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The Best Bounds for Toader Mean in Terms of the Centroidal and Arithmetic Means

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Abstract. In the paper, the authors discover the best constants α_1, α_2, β_1, and β_2 for the double inequalities

$$\alpha_1\overline{C}(a,b)+(1-\alpha_1)A(a,b)<T(a,b)<\beta_1\overline{C}(a,b)+(1-\beta_1)A(a,b)$$

and

$$\frac{\alpha_2}{A(a,b)}+\frac{1-\alpha_2}{\overline{C}(a,b)}<\frac{1}{T(a,b)}<\frac{\beta_2}{A(a,b)}+\frac{1-\beta_2}{\overline{C}(a,b)}$$

to be valid for all $a,b>0$ with $a\ne b$, where

$$\overline{C}(a,b)=\frac{2(a^2+ab+b^2)}{3(a+b)},\quad A(a,b)=\frac{a+b}{2},\quad \text{and}\quad T(a,b)=\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\,\mathrm{d}\,\theta$$

are respectively the centroidal, arithmetic, and Toader means of two positive numbers a and b. As an application of the above inequalities, the authors also find some new bounds for the complete elliptic integral of the second kind.

1. Introduction

In [24], Toader introduced a mean

$$T(a,b)=\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\,\mathrm{d}\,\theta=\begin{cases}\dfrac{2a}{\pi}\mathcal{E}\left(\sqrt{1-\left(\dfrac{b}{a}\right)^2}\right), & a>b,\\[2ex] \dfrac{2b}{\pi}\mathcal{E}\left(\sqrt{1-\left(\dfrac{b}{a}\right)^2}\right), & a<b,\\[1ex] a, & a=b,\end{cases}$$

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where

$$\mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2 \theta}\, \mathrm{d}\,\theta$$

for $r \in [0, 1]$ is the complete elliptic integral of the second kind. The quantities

$$\overline{C}(a,b) = \frac{2(a^2 + ab + b^2)}{3(a+b)}, \quad A(a,b) = \frac{a+b}{2}, \quad S(a,b) = \sqrt{\frac{a^2+b^2}{2}}$$

are called in the literature the centroidal, arithmetic, and quadratic means of two positive real numbers a and b with $a \neq b$. For $p \in \mathbb{R}$ and $a, b > 0$ with $a \neq b$, the p-th power mean $M_p(a,b)$ is defined by

$$M_p(a,b) = \begin{cases} \left(\dfrac{a^p + a^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases} \tag{1.1}$$

It is well known that

$$M_{-1}(a,b) < A(a,b) = M_1(a,b) < \overline{C}(a,b) < S(a,b) = M_2(a,b)$$

for all $a, b > 0$ with $a \neq b$.

In [25], Vuorinen conjectured that

$$M_{3/2}(a,b) < T(a,b) \tag{1.2}$$

for all $a, b > 0$ with $a \neq b$. This conjecture was verified by Qiu and Shen [23] and by Barnard, Pearce, and Richards [7]. In [1], Alzer and Qiu presented that

$$T(a,b) < M_{(\ln 2)/\ln(\pi/2)}(a,b) \tag{1.3}$$

for all $a, b > 0$ with $a \neq b$, which gives a best possible upper bound for Toader mean in terms of the power mean. From (1.2) and (1.3), one concludes that

$$A(a,b) < T(a,b) < S(a,b) \tag{1.4}$$

for all $a, b > 0$ with $a \neq b$. In [12], the authors demonstrated that the double inequality

$$\alpha S(a,b) + (1-\alpha)A(a,b) < T(a,b) < \beta S(a,b) + (1-\beta)A(a,b) \tag{1.5}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq \frac{1}{2}$ and $\beta \geq \frac{4-\pi}{(\sqrt{2}-1)\pi}$.

Motivated by the double inequality (1.5), we naturally ask a question: What are the best constants $\alpha_1, \alpha_2, \beta_1, \beta_2 \in (0,1)$ such that the double inequalities

$$\alpha_1 \overline{C}(a,b) + (1-\alpha_1)A(a,b) < T(a,b) < \beta_1 \overline{C}(a,b) + (1-\beta_1)A(a,b) \tag{1.6}$$

and

$$\frac{\alpha_2}{A(a,b)} + \frac{1-\alpha_2}{\overline{C}(a,b)} < \frac{1}{T(a,b)} < \frac{\beta_2}{A(a,b)} + \frac{1-\beta_2}{\overline{C}(a,b)} \tag{1.7}$$

hold for all $a, b > 0$ with $a \neq b$?

The main aim of this paper is to affirmatively answer the above question.

Theorem 1.1. *The double inequality* (1.6) *holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq \frac{3}{4}$ and $\beta_1 \geq \frac{12}{\pi} - 3$.*

Theorem 1.2. *The double inequality* (1.7) *holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \le \pi - 3$ and $\beta_1 \ge \frac{1}{4}$.*

As an immediate applications of Theorem 1.1, we will derive a new bounds in terms of elementary functions for the complete elliptic integral of the second kind.

Theorem 1.3. *For $r \in (0,1)$ and $r' = \sqrt{1-r^2}$, we have*

$$\frac{\pi}{2}\left[\frac{1+r'+(r')^2}{2(1+r')}+\frac{1+r'}{8}\right] < \mathcal{E}(r) < \frac{\pi}{2}\left[\left(\frac{8}{\pi}-2\right)\frac{1+r'+(r')^2}{1+r'}+\left(2-\frac{6}{\pi}\right)(1+r')\right]. \tag{1.8}$$

In Section 4 we will compare the above main results with some well-known ones.

Remark 1.1. Some estimates for the three kinds of complete elliptic integrals were established in [1–3, 5, 7, 10, 13, 19, 20, 22, 25–27]. and there is a short review and survey in [21, pp. 40–46] for these estimates.

2. Lemmas

For proving our main results, we need the following lemmas.

For $0 < r < 1$, denote $r' = \sqrt{1-r^2}$. It is known that Legendre's complete elliptic integrals of the first and second kind are defined respectively by

$$\begin{cases} \mathcal{K} = \mathcal{K}(r) = \displaystyle\int_0^{\pi/2} \frac{1}{\sqrt{1-r^2\sin^2\theta}}\,\mathrm{d}\,\theta, \\ \mathcal{K}' = \mathcal{K}'(r) = \mathcal{K}(r'), \\ \mathcal{K}(0) = \dfrac{\pi}{2}, \\ \mathcal{K}(1) = \infty \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{E} = \mathcal{E}(r) = \displaystyle\int_0^{\pi/2} \sqrt{1-r^2\sin^2\theta}\,\mathrm{d}\,\theta, \\ \mathcal{E}' = \mathcal{E}'(r) = \mathcal{E}(r'), \\ \mathcal{E}(0) = \dfrac{\pi}{2}, \\ \mathcal{E}(1) = 1. \end{cases}$$

See [9, 11]. For $0 < r < 1$, the following formulas were presented in [3, Appendix E, pp. 474–475]:

$$\frac{\mathrm{d}\,\mathcal{K}}{\mathrm{d}\,r} = \frac{\mathcal{E}-(r')^2\mathcal{K}}{r(r')^2}, \quad \frac{\mathrm{d}\,\mathcal{E}}{\mathrm{d}\,r} = \frac{\mathcal{E}-\mathcal{K}}{r}, \quad \frac{\mathrm{d}(\mathcal{E}-(r')^2\mathcal{K})}{\mathrm{d}\,r} = r\mathcal{K},$$

$$\frac{\mathrm{d}(\mathcal{K}-\mathcal{E})}{\mathrm{d}\,r} = \frac{r\mathcal{E}}{(r')^2}, \quad \mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right) = \frac{2\mathcal{E}-(r')^2\mathcal{K}}{1+r}.$$

Lemma 2.1 ([3, Theorem 3.21])**.** *The function $\frac{\mathcal{E}-(r')^2\mathcal{K}}{r^2}$ is strictly increasing from $(0,1)$ onto $\left(\frac{\pi}{4},1\right)$.*

Lemma 2.2. *The function $5\mathcal{E}(r) - 3(r')^2\mathcal{K}(r)$ is positive and strictly increasing on $(0,1)$.*

Proof. Let $f(r) = 5\mathcal{E}(r) - 3(r')^2\mathcal{K}(r)$ for $r \in (0,1)$ and $r' = \sqrt{1-r^2}$. A simple computation leads to

$$f'(r) = \frac{2\mathcal{E}(r) - 2\mathcal{K}(r) + 3r^2\mathcal{K}(r)}{r} \triangleq \frac{g(r)}{r}.$$

A direct differentiation yields $g'(r) = \frac{r(\mathcal{E}(r))+3(r')^2\mathcal{K}(r)}{(r')^2} > 0$ for all $r \in (0,1)$, that is, the function $g(r)$ is strictly increasing on $(0,1)$. Hence, it is derived that $g(r) > g(0) = 0$, that $f'(r) > 0$, that $f(r)$ is increasing on $(0,1)$, and that $f(x) > f(0) = \pi > 0$. □

Lemma 2.3 ([3, Theorem 1.25])**.** *For $-\infty < a < b < \infty$, let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b), and $g'(x) \neq 0$ on (a,b). If $\frac{f'(x)}{g'(x)}$ is increasing (or decreasing respectively) on (a,b), so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \textit{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

Remark 2.1. Lemma 2.3 and its variants have been extensively applied in, for example, [6, 15, 18, 22] and many references listed in [21], and have been generalized in, for example, [14, Lemma 2.2] and [15] and closely related references therein. For more information, please read the first sentence after [6, p. 582, Lemma 2.1], the references [4, 8, 16, 17] and [18, Remark 2.2]

3. Proofs of Main Results

Now we are in a position to prove our main results.

Proof. [Proof of Theorem 1.1] Without loss of generality, we assume that $a > b$. Let $t = \frac{b}{a} \in (0,1)$ and $r = \frac{1-t}{1+t}$. Then

$$\frac{T(a,b) - A(a,b)}{\overline{C}(a,b) - A(a,b)} = \frac{\frac{\pi}{2}\mathcal{E}'(t) - \frac{1+t}{2}}{\frac{2}{3}\frac{1+t+t^2}{1+t} - \frac{1+t}{2}} = \frac{\frac{2}{\pi}\mathcal{E}\Big(\frac{2\sqrt{r}}{1+r}\Big) - \frac{1}{1+r}}{\frac{1}{3}\frac{r^2}{1+r}} = 3\frac{\frac{2}{\pi}[2\mathcal{E} - (r')^2\mathcal{K}] - 1}{r^2}.$$

Let $f_1(r) = \frac{2}{\pi}[2\mathcal{E} - (r')^2\mathcal{K}] - 1$, $f_2(r) = r^2$, and $f(r) = 3\frac{f_1(r)}{f_2(r)} = 3\frac{\frac{2}{\pi}[2\mathcal{E}-(r')^2\mathcal{K}]-1}{r^2}$. Simple computations lead to

$$f_1(0) = f_2(0) = 0, \quad f_1'(r) = \frac{2}{\pi}\frac{\mathcal{E} - (r')^2\mathcal{K}}{r}, \quad f_2'(r) = 2r, \quad \frac{f_1'(r)}{f_2'(r)} = \frac{1}{\pi}\frac{\mathcal{E} - (r')^2\mathcal{K}}{r^2}.$$

Combining this with Lemmas 2.1 and 2.3 reveals that the function $f(r)$ is strictly increasing on $(0,1)$. Further making use of L'Hôpital's rule gives $\lim_{r\to 0^+} f(r) = \frac{3}{4}$ and $\lim_{r\to 1^-} f(r) = \frac{12}{\pi} - 3$. Theorem 1.1 is thus proved. □

Proof. [Proof of Theorem 1.2] Without loss of generality, we assume that $a > b$. Let $t = \frac{b}{a} \in (0,1)$ and $r = \frac{1-t}{1+t}$. Then

$$\frac{1/T(a,b) - 1/\overline{C}(a,b)}{1/A(a,b) - 1/\overline{C}(a,b)} = \frac{1 - T(a,b)/\overline{C}(a,b)}{T(a,b)\Big[\frac{1}{A(a,b)} - \frac{1}{\overline{C}(a,b)}\Big]} = \frac{1 - \frac{2}{\pi}\mathcal{E}'(t)\Big/\frac{2}{3}\frac{1+t+t^2}{1+t}}{\frac{2}{\pi}\mathcal{E}'(t)\Big[\frac{2}{1+t} - \frac{3(1+t)}{2(1+t+t^2)}\Big]} = \frac{3 + r^2 - \frac{6}{\pi}[2\mathcal{E} - (r')^2\mathcal{K}]}{\frac{2}{\pi}r^2[2\mathcal{E} - (r')^2\mathcal{K}]}.$$

Let $f_1(r) = 3 + r^2 - \frac{6}{\pi}[2\mathcal{E} - (r')^2\mathcal{K}]$, $f_2(r) = \frac{2}{\pi}r^2[2\mathcal{E} - (r')^2\mathcal{K}]$, and $f(r) = \frac{f_1(r)}{f_2(r)} = \frac{3+r^2-\frac{6}{\pi}[2\mathcal{E}-(r')^2\mathcal{K}]}{\frac{2}{\pi}r^2[2\mathcal{E}-(r')^2\mathcal{K}]}$. Simple computations lead to

$$f_1(0) = f_2(0) = 0, \quad f_1'(r) = 2r - \frac{6}{\pi}\frac{\mathcal{E} - (r')^2\mathcal{K}}{r}, \quad f_2'(r) = \frac{2}{\pi}[5\mathcal{E} - 3(r')^2]\mathcal{K}r, \quad \text{and} \quad \frac{f_1'(r)}{f_2'(r)} = \frac{2 - \frac{6}{\pi}\frac{[\mathcal{E}-(r')^2\mathcal{K}]}{r^2}}{\frac{2}{\pi}[5\mathcal{E} - 3(r')^2\mathcal{K}]}.$$

Combining this with Lemmas 2.1, 2.2, and 2.3 yields that the function $f(r)$ is strictly decreasing on $(0,1)$. Making use of L'Hôpital's rule shows that $\lim_{r\to 0^+} f(r) = \frac{1}{4}$ and $\lim_{r\to 1^-} f(r) = \pi - 3$. Thus, Theorem 1.2 is proved. □

Proof. [Proof of Theorem 1.3] Without loss of generality, assume that $a > b$. Substituting $r' = \frac{b}{a}$, $\alpha_1 = \frac{3}{4}$, and $\beta_1 = \frac{12}{\pi} - 3$ into Theorem 1.1 produces Thorem 1.3. □

4. Comparisons with some Known Results

In [12], it was obtained that

$$\frac{\pi}{2}\left[\frac{1}{2}\sqrt{\frac{1 + (r')^2}{2}} + \frac{1 + r'}{4}\right] < \mathcal{E}(r) < \frac{\pi}{2}\left[\frac{4 - \pi}{(\sqrt{2} - 1)\pi}\sqrt{\frac{1 + (r')^2}{2}} + \frac{(\sqrt{2}\pi - 4)(1 + r')}{2(\sqrt{2} - 1)\pi}\right] \tag{4.1}$$

for all $r \in (0,1)$ and $r' = \sqrt{1 - r^2}$. Guo and Qi proved in [13] that

$$\frac{\pi}{2} - \frac{1}{2}\ln\frac{(1 + r)^{1-r}}{(1 - r)^{1+r}} < \mathcal{E}(r) < \frac{\pi - 1}{2} + \frac{1 - r^2}{4r}\ln\frac{1 + r}{1 - r} \tag{4.2}$$

for all $r \in (0,1)$. It was pointed out in [12] that the bounds in (4.1) for $\mathcal{E}(r)$ are better than those in (4.2) for some $r \in (0,1)$. Very recently, Yin and Qi obtained in [27] that

$$\frac{\pi}{2}\frac{\sqrt{6+2\sqrt{1-r^2}-3r^2}}{2\sqrt{2}} \le \mathcal{E}(r) \le \frac{\pi}{2}\frac{\sqrt{10-2\sqrt{1-r^2}-5r^2}}{2\sqrt{2}}. \tag{4.3}$$

Let

$$g(x) = \frac{1+x+x^2}{2(1+x)} + \frac{1+x}{8} - \left(\frac{1}{2}\sqrt{\frac{1+x^2}{2}} + \frac{1+x}{4}\right)$$

for $x \in (0,1)$. Then a simplification leads to

$$g(x) = \frac{3x^2+2x+3-2(1+x)\sqrt{2(1+x^2)}}{8(1+x)}.$$

Since

$$\left(3x^2+2x+3\right)^2 - \left[2(1+x)\sqrt{2(1+x^2)}\right]^2 = (1-x)^4 > 0,$$

the lower bound in (1.8) for $\mathcal{E}(r)$ is better than the one in (4.1).

Since

$$\frac{1+x+x^2}{2(1+x)} + \frac{1+x}{8} > \frac{\sqrt{6+2x-3(1-x^2)}}{2\sqrt{2}}$$

is equivalent to

$$\left(5x^2+6x+5\right)^2 > 8(x+1)^2\left(3x^2+2x+3\right)$$

and $(x-1)^4 > 0$, the lower bound in (1.8) for $\mathcal{E}(r)$ is better than the one in (4.3).

Let

$$J(r) = \frac{\pi}{2}\left[\left(\frac{8}{\pi}-2\right)\frac{1+r'+(r')^2}{1+r'} + \left(2-\frac{6}{\pi}\right)(1+r')\right], \qquad Q(r) = \frac{\pi-1}{2} + \frac{1-r^2}{4r}\ln\frac{1+r}{1-r},$$

$$D(r) = \frac{\pi}{2}\left[\frac{4-\pi}{\left(\sqrt{2}-1\right)\pi}\sqrt{\frac{1+(r')^2}{2}} + \frac{(\sqrt{2}\,\pi-4)(1+r')}{2\left(\sqrt{2}-1\right)\pi}\right], \quad Y(r) = \frac{\pi}{2}\frac{\sqrt{10-2\sqrt{1-r^2}-5r^2}}{2\sqrt{2}}.$$

The values of these functions at points 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 can be numerical computed and listed in Table 1. This implies that the upper bound in (1.8) for $\mathcal{E}(r)$ are better than those in (4.1), (4.2), and (4.3) for some $r \in (0,1)$.

In conclusion, the double inequality (1.8) is better than some known results in [12, 13, 27] somewhere.

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Table 1: Values of $J(r)$, $D(r)$, $Q(r)$, and $Y(r)$

r	$J(r)$	$D(r)$	$Q(r)$	$Y(r)$
0.1	$1.566862174\cdots$	$1.566862736\cdots$	$1.567456298\cdots$	$1.566866887\cdots$
0.2	$1.554972309\cdots$	$1.554981471\cdots$	$1.557354457\cdots$	$1.555049510\cdots$
0.3	$1.534853276\cdots$	$1.534901499\cdots$	$1.540234393\cdots$	$1.535259718\cdots$
0.4	$1.506007907\cdots$	$1.506169094\cdots$	$1.515627704\cdots$	$1.507368120\cdots$
0.5	$1.467637170\cdots$	$1.468061483\cdots$	$1.482775936\cdots$	$1.471228040\cdots$
0.6	$1.418485626\cdots$	$1.419455645\cdots$	$1.440474824\cdots$	$1.426746617\cdots$
0.7	$1.356514851\cdots$	$1.358548915\cdots$	$1.386741519\cdots$	$1.374078083\cdots$
0.8	$1.278097245\cdots$	$1.282149209\cdots$	$1.317984092\cdots$	$1.314222496\cdots$
0.9	$1.175305090\cdots$	$1.183095913\cdots$	$1.226197273\cdots$	$1.251499407\cdots$

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