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Generalized Hyers-Ulam Stability of General Cubic Functional Equation in Random Normed Spaces

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Abstract. In this paper, we investigate the generalized Hyers-Ulam stability of a general cubic functional equation:

$$f(x+ky) - kf(x+y) + kf(x-y) - f(x-ky) = 2k(k^2-1)f(y)$$

for fixed $k \in \mathbb{Z}^+$ with $k \geq 2$ in random normed spaces.

1. Introduction

A basic question in the theory of functional equations is as follows:

When is it true that a function that approximately satisfies a functional equation must be close to an exact solution of the equation?

If the problem accepts a solution, we say that the equation is *stable*. The first stability problem concerning of group homomorphisms was introduced by Ulam [20] in 1940. The famous Ulam stability problem was partially solved by Hyers [12] for linear functional equation of Banach spaces. Hyers theorem was generalized by Aoki [3] for additive mappings and by Rassias [17] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [10] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach. Cădariu and Radu [5] applied the *fixed point method* to investigation of the Jensen functional equation. They could present a short and a simple proof (different from the *direct method* initiated by Hyers in 1941) for the generalized Hyers-Ulam stability of Jensen functional equation and for quadratic functional equation. Their methods are a powerful tool for studying the stability of several functional equations.

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The theory of random normed spaces (briefly, *RN*-spaces) is important as a generalization of deterministic result of normed spaces (see [1]) and also in the study of random operator equations. The notion of an *RN*-space corresponds to the situations when we do not know exactly the norm of the point and we know only probabilities of passible values of this norm. The *RN*-spaces may provide us the appropriate tools to study the geometry of nuclear physics and have usefully application in quantum particle physics. A number of papers and research monographs have been published on generalizations of the stability of different functional equations in *RN*-spaces [4, 7, 14–16, 22].

In the sequel, we use the definitions and notations of a random normed space as in [2, 18, 19]. A function $F : \mathbb{R} \cup \{-\infty, +\infty\} \to [0, 1]$ is called a *distribution function* if it is nondecreasing and left-continuous, with $F(0) = 0$ and $F(+\infty) = 1$. The class of all probability distribution functions F is denoted by Λ. D^+ is a subset of Λ consisting of all functions $F \in \Lambda$ for which $l^- F(+\infty) = 1$, where $l^- F(x) = \lim_{t\to x^-} F(t)$. The space Λ is partially ordered by the usual pointwise ordering of functions, that is, $F \le G$ if and only if $F(t) \le G(t)$ for all $t \in \mathbb{R}$. For any $a \ge 0$, ϵ_a is the element of D^+, which is defined by

$$\epsilon_a(t) = \begin{cases} 0 & \text{if } t \le a, \\ 1 & \text{if } t > a. \end{cases}$$

Definition 1.1. ([18]) A function $T : [0, 1] \times [0, 1] \to [0, 1]$ is a *continuous triangular norm* (briefly, a *continuous t-norm*) if T satisfies the following conditions:

(1) T is commutative and associative;
(2) T is continuous;
(3) $T(a, 1) = a$ for all $a \in [0, 1]$;
(4) $T(a, b) \le T(c, d)$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

Three typical examples of continuous t-norms are as follows:

$$T(a, b) = ab, \quad T(a, b) = \max(a + b - 1, 0), \quad T(a, b) = \min(a, b).$$

Recall that, if T is a t-norm and $\{x_n\}$ is a sequence of numbers in $[0, 1]$, then $T^n_{i=1} x_i$ is defined recurrently by

$$T^1_{i=1} x_i = x_1, \quad T^n_{i=1} x_i = T(T^{n-1}_{i=1} x_i, x_n) = T(x_1, \cdots, x_n)$$

for all $n \ge 2$, where $T^\infty_{i=n} x_i$ is defined as $T^\infty_{i=1} x_{n+i}$ ([11]).

Definition 1.2. ([19]) Let X be a real linear space, μ be a mapping from X into D^+ (for any $x \in X$, $\mu(x)$ is denoted by μ_x) and T be a continuous t-norm. The triple (X, μ, T) is called a *random normed space* (briefly, *RN-space*) if μ satisfies the following conditions:

(RN1) $\mu_x(t) = \epsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
(RN2) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X$, $\alpha \ne 0$ and $t \ge 0$;
(RN3) $\mu_{x+y}(t + s) \ge T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \ge 0$.

Every normed space $(X, \|\cdot\|)$ defines a random normed space (X, μ, T_M), where

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $x \in X$, $t > 0$ and T_M is the minimum continuous t-norm. This space is called the *induced random normed space*.

Definition 1.3. Let (X, μ, T) be an RN-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ if, for all $t > 0$ and $\lambda > 0$, there exists a positive integer N such that

$$\mu_{x_n - x}(t) > 1 - \lambda$$

whenever $n \geq N$. In this case, x is called the *limit* of the sequence $\{x_n\}$, which is denoted by $\lim_{n\to\infty} \mu_{x_n - x} = 1$.

(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for all $t > 0$ and $\lambda > 0$, there exists a positive integer N such that

$$\mu_{x_n - x_m}(t) > 1 - \lambda$$

whenever $n \geq m \geq N$.

(3) The RN-space (X, μ, T) is said to be *complete* if every Cauchy sequence in X is convergent to a point in X.

Theorem 1.4. ([18]) *If (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence of X such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$.*

Now, we consider a mapping $f : X \to Y$ satisfying the following functional equation:

$$f(x + ky) - kf(x + y) + kf(x - y) - f(x - ky) = 2k(k^2 - 1)f(y) \tag{1}$$

for fixed $k \in \mathbb{Z}^+$ with $k \geq 3$. Then the equation (1) is called the *general cubic functional equation* since the function $f(x) = x^3$ is its solution. Every solution of the cubic functional equation is called a *cubic mapping*.

Note that, if we put $x = 0$ and $y = x$ in the equation (1), then $f(kx) = k^3 f(x)$ and $f(k^n x) = k^{3n} f(x)$ for all $n \in \mathbb{Z}^+$.

In the case $k = 3$, Wiwatwanich et al. [21] established the general solution and the general Hyers-Ulam-Rassias stability of cubic functional equation on Banach spaces. The stability of the functional equation (1) in quasi-β-normed spaces and fuzzy normed spaces were investigated by Eskandani et al. [9] and Javdian et al. [13], respectively.

In this paper, using the direct and fixed point methods, we prove the generalized Hyers-Ulam stability problem of the general cubic functional equation (1) in random normed spaces in the sense of Scherstnev under the minimum continuous t-norm T_M.

Throughout this paper, let X be a real linear space, (Z, μ', T_M) be an RN-space and (Y, μ, T_M) be a complete RN-space. For any mapping $f : X \to Y$, we define

$$\Delta f(x, y) = f(x + ky) - kf(x + y) + kf(x - y) - f(x - ky) - 2k(k^2 - 1)f(y)$$

for all $x, y \in X$ and $k \in \mathbb{Z}^+$ with $k \geq 2$.

2. Random Stability of Functional Equation (1)

In this section, we investigate the generalized Hyers-Ulam stability of the general cubic functional equation $\Delta f(x, y) = 0$ in random spaces via *direct and fixed point methods*.

2.1 The direct Method

Theorem 2.1. *Let $\phi : X^2 \to Z$ be an even function such that, for some $0 < \alpha < k^3$,*

$$\mu'_{\phi(kx,ky)}(t) \geq \mu'_{\alpha\phi(x,y)}(t) \tag{2}$$

and $\lim_{n\to\infty} \mu'_{\phi(k^n x, k^n y)}(k^{3n}t) = 1$ *for all $x, y \in X$ and $t > 0$. If $f : X \to Y$ is a mapping with $f(0) = 0$ such that*

$$\mu_{\Delta f(x,y)}(t) \geq \mu'_{\phi(x,y)}(t) \tag{3}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(y)-C(y)}(t) \geq \mu'_{\phi(0,y)}\left(\frac{2k(k^2-1)(k^3-\alpha)t}{k^3+\alpha}\right) \tag{4}$$

for all $y \in X$ and $t > 0$.

Proof. Substituting $x = 0$ in (3), we have

$$\mu_{\Delta f(0,y)}(t) \geq \mu'_{\phi(0,y)}(t) \tag{5}$$

and replacing $y = -y$ in (5), we have

$$\mu_{\Delta f(0,-y)}(t) \geq \mu'_{\phi(0,y)}(t) \tag{6}$$

for all $y \in X$ and $t > 0$. It follows from (5) and (6) that

$$\mu_{f(y)+f(-y)}(t) \geq \mu'_{\phi(0,y)}(k(k^2-1)t).$$

Since $\mu_{2f(ky)-2k^3 f(y)}(t) = \mu_{f(ky)+f(-ky)-k(f(y)+f(-y))+\Delta f(0,y)}(t)$, we have

$$\begin{aligned}
&\mu_{2f(ky)-2k^3 f(y)}\left(\frac{k^3+\alpha}{k(k^2-1)}t\right)\\
&\geq T_M\left(\mu_{f(ky)+f(-ky)}\Big(\frac{\alpha}{k(k^2-1)}t\Big), \mu_{k(f(y)-f(-y))}\Big(\frac{t}{k^2-1}\Big), \mu_{\Delta f(0,y)}(t)\right)\\
&\geq T_M\Big(\mu'_{\phi(0,ky)}(\alpha t), \mu'_{\phi(0,y)}(t), \mu'_{\phi(0,y)}(t)\Big)\\
&= \mu'_{\phi(0,y)}(t)
\end{aligned}$$

for all $y \in X$ and $t > 0$. Thus we have

$$\mu_{\frac{f(ky)}{k^3}-f(y)}(t) \geq \mu'_{\phi(0,y)}\Big(\frac{2k^4(k^2-1)}{k^3+\alpha}t\Big) \tag{7}$$

for all $y \in X$ and $t > 0$. Replacing y by $k^n y$ in (7), we have

$$\mu_{\frac{f(k^{n+1}y)}{k^{3(n+1)}}-\frac{f(k^n y)}{k^{3n}}}(t) \geq \mu'_{\phi(0,y)}\Big(\frac{2k^4(k^2-1)}{k^3+\alpha}\Big(\frac{k^3}{\alpha}\Big)^n t\Big)$$

for all $y \in X$ and $t > 0$. Since $\frac{f(k^n y)}{k^{3n}} - f(y) = \sum_{j=0}^{n-1}\left(\frac{f(k^{j+1}y)}{k^{3(j+1)}} - \frac{f(k^j y)}{k^{3j}}\right)$, we have

$$\mu_{\frac{f(k^n y)}{k^{3n}}-f(y)}\Big(\sum_{j=0}^{n-1}\frac{k^3+\alpha}{2k^4(k^2-1)}\Big(\frac{\alpha}{k^3}\Big)^j t\Big) \geq T_{M\,j=0}^{\;n-1}(\mu'_{\phi(0,y)}(t)) = \mu'_{\phi(0,y)}(t) \tag{8}$$

for all $y \in X$ and $t > 0$. Replacing y by $k^m y$ in (8), we obtain

$$\mu_{\frac{f(k^{n+m}y)}{k^{3(n+m)}} - \frac{f(k^m y)}{k^{3m}}}(t) \geq \mu'_{\phi(0,y)}\left(\frac{2k^4(k^2-1)t}{(k^3+\alpha)\sum_{j=m}^{n+m-1}\left(\frac{\alpha}{k^3}\right)^j}\right) \tag{9}$$

for all $y \in X$ and $m, n \in \mathbb{Z}^+$ with $n > m$. Since $\alpha < k^3$, the sequence $\{\frac{f(k^n y)}{k^{3n}}\}$ is a Cauchy sequence in a complete RN-space (Y, μ, T_M) and so it converges to some point $C(y) \in Y$. Fix $y \in X$ and put $m = 0$ in (9). Then we obtain

$$\mu_{\frac{f(k^n y)}{k^{3n}} - f(y)}(t) \geq \mu'_{\phi(0,y)}\left(\frac{2k^4(k^2-1)t}{(k^3+\alpha)\sum_{j=0}^{n-1}\left(\frac{\alpha}{k^3}\right)^j}\right)$$

and so, for any $\delta > 0$, it follows that

$$\begin{aligned}
&\mu_{C(y)-f(y)}(\delta + t) \\
&\geq T_M\Big(\mu_{C(y)-\frac{f(k^n y)}{k^{3n}}}(\delta), \mu_{\frac{f(k^n y)}{k^{3n}} - f(y)}(t)\Big) \\
&\geq T_M\left(\mu_{C(y)-\frac{f(k^n y)}{k^{3n}}}(\delta), \mu'_{\phi(0,y)}\Big(\frac{2k^4(k^2-1)t}{(k^3+\alpha)\sum_{j=0}^{n-1}\left(\frac{\alpha}{k^3}\right)^j}\Big)\right)
\end{aligned} \tag{10}$$

for all $y \in X$ and $t > 0$. Taking the limit as $n \to \infty$ in (10), we obtain

$$\mu_{C(y)-f(y)}(\delta + t) \geq \mu'_{\phi(0,y)}\left(\frac{2k(k^2-1)(k^3-\alpha)t}{k^3+\alpha}\right). \tag{11}$$

Since δ is arbitrary, by taking $\delta \to 0$ in (11), we have

$$\mu_{C(y)-f(y)}(t) \geq \mu'_{\phi(0,y)}\left(\frac{2k(k^2-1)(k^3-\alpha)t}{k^3+\alpha}\right) \tag{12}$$

for all $y \in X$ and $t > 0$. Therefore, we conclude that the condition (4) holds. Replacing x and y by $k^n x$ and $k^n y$ in (3), respectively, we have

$$\begin{aligned}
&\mu_{\frac{f(k^n x + k^{n+1} y)}{k^{3n}} - \frac{kf(k^n x + k^n y)}{k^{3n}} + \frac{kf(k^n x - k^n y)}{k^{3n}} - \frac{f(k^n x + k^{n+1} y)}{k^{3n}} - \frac{2k(k^2-1)f(k^n y)}{k^{3n}}}(t) \\
&\geq \mu'_{\phi(k^n x, k^n y)}(k^{3n} t)
\end{aligned}$$

for all $x, y \in X$ and $t > 0$. Since $\lim_{n\to\infty} \mu'_{\phi(k^n x, k^n y)}(k^{3n} t) = 1$, it follows that C satisfies the equation (1), which implies that C is a cubic mapping.

To prove the uniqueness of the cubic mapping C, let us assume that there exists another mapping $D : X \to Y$ which satisfies (4). Fix $y \in X$. Then $C(k^n y) = k^{3n} C(y)$ and $D(k^n y) = k^{3n} D(y)$ for all $n \in \mathbb{Z}^+$. It follows from (2.3) that

$$\begin{aligned}
\mu_{C(y)-D(y)}(t) &= \mu_{\frac{C(k^n y)}{k^{3n}} - \frac{D(k^n y)}{k^{3n}}}(t) \\
&\geq T_M\Big(\mu_{\frac{C(k^n y)}{k^{3n}} - \frac{f(k^n y)}{k^{3n}}}\Big(\frac{t}{2}\Big), \mu_{\frac{f(k^n y)}{k^{3n}} - \frac{D(k^n y)}{k^{3n}}}\Big(\frac{t}{2}\Big)\Big) \\
&\geq \mu'_{\phi(0,y)}\left(\frac{2k(k^2-1)(k^3-\alpha)k^{3n} t}{(k^3+\alpha)\alpha^n}\right).
\end{aligned} \tag{13}$$

Since $\lim_{n\to\infty} \frac{2k(k^2-1)(k^3-\alpha)k^{3n} t}{(k^3+\alpha)\alpha^n} = \infty$, we have $\mu_{C(y)-D(y)}(t) = 1$ for all $t > 0$. Thus the cubic mapping C is unique. This completes the proof. □

Theorem 2.2. *Let $\phi : X^2 \to Z$ be an even function such that, for some $0 < k^3 < \alpha$,*

$$\mu'_{\phi(\frac{x}{k},\frac{y}{k})}(t) \geq \mu'_{\phi(x,y)}(\alpha t) \tag{14}$$

and $\lim_{n\to\infty} \mu'_{k^{3n}\phi(\frac{x}{k^n},\frac{y}{k^n})}(t) = 1$ for all $x, y \in X$ and $t > 0$. If $f : X \to Y$ is a mapping with $f(0) = 0$ which satisfies (3), then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(y)-C(y)}(t) \geq \mu'_{\phi}(0, y)\left(\frac{2k(k^2-1)(\alpha - k^3)t}{k^3+\alpha}\right) \tag{15}$$

for all $y \in X$ and $t > 0$.

Proof. It follows from (7) that

$$\mu_{f(y)-k^3 f(\frac{y}{k})}(t) \geq \mu'_{\phi(0,y)}\left(\frac{2\alpha k(k^2-1)t}{k^3+\alpha}\right) \tag{16}$$

for all $y \in X$ and $t > 0$. Applying the triangle inequality and (16), we have

$$\mu_{f(y)-k^{3n} f(\frac{y}{k^n})}(t) \geq \mu'_{\phi(0,y)}\left(\frac{2\alpha k(k^2-1)t}{(k^3+\alpha)\sum_{j=m}^{n+m-1}\left(\frac{k^3}{\alpha}\right)^j}\right) \tag{17}$$

for all $y \in X$ and $m, n \in \mathbb{Z}^+$ with $n > m \geq 0$. Then the sequence $\{k^{3n} f(\frac{y}{k^n})\}$ is a Cauchy sequence in a complete RN-space (Y, μ, T_M) and so it converges to some point $C(y) \in Y$. We can define a mapping $C : X \to Y$ by

$$C(y) = \lim_{n\to\infty} k^{3n} f(\frac{y}{k^n})$$

for all $y \in X$. Then the mapping C satisfies (1) and (15). The remaining assertion goes through in the similar method to the corresponding part of Theorem 2.1. This complete the proof. □

Corollary 2.3. *Let $p \in \mathbb{R}$ be positive real number with $p \neq 3$ and a fixed unit point of $z_0 \in Z$. If $f : X \to Y$ is a mapping with $f(0) = 0$ and satisfying*

$$\mu_{\Delta f(x,y)}(t) \geq \mu'_{(\|x\|^p+\|y\|^p)z_o}(t) \tag{18}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(y)-C(y)}(t) \geq \mu'_{\|y\|^p z_o}\left(\frac{2k(k^2-1)|k^3-k^{3p}|t}{k^3+k^{3p}}\right) \tag{19}$$

for all $x, y \in X$ and $t > 0$.

Proof. Let $\phi : X^2 \to Z$ be defined by $\phi(x, y) = (\|x\|^p + \|y\|^p)z_o$. Then, by Theorem 2.1, we obtain the desired result, where $\alpha = k^{3p}$. □

Remark. (1) An example to illustrate that the functional equation (1) is not stable for $p = 3$ in Corollary 2.3 (see [13]).

(2) In Corollary 2.3, if we assume that

$$\phi(x, y) = \|x\|^p\|y\|^p z_0$$

or

$$\phi(x, y) = (\|x\|^p\|y\|^q + \|x\|^{p+q} + \|y\|^{p+q})z_0,$$

then we have the product stability of Ulam-Gavuta-Rassias and the mixed product-sum stability of Rassias, respectively.

Example 2.4. Let $(X, \|\cdot\|)$ be a Banach normed space and

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $x \in X$ and $t > 0$. Then $(X, \mu, \min)$ is a complete RN-space. Also, let

$$\mu'_{\phi(x,y)}(t) = \frac{t}{t + \|\phi(x, y)\|}$$

for all $x, y \in X$ and $t > 0$. Then $(X, \mu', \min)$ is a RN-space.

Define a mapping $f : X \to X$ by $f(x) = x^3 + \|x\|z_0$, where z_0 is a unit point in X. By a simple calculation, we have

$$\begin{aligned}
&\|\Delta f(x, y)\| \\
&= \|f(x + ky) - kf(x + y) + kf(x - y) - f(x - ky) - 2k(k^2 - 1)f(y)\| \\
&\le 2k(k^2 - 1)\|y\| \le 2k(k^2 - 1)(\|x\| + \|y\|).
\end{aligned}$$

Then it follows that

$$\mu_{\Delta f(x,y)}(t) \ge \mu'_{\phi(x,y)}(t)$$

for all $x, y \in X$ and $t > 0$, where $\phi(x, y) = 2k(k^2 - 1)(\|x\| + \|y\|)$. Also, we obtain

$$\mu'_{\phi(0,k^n y)}(k^{3n}(k^3 - \alpha)t) = \frac{k^{3n}(k^3 - \alpha)t}{t + 2k(k^2 - 1)k^n\|y\|},$$

where $0 < \alpha < k^3$, and

$$\lim_{n\to\infty} \mu'_{\phi(0,k^n y)}(k^{3n}(k^3 - \alpha)t) = 1.$$

Thus all the conditions of Theorem 2.1 hold. Therefore, there exists a unique cubic mapping $C : X \to X$ such that

$$\mu_{f(y)-C(y)}(t) \ge \frac{(k^3 - \alpha)t}{(k^3 - \alpha)t + (k^3 + \alpha)\|y\|}$$

and $C(y) = y^3$ for all $y \in X$ and $t > 0$.

2.2 The fixed point method

Recall that a mapping $d : X^2 \to [0, +\infty]$ is called a *generalized metric* on a nonempty set X if

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

A set X with the generalized metric d is called a *generalized metric space*.

The following fixed point theorem proved by Diaz and Margolis [8] plays an important role in proving our theorem:

Theorem 2.5. ([8]) *Suppose that (Ω, d) is a complete generalized metric space and $J : \Omega \to \Omega$ is a strictly contractive mapping with Lipshitz constant $L < 1$. Then, for each $x \in \Omega$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integers $n \ge 0$ or there exists a natural number n_0 such that*

(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \ge n_0$;*

(2) *the sequence $\{J^n x\}$ is convergent to a fixed point $y*$ of J;*

(3) *$y*$ is the unique fixed point of J in the set $\Lambda = \{y \in \Omega : d(J^{n_0}x, y) < \infty\}$;*

(4) *$d(y, y*) \le \frac{1}{1-L} d(y, Jy)$ for all $y \in \Lambda$.*

Theorem 2.6. *Let $\psi : X^2 \to D^+$ ($\psi(x, y)$ is denoted by $\psi_{x,y}$) be a even function such that, for some $0 < \alpha < k^3$,*

$$\psi_{x,y}(t) \le \psi_{kx,ky}(\alpha t)$$

for all $x, y \in X$ and $t > 0$. If $f : X \to Y$ is a mapping with $f(0) = 0$ which satisfies

$$\mu_{\Delta f(x,y)}(t) \ge \psi_{x,y}(t) \tag{20}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(y)-C(y)}(t) \ge \psi_{0,y}\left(\frac{2k(k^2-1)(k^3-\alpha)t}{k^3+\alpha}\right) \tag{21}$$

for all $y \in X$ and $t > 0$.

Proof. It follows from (20) and the similar methods in the proof of Theorem 2.1 that

$$\mu_{\frac{f(ky)}{k^3}-f(y)}(t) \ge \psi_{0,y}\Big(\frac{2k^4(k^2-1)t}{k^3+\alpha}\Big). \tag{22}$$

Let Ω be a set of all mappings from X into Y and introduce a generalized metric on Ω as follows:

$$d(g, h) = \inf\{c \in [0, \infty) : \mu_{g(y)-h(y)}(ct) \ge \psi_{0,y}(t),\ \forall\ y \in X\}.$$

where, as usual, $\inf \emptyset = -\infty$. It is easy to show that (Ω, d) is a generalized complete metric space ([6]).

Now, let us consider the mapping $J : \Omega \to \Omega$ defined by

$$Jg(y) = \frac{g(ky)}{k^3}$$

for all $g \in \Omega$ and $y \in X$. Let g, h in Ω and $c \in [0, \infty)$ be an arbitrary constant with $d(g, h) < c$. Then we have

$$\mu_{g(y)-h(y)}(ct) \ge \psi_{0,y}(t)$$

for all $y \in X$ and $t > 0$, whence

$$\mu_{Jg(y)-Jh(y)}\Big(\frac{\alpha}{k^3}ct\Big) \ge \psi_{0,y}(t) \tag{23}$$

for all $y \in X$ and $t > 0$ and so

$$d(Jg, Jh) \le \frac{\alpha c}{k^3} \le \frac{\alpha}{k^3} d(g, h)$$

for all $g, h \in \Omega$. Then J is a strictly contractive self-mapping on Ω with the Lipschitz constant $\frac{\alpha}{k^3} < 1$. It follows from (22) that

$$d(f, Jf) \le \frac{k^3+\alpha}{2k^4(k^2-1)}.$$

Due to Theorem 2.5, there exists a mapping $C : X \to Y$, which is a unique fixed point of J in the set $\Omega_1 = \{g \in \Omega : d(f, g) < \infty\}$ such that

$$C(y) = \lim_{n\to\infty} \frac{f(k^n y)}{k^{3n}}$$

for all $y \in X$ since $\lim_{n\to\infty} d(J^n f, C) = 0$. Again, it follows from Theorem 2.5 that

$$d(f,C) \le \frac{1}{1-L} d(f,Jf) \le \frac{k^3+\alpha}{2k^4(k^2-1)(1-L)}.$$

Then we conclude

$$\mu_{f(y)-C(y)}(t) \ge \psi_{0,y}\left(\frac{2k(k^2-1)(k^3-\alpha)t}{k^3+\alpha}\right)$$

for all $y \in X$ and $t > 0$, where $L = \frac{\alpha}{k^3}$. The remaining assertion goes through in the similar method to the corresponding part of Theorem 2.1. This completes the proof. □

Theorem 2.7. *Let $\psi : X^2 \to Z$ be an even function such that, for some $0 < k^3 < \alpha$,*

$$\psi_{\frac{x}{k},\frac{y}{k}}(t) \ge \psi_{x,y}(\alpha t)$$

for all $x, y \in X$ and $t > 0$. If $f : X \to Y$ is a mapping with $f(0) = 0$ which satisfies (20), then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mu_{f(y)-C(y)}(t) \ge \psi_{0,y}\left(\frac{2k(k^2-1)(\alpha-k^3)t}{k^3+\alpha}\right) \tag{24}$$

for all $y \in X$ and $t > 0$.

Proof. It follows from (20) that

$$\mu_{f(y)-k^3 f(\frac{y}{k})}(t) \ge \psi_{0,y}\left(\frac{2\alpha k(k^2-1)t}{k^3+\alpha}\right) \tag{25}$$

for all $y \in X$ and $t > 0$. Let Ω and d be as in the proof of Theorem 2.6. Then (Ω, d) is a generalized complete metric space ([6]) and we consider the mapping $J : \Omega \to \Omega$ defined by

$$Jg(y) = k^3 g(\frac{y}{k})$$

for all $g \in \Omega$ and $y \in X$. So, we have

$$d(Jg, Jh) \le \frac{k^3 c}{\alpha} \le \frac{k^3}{\alpha} d(g,h)$$

for all $g, h \in \Omega$. Then J is a strictly contractive self-mapping on Ω with the Lipschitz constant $\frac{k^3}{\alpha} < 1$. It follows from (25) that

$$d(f, Jf) \le \frac{k^3+\alpha}{\alpha(2k(k^2-1))}.$$

Due to Theorem 2.5, there exists a mapping $C : X \to Y$, which is a unique fixed point of J in the set $\Omega_1 = \{g \in \Omega : d(f,g) < \infty\}$, such that

$$C(y) = \lim_{n\to\infty} k^{3n} f(\frac{y}{k^n y})$$

for all $y \in X$ since $\lim_{n\to\infty} d(J^n f, C) = 0$. Again, it follows from Theorem 2.5 that

$$d(f,C) \le \frac{1}{1-L} d(f,Jf) \le \frac{k^3+\alpha}{2k(k^2-1)(\alpha-k^3)}$$

for all $y \in X$ and $t > 0$, where $L = \frac{k^3}{\alpha}$. The remaining assertion goes through in the similar method to the corresponding part of Theorem 2.2. This completes the proof. □

Now, we present a corollary that is an application of Theorem 2.6 in the classical case.

Corollary 2.8. *Let X be a real normed space, $\theta \geq 0$ and p be a real number with $0 < p < 1$. Assume that $f : X \to X$ is a mapping with $f(0) = 0$ which satisfies*

$$\mu_{\Delta f(x,y)}(t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}$$

for all $x, y \in X$ and $t > 0$. Then the limit $C(x) = \lim_{n\to 0} \frac{f(k^n y)}{k^{3n}}$ exists for all $y \in X$ and $C : X \to X$ is a unique cubic mapping such that

$$\mu_{f(y)-C(y)}(t) \geq \frac{2k(k^2-1)(k^3-k^p)t}{2k(k^2-1)(k^3-k^p)t + \theta(k^3+k^p)\|y\|^p}$$

for all $y \in X$ and $t > 0$.

Proof. Let $\psi_{x,y}(t) = \frac{t}{t+\theta(\|x\|^p+\|y\|^p)}$ for all $x, y \in X$ and $t > 0$ and $\alpha = k^p$. Then we obtain the desired result. □

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