Filomat 30:1 (2016), 105–112
DOI 10.2298/FIL1601105D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Convolutions of the Generalized Pell and Pell-Lucas Numbers

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Abstract. We consider the convolution of the generalized Pell numbers–$P^{(s)}_{n,m}$ and the convolution of the generalized Pell–Lucas numbers–$Q^{(s)}_{n,m}$. For $s = 0$, the sequence $P^{(0)}_{n,m}$ represents the generalized Pell numbers $P_{n,m}$, and the sequence $Q^{(0)}_{n,m}$ represents the generalized Pell–Lucas numbers $Q_{n,m}$ ([1], [2]). For $m = 2$ and $s = 0$, the numbers $P^{(0)}_{n,2}$ and $Q^{(0)}_{n,2}$ are Pell and Pell-Lucas numbers, respectively.

1. Convolution of the Pell–Numbers

In [8], A F. Horadam considered sequences of numbers $P^{(s)}_n$ and $Q^{(s)}_n$. The first sequence, $P^{(s)}_n$, is the covolution of the Pell numbers P_n, and the second one, $Q^{(s)}_n$, is the convolution of the Pell–Lucas numbers Q_n. In [5] the polynomials $U^k_{n,m}(x)$ and $V^k_{n,m}(x)$ are considered.

In this paper we consider the convolution of the generalized Pell numbers–$P^{(s)}_{n,m}$ and the convolution of the generalized Pell–Lucas numbers–$Q^{(s)}_{n,m}$. For $s = 0$, the sequence $P^{(0)}_{n,m}$ represents the generalized Pell numbers $P_{n,m}$, and the sequence $Q^{(0)}_{n,m}$ represents the generalized Pell–Lucas numbers $Q_{n,m}$ ([1], [2], [5]). The sequences $P^{(s)}_{n,m}$ and $Q^{(s)}_{n,m}$ are given as:

$$(1 - 2t - t^m)^{-(s+1)} = \sum_{n=0}^{\infty} P^{(s)}_{n+1,m} t^n, \tag{0.1}$$

and

$$\left(\frac{2 + 2t^{m-1}}{1 - 2t - t^m}\right)^{s+1} = \sum_{n=0}^{\infty} Q^{(s)}_{n+1,m} t^n, \tag{0.2}$$

where $n \in \mathbb{N}$, $s \in \mathbb{N}$ and $m \geq 1$.

Throughout in this paper we use $\mathbb{N}$ to denote set of all nonnegative integers.

This paper is motivated by papers [3], [4], [5], [6], [7], [8], [9].

2010 *Mathematics Subject Classification*. Primary 11B83; 11B37, 11B39.
Keywords. Generalized Pell and Pell–Lucas numbers, Convolution, Recurrence relation; Explicit representation, Generalized Fibonacci numbers.
Received: 17 February 2014; Accepted: 06 April 2014
Communicated by Hari M. Srivastava
The author is supported by the Ministry of Science, Republic of Serbia, grant no. 174007
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Using the standard method, starting with recurrence relation (0.1), we get the following explicit formula

$$P^{(s)}_{n+1,m} = \sum_{k=0}^{[n/m]} \frac{(s+1)_{n-(m-1)k}}{k!(n-mk)!} \cdot 2^{n-mk}. \tag{1.1}$$

For $s = 0$ in (1.1), we have $P_{n+1,m}$– generalized Pell numbers:

$$P_{n+1,m} = \sum_{k=0}^{[n/m]} \frac{(n-(m-1)k)!}{k!(n-mk)!} \cdot 2^{n-mk}.$$

Remark. Using the known relations (see [10]):

$$(\alpha)_{n+k} = (\alpha)_n(\alpha+n)_k; \quad \frac{(-1)^k}{(n-k)!} = \frac{(-n)_k}{n!}; \quad (\alpha)_{n-k} = \frac{(-1)^k(\alpha)_n}{(1-\alpha-n)_k},$$

the explicit formula (1.1) can be written in the following form (see [3])

$$P^{(s)}_{n+1,m} = \frac{2^n(s+1)_n}{n!} {}_mF_{m-1}\left[\begin{matrix} \frac{-n}{m}, \frac{1-n}{m}, \ldots, \frac{m-1-n}{m}; & \frac{-(m/2)^m}{(m-1)^{m-1}} \\ \frac{-s-n}{m-1}, \frac{1-s-n}{m-1}, \ldots, \frac{m-2-s-n}{m-1} & \end{matrix}\right].$$

So, for $m = 2$ and $m = 3$, the last relation yields:

$$P^{(s)}_{n+1,2} = \frac{2^n(s+1)_n}{n!} {}_2F_1\left[\begin{matrix} \frac{-n}{2}, \frac{1-n}{2}; & -1 \\ \frac{-s-n}{1}; & \end{matrix}\right]$$

and

$$P^{(s)}_{n+1,3} = \frac{2^n(s+1)_n}{n!} {}_3F_2\left[\begin{matrix} \frac{-n}{3}, \frac{1-n}{3}, \frac{2-n}{3}; & \frac{-(3/2)^3}{4} \\ \frac{-s-n}{2}, \frac{1-s-n}{2}; & \end{matrix}\right].$$

Theorem 1. The sequence of numbers $P^{(s)}_{n,m}$ satisfies the following recurrence relation ($P^{(s)}_{0,m} = 0$):

$$2P^{(s)}_{n-1,m} + P^{(s)}_{n-m,m} + P^{(s-1)}_{n,m} = P^{(s)}_{n,m}, \quad n \geq m, \ s \geq 1. \tag{1.2}$$

Proof. Using the explicit representation (1.1) on the left side of (1.2), we get:

$$\begin{aligned}
&2 \cdot \frac{2^{n-2-mk}(s+1)_{n-2-(m-1)k}}{k!(n-2-mk)!} + \frac{(s+1)_{n-1-m-(m-1)k}}{k!(n-1-m-mk)!} \cdot 2^{n-1-m-mk} \\
&\quad + \frac{(s)_{n-1-(m-1)k}}{k!(n-1-mk)!} \cdot 2^{n-1-mk} \\
&= 2^{n-1-mk}\frac{(s+1)_{n-2-(m-1)k}}{k!(n-2-mk)!} + \frac{(s+1)_{n-2-(m-1)k}}{(k-1)!(n-1-mk)!} \cdot 2^{n-1-mk} \\
&\quad + \frac{(s)_{n-1-(m-1)k}}{k!(n-1-mk)!} \cdot 2^{n-1-mk} = \frac{2^{n-1-mk}}{k!(n-1-mk)!} \times \\
&\Big((n-1-mk)(s+1)_{n-2-(m-1)k} + k(s+1)_{n-2-(m-1)k} + (s)_{n-1-(m-1)k}\Big) \\
&\quad = \frac{2^{n-1-mk}}{k!(n-1-mk)!} \cdot (s+1)_{n-1-(m-1)k}.
\end{aligned}$$

□

If $m = 2$, then (1.2) becomes (see [4])

$$P_n^{(s)} = 2P_{n-1}^{(s)} + P_{n-2}^{(s)} + P_n^{(s-1)}, \quad n \geq 2, \;\; s \geq 1. \tag{1.3}$$

Using the relation (1.3), we find some first members of sequence $P_n^{(s)}$:

$$P_0^{(s)} = 0, \quad P_1^{(s)} = 1, \quad P_2^{(s)} = 2(s+1),$$

$$P_3^{(s)} = 2(s+1)_2, \quad P_4^{(s)} = \frac{4(s+1)_3}{3} + (s+1).$$

For $m = 3$, the relation (1.2) yields:

$$P_{n,3}^{(s)} = 2P_{n-1,3}^{(s)} + P_{n-3,3}^{(s)} + P_{n,3}^{(s-1)}, \;\; n \geq 3, \;\; s \geq 1. \tag{1.4}$$

Now, from (1.1), for $m = 3$ and $s = 0, 1, 2, 3, 4$, we get:

Table 1: Numbers $P_{n,3}^{(s)}$

n	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$
1	1	1	1	1	1
2	2	4	6	8	10
3	4	12	24	40	60
4	9	34	83	164	285
5	20	92	264	600	1180
6	44	240	792	2032	4452

Again from (1.2), for $m = 4$, we find that numbers $P_{n,4}^{(s)}$ satisfy the following recurrence relation:

$$P_{n,4}^{(s)} = 2P_{n-1,4}^{(s)} + P_{n-4,4}^{(s)} + P_{n,4}^{s-1)}, \quad n \geq 4, \;\; s \geq 1. \tag{1.5}$$

For $m = 4$, and from (1.1), we get the following members of the sequence $\{P_{n,4}^{(s)}\}$:

Table 2: Numbers $P_{n,4}^{(s)}$

n	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$
1	1	1	1	1	1
2	2	4	6	8	10
3	4	12	24	40	60
4	8	32	80	160	280
5	17	82	243	564	1125
6	36	204	696	1832	4092
7	76	496	1912	5616	13860

Theorem 2. The following relation holds:

$$\sum_{k=0}^{s} P_{n,m}^{(k)} = \frac{1}{2}\left\{P_{n+1,m}^{(s)} - \sum_{k=0}^{s} P_{n+1-m,m}^{(k)}\right\}, \tag{1.6}$$

for $n \geq m - 1$.

Proof. From the relation (1.2), for successive values $s = 0, 1, \ldots, k$, we get:

$$\sum_{k=0}^{s} P_{n,m}^{(k)} = 2\sum_{k=0}^{s} P_{n-1,m}^{(k)} + \sum_{k=0}^{s} P_{n-m,m}^{(k)} + \sum_{k=0}^{s-1} P_{n,m}^{(k)},$$

$$P_{n,m}^{(s)} + \sum_{k=0}^{s-1} P_{n,m}^{(k)} = 2\sum_{k=0}^{s} P_{n-1}^{(k)} + \sum_{k=0}^{s} P_{n-m,m}^{(k)} + \sum_{k=0}^{s-1} P_{n,m}^{(k)},$$

$$P_{n,m}^{(s)} = 2\sum_{k=0}^{s} P_{n-1,m}^{(k)} + \sum_{k=0}^{s} P_{n-m,m}^{(k)}.$$

Hence, substituting n by $n+1$, we obtain

$$\sum_{k=0}^{s} P_{n,m}^{(k)} = \frac{1}{2}\left\{P_{n+1,m}^{(s)} - \sum_{k=0}^{s} P_{n+1-m,m}^{(k)}\right\}.$$

□

The relation (1.6), for $m = 2$, is given in [8].

We give some examples.

Example 1. Let $n = 4$, $s = 3$ and $m = 3$. From Table 1 and using (1.6), we get:

$$\sum_{k=0}^{3} P_{4,3}^{(k)} = \frac{1}{2}\left\{P_{5,3}^{(3)} - \sum_{k=0}^{3} P_{2,3}^{(k)}\right\};$$

$$\Leftrightarrow 9 + 34 + 83 + 164 = \frac{1}{2}(600 - 2 - 4 - 6 - 8) \Leftrightarrow 290 = 290.$$

Example 2. Let $n = 5$, $s = 3$ and $m = 3$. Again from Table 1 and using (1.6), we find:

$$\sum_{k=0}^{3} P_{5,3}^{(k)} = \frac{1}{2}\left\{P_{6,3}^{(3)} - \sum_{k=0}^{3} P_{3,3}^{(k)}\right\}$$

$$\Leftrightarrow 20 + 92 + 264 + 600 = \frac{1}{2}(2032 - 4 - 12 - 24 - 40) \Leftrightarrow 976 = 976.$$

Example 3. Let $n = 4$ and $s = 4$. Then, from Table 2 and (1.6), we have:

$$\sum_{k=0}^{3} P_{4,4}^{(k)} = \frac{1}{2}\left\{P_{5,4}^{(3)} - \sum_{k=0}^{3} P_{1,4}^{(k)}\right\}$$

$$\Leftrightarrow 8 + 32 + 80 + 160 = \frac{1}{2}(564 - 4) \Leftrightarrow 280 = 280.$$

Example 4. From Table 2, using (1.6), for $n = 6$, $s = 4$, we get:

$$\sum_{k=0}^{4} P_{6,4}^{(k)} = \frac{1}{2}\left\{P_{7,4}^{(4)} - \sum_{k=0}^{4} P_{3,4}^{(k)}\right\} \Leftrightarrow$$

$$36 + 204 + 696 + 1832 + 4092 = \frac{1}{2}(13860 - 4 - 12 - 24 - 40 - 60)$$

$$\Leftrightarrow 6860 = 6860.$$

Theorem 3. The numbers $P^{(s)}_{n,m}$ satisfy the following relation

$$2\sum_{i=1}^{n} P^{(s)}_{i,m} = P^{(s)}_{n+1,m} + P^{(s)}_{n,m} + \cdots + P^{(s)}_{n+2-m,m} - \sum_{i=1}^{n+1} P^{(s-1)}_{i,m}, \tag{1.7}$$

where $s \geq 1$, $n \geq m-2$, $m \geq 2$.

Proof. Taking $n = 1, 2, \ldots$, and from the relation (1.2), it follows:

$$\begin{aligned}
&P^{(s)}_{1,m} = 2P^{(s)}_{0,m} + P^{(s)}_{1-m,m} + P^{(s-1)}_{1,m}\\
&P^{(s)}_{2,m} = 2P^{(s)}_{1,m} + P^{(s)}_{2-m,m} + P^{(s-1)}_{2,m}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&P^{(s)}_{n-m,m} = 2P^{(s)}_{n-1-m,m} + P^{(s)}_{n-2m,m} + P^{(s-1)}_{n-m,m}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&P^{(s)}_{n-1,m} = 2P^{(s)}_{n-2,m} + P^{(s)}_{n-1-m,m} + P^{(s-1)}_{n-1,m}\\
&P^{(s)}_{n,m} = 2P^{(s)}_{n-1,m} + P^{(s)}_{n-m,m} + P^{(s-1)}_{n,m}\\
&P^{(s)}_{n+1,m} = 2P^{(s)}_{n,m} + P^{(s)}_{n+1-m,m} + P^{(s-1)}_{n+1,m}.
\end{aligned}$$

So, summing the last equalities, we find that

$$P^{(s)}_{n+2-m,m} + P^{(s)}_{n+3-m,m} + \cdots + P^{(s)}_{n-1,m} + P^{(s)}_{n,m} + P^{(s)}_{n+1,m} = 2\sum_{i=1}^{n} P^{(s)}_{i,m} + \sum_{i=1}^{n+1} P^{(s-1)}_{i,m}.$$

□

Example 5. For $m = 3$, $s = 3$ and $n = 4$, by (1.7), and from Table 1, we have

$$\begin{aligned}
&2\sum_{i=1}^{4} P^{(3)}_{i,3} = P^{(3)}_{5,3} + P^{(3)}_{4,3} + P^{(3)}_{3,3} - \sum_{i=1}^{5} P^{(2)}_{i,3}\\
&\Leftrightarrow 2(1 + 8 + 40 + 164) = 600 + 164 + 40 - 1 - 6 - 24 - 83 - 264\\
&\Leftrightarrow 426 = 426.
\end{aligned}$$

Example 6. For $n = 5$, $m = 4$ and $s = 4$, using (1.7), and from Table 2, we have:

$$\begin{aligned}
&2\sum_{i=1}^{5} P^{(4)}_{i,4} = P^{(4)}_{6,4} + P^{(4)}_{5,4} + P^{(4)}_{4,4} + P^{(4)}_{3,4} - \sum_{i=1}^{6} P^{(3)}_{i,4}\\
&\Leftrightarrow 2(1 + 10 + 60 + 280 + 1125) = 4092 + 1125 + 280 + 60 - 2605\\
&\Leftrightarrow 2952 = 2952.
\end{aligned}$$

2. Connection with Generalized Fibonacci Numbers

In this section we find relations between the sequence $\{P^{(s)}_{n,m}\}$ and the generalized Fibonacci numbers $F_{n,m}(3)$, where $F_{n,m}(x)$ are the generalized Fibonacci polynomials (see [1], [2], [5], [7]).

Namely, the numbers $F_{n,m}(3)$ are given as

$$F_{n,m}(3) = 3F_{n-1,m}(3) + F_{n-m,m}(3), \tag{2.1}$$

$$(n \in \mathbb{N},\ m \geq 1,\ n \geq m;\ F_{0,m}(3) = 0,\ F_{n,m}(3) = 3^{n-1},\ n = 1, \ldots, m-1.)$$

Theorem 4. For the numbers $P_{n,m}^{(s)}$ it holds:

$$\sum_{i=1}^{n} P_{i,m}^{(n-i)} = F_{n,m}(3). \tag{2.2}$$

Proof. In the proof we use induction on n. For $m = 2$ this theorem is proved in [8]. It is easily to prove this theorem for $m = 3$ and $m = 4$, for $n = 1, 2, 3$. Suppose that (2.2) is correct for N:

$$P_{1,m}^{(N-1)} + P_{2,m}^{(N-2)} + \cdots + P_{N-1,m}^{(1)} + P_{N,m}^{(0)} = F_{N,m}(3),$$

then, for $N + 1$, and taking $P_{n,m}^{(s)} = 0$ for $s < 0$ and $n \leq 0$, we get:

$$\begin{aligned}
&P_{1,m}^{(N)} + P_{2,m}^{(N-1)} + \cdots + P_{N,m}^{(1)} + P_{N+1,m}^{(0)} \quad \text{(by Th. 1)}\\
&= 2P_{0,m}^{(N)} + P_{1-m,m}^{(N)} + P_{1,m}^{(N-1)} + 2P_{1,m}^{(N-1)} + P_{2-m,m}^{(N-1)} + P_{2,m}^{(N-2)} +\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\ + 2P_{N-1,m}^{(1)} + P_{N+1-m,m}^{(1)} + P_{N,m}^{(0)} + 2P_{N,m}^{(0)} + P_{N+1-m,m}^{(0)} + P_{N+1,m}^{(-1)}\\
&= 3\left(P_{1,m}^{(N-1)} + P_{2,m}^{(N-2)} + \cdots + P_{N-m,m}^{(1)} + P_{N,m}^{(0)}\right)\\
&\ + P_{1-m,m}^{(N)} + P_{2-m,m}^{(N-1)} + \cdots + P_{N-m,m}^{(1)} + P_{N+1-m,m}^{(0)}\\
&= 3F_{N,m}(3) + F_{N+1-m,m}(3) = F_{N+1,m}(3).
\end{aligned}$$

□

3. Generalized Pell-Lucas Convolutions

The s^{th}-convolution $Q_{n,m}^{(s)}$ of the generalized Pell–Lucas numbers is given by

$$\left(\frac{2 + 2t^{m-1}}{1 - 2t - t^m}\right)^{s+1} = \sum_{n=0}^{\infty} Q_{n+1,m}^{(s)} t^n, \qquad s \geq 1. \tag{3.1}$$

If $m = 2$ and $s = 0$, then $Q_{n,2}^{(0)}$ is one modification of L_n ([9]).

Now, from (3.1), we get the following recurrence relation ($Q_{0,m}^{(s)} = 0$):

$$\begin{aligned}
nQ_{n+1,m}^{(s)} &= 2(s+1)(m-1)Q_{n+2-m,m}^{(s-1)} + 2(n+s)Q_{n,m}^{(s)}\\
&\quad + (ms+n)Q_{n+1-m,m}^{(s)}, \quad n \geq m, \quad s \geq 1.
\end{aligned} \tag{3.2}$$

Next, by the generating function (3.1), from (0.1), we find the following formula

$$Q_{n+1,m}^{(s)} = 2^{s+1} \sum_{k=0}^{[n/(m-1)]} \binom{s+1}{k} P_{n+1-(m-1)k,m}^{(s)}. \tag{3.3}$$

Hence, from (1.1) and (3.3), we get the formula

$$\begin{aligned}
Q_{n+1,m}^{(s)} &= 2^{s+1} \sum_{k=0}^{[n/(m-1)]} \binom{s+1}{k} \times\\
&\sum_{j=0}^{[(n-(m-1)k)/m]} \frac{(s+1)_{n-(m-1)k-(m-1)j}}{j!(n-(m-1)k-mj)!} 2^{n-(m-1)k-mj},
\end{aligned} \tag{3.4}$$

which is an explicit formula corresponding to numbers $Q^{(s)}_{n+1,m}$.

If $s = 0$, then $Q^{(0)}_{n,m} = Q_{n,m}$ are the generalized Pell–Lucas numbers.

The recurrence relation (3.2), for $m = 3$, yields ($Q^{(s)}_{0,3} = 0$):

$$nQ^{(s)}_{n+1,3} = 4(s+1)Q^{(s-1)}_{n-1,3} + 2(n+s)Q^{(s)}_{n,3} + (3s+n)Q^{(s)}_{n-2,3}, \ n \geq 2. \tag{3.5}$$

Using the relation (3.5), for $n = 0, 1, 2, 3, 4, 5, 6$ and $s = 0, 1, 2, 3, 4$, we get:

Table 3: Numbers $Q^{(s)}_{n,3}$

n	$s = 0$	$s = 1$	$s = 2$	$s = 3$	$s = 4$
1	2	4	8	16	32
2	4	16	48	128	320
3	10	56	216	704	2080
4	22	168	808	3136	10720
5	48	468	2712	12256	47680
6	106	1248	8472	43776	191264

Next, the recurrence relation (3.2), for $m = 4$, becomes ($Q^{(s)}_{0,4} = 0$):

$$nQ^{(s)}_{n+1,4} = 6(s+1)Q^{(s-1)}_{n-2,4} + 2(n+s)Q^{(s)}_{n,4} + (4s+n)Q^{(s)}_{n-3,4}. \tag{3.6}$$

Next, from (3.6), for $n = 1, 2, 3, 4, 5, 6$ and $s = 0, 1, 2, 3, 4$, we get:

Table 4: Numbers $Q^{(s)}_{n,4}$

n	$s = 0$	$s = 1$	$s = 2$	$s = 3$	$s = 4$
1	2	4	8	16	32
2	4	16	48	128	320
3	8	48	192	640	1920
4	18	136	664	2624	9120
5	38	360	2088	9536	37600
6	80	912	6144	31872	140544

Theorem 5. For $n \geq m$, $s \geq 1$ and $Q^{(s)}_{0,m} = 0$, the following relation

$$Q^{(s)}_{n,m} = 2Q^{(s)}_{n-1,m} + Q^{(s)}_{n-m,m} + 2Q^{(s-1)}_{n,m} + 2Q^{(s-1)}_{n+1-m,m} \tag{3.7}$$

holds.

Theorem 6. For $n \geq m$ and $s \geq 1$, the numbers $Q^{(s)}_{n,m}$ satisfy

$$Q^{(s)}_{n,m} = 2\sum_{k=0}^{s} Q^{(k)}_{n-1,m} + \sum_{k=0}^{s} Q^{(k)}_{n-m,m} + \sum_{k=0}^{s-1} Q^{(k)}_{n,m} + 2\sum_{k=0}^{s-1} Q^{(k)}_{n+1-m,m}. \tag{3.8}$$

Proof. Using the relation (3.7), we get:

$$Q^{(1)}_{n,m} = 2Q^{(1)}_{n-1,m} + Q^{(1)}_{n-m,m} + 2Q^{(0)}_{n,m} + 2Q^{(0)}_{n+1-m,m}$$
$$Q^{(2)}_{n,m} = 2Q^{(2)}_{n-1,m} + Q^{(2)}_{n-m,m} + 2Q^{(1)}_{n,m} + 2Q^{(1)}_{n+1-m,m}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$Q^{(s)}_{n,m} = 2Q^{(s)}_{n-1,m} + Q^{(s)}_{n-m,m} + 2Q^{(s-1)}_{n,m} + 2Q^{(s-1)}_{n+1-m,m}.$$

Summing the last equalities, it follows

$$Q^{(s)}_{n,m} = 2\sum_{k=0}^{s} Q^{(k)}_{n-1,m} + \sum_{k=0}^{s} Q^{(k)}_{n-m,m} + \sum_{k=0}^{s-1} Q^{(k)}_{n,m} + 2\sum_{k=0}^{s-1} Q^{(k)}_{n+1-m,m}.$$

So, the relation (3.8) is proved. □

Finally, we give the following two examples.

Example 7. From Table 3 and using (3.8), we obtain:

$$Q^{(3)}_{6,3} = 2\sum_{k=0}^{3} Q^{(k)}_{5,3} + \sum_{k=0}^{3} Q^{(k)}_{3,3} + \sum_{k=0}^{2} Q^{(k)}_{6,3} + 2\sum_{k=0}^{2} Q^{(k)}_{4,3}$$
$$\Leftrightarrow \quad 43776 = 2(12256 + 2712 + 468 + 48) + (10 + 56 + 216 + 704) +$$
$$(106 + 1248 + 8472) + 2(22 + 168 + 808)$$
$$\Leftrightarrow \quad 43776 = 43776.$$

Example 8. From (3.8) and Table 4, we have:

$$Q^{(3)}_{5,4} = 2\sum_{k=0}^{3} Q^{(k)}_{4,4} + \sum_{k=0}^{3} Q^{(k)}_{1,4} + \sum_{k=0}^{2} Q^{(k)}_{5,4} + 2\sum_{k=0}^{2} Q^{(k)}_{2,4}$$
$$\Leftrightarrow 9536 = 2(18 + 136 + 664 + 2524) + (2 + 4 + 8 + 16) + (38 + 360 + 2088)$$
$$+ 2(4 + 48 + 16)$$
$$\Leftrightarrow 9536 = 9536.$$

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