays in fact a fundamental role in several aspects of the theory of groups. Therefore:

Open Question 7.3. What is the permutability degree of the symmetric group S_n on n objects?

Finally, the classification of (finite) simple groups may provide interesting aspects of study. For the (finite) simple (and almost simple) groups of sporadic type a lot is known about their subgroup lattices, see [7]. Therefore

Open Question 7.4. What is the permutability degree of (finite) simple groups?

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