



Generalized Reflexive and Anti-Reflexive Solution for a System of Equations

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Abstract. In this article we find necessary and sufficient conditions for the generalized reflexive and antireflexive solution of the system of equations $ax = b$ and $xc = d$ in a ring with involution. As corollaries, among other results, we obtain some recent results from (A. Dajić and J. Koliha, Linear Algebra Appl. (2008)), and Li Fanliang, Hu Xiuyan and Zhang Lei, Acta Math. Scientia 28B(1) (2008)).

1. Introduction and preliminaries

Various matrix and operator equations play important roles in many applications. The system of matrix equations that is often considered is

$$AX = B, \quad XC = D, \tag{1}$$

and there are many significant results for its solution. These solutions are obtained using rather complicated methods, such as singular value decompositions, theory of rank etc. We found the motivation for this work in [8], where the authors considered the generalized reflexive and anti-reflexive solution of (1). Here we use algebraic methods in rings with involution, to obtain the generalization of some results from [8]. A general solution of that kind of a problem is offered in ring with involution. Some results from [5] are generalized. Related results can be found in [1],[2], [3], [4] and [6]. Usually, one is interested in finding hermitian and anti-hermitian, then positive, reflexive and anti-reflexive solutions of the system (1).

Let \mathcal{R} be a ring with the unit 1 and the involution $x \mapsto x^*$. We use \mathcal{R}^{-1} to denote the set of all invertible elements of \mathcal{R} .

Recall that an element $a \in \mathcal{R}$ is hermitian if $a = a^*$, and a is normal if $aa^* = a^*a$. An element $a \in \mathcal{R}$ is regular (inner invertible, von Neumann regular) if there exists $b \in \mathcal{R}$ such that $aba = a$. Any such b is called an inner inverse of a . In general, inner inverse of a is not unique, and the set of all inner inverses of the element $a \in \mathcal{R}$ is denoted by

$$a\{1\} = \{b \in \mathcal{R} \mid aba = a\}.$$

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We denote by a^\dagger the (unique when it exists) Moore-Penrose inverse of an element a , satisfying

$$aa^\dagger a = a \quad a^\dagger aa^\dagger = a^\dagger \quad (aa^\dagger)^* = aa^\dagger \quad (a^\dagger a)^* = a^\dagger a.$$

If \mathcal{R} is a C^* -algebra, then it is well known fact that the following hold ([7]; see also [9]):

$$a \text{ is regular} \iff a\mathcal{R} \text{ is closed} \iff a^\dagger \text{ exists}.$$

We say that $p \in \mathcal{R}$ is a projection, if p is hermitian and idempotent, that is $p = p^2 = p^*$.

For further investigations we assume that $2 \in \mathcal{R}^{-1}$. The motivation comes from the following: for a Hilbert space H , let $W \in \mathcal{B}(H)$ be the reflection with respect to the closed subspace M of H . Actually, the reflection of a point $x \in H$ in M , is the point Wx such that the orthogonal projection Px of x onto M is the midpoint between x and Wx . Since $Px = \frac{1}{2}(x + Wx)$, we get $W = 2P - I$, and because $P^* = P = P^2$, one can easily get $W^* = W$ and $W^2 = I$.

In the following we generalize this concept in rings with involution. First, we give definitions and some basic results.

Definition 1.1. An element $w \in \mathcal{R}$ is said to be a generalized reflection element if $w = w^*$ and $w^2 = 1$.

In other words, a generalized reflection is a unitary and hermitian element in \mathcal{R} . Moreover, w is generalized reflection if and only if there exists a projection $p \in \mathcal{R}$, such that $w = 2p - 1$ (or $p = \frac{1}{2}(w + 1)$, since $2 \in \mathcal{R}^{-1}$). As a special case, for $w = 1$ we have $p = 1$.

Definition 1.2. Let $w, v \in \mathcal{R}$ be generalized reflection elements. An element $x \in \mathcal{R}$ is called a generalized reflexive element (according to w and v), if $x = wxv$ holds. Denote this class of elements by $\mathcal{R}_r(w, v)$.

An element $x \in \mathcal{R}$ is called a generalized anti-reflexive element (according to w and v), if $x = -wxv$ holds. Denote this class by $\mathcal{R}_{ar}(w, v)$.

For given generalized reflection elements w and v in \mathcal{R} it is true that $\mathcal{R}_r(w, v)$ and $\mathcal{R}_{ar}(w, v)$ are additive subgroups of \mathcal{R} . Moreover, if \mathcal{R} is a C^* -algebra, then $\mathcal{R}_r(w, v)$ and $\mathcal{R}_{ar}(w, v)$ are closed subspaces of \mathcal{R} . Also, the following hold:

$$\begin{aligned} x \in \mathcal{R}_r(w, v) &\iff x^* \in \mathcal{R}_r(v, w) \\ x \in \mathcal{R}_{ar}(w, v) &\iff x^* \in \mathcal{R}_{ar}(v, w). \end{aligned}$$

Let $w, v \in \mathcal{R}$ be generalized reflection elements. Special types of generalized reflexive elements are defined as

$$\mathcal{R}_r(w, 1) = \{x \mid x = wx\} \text{ and } \mathcal{R}_{ar}(w, 1) = \{x \mid x = -wx\};$$

$$\mathcal{R}_r(1, v) = \{x \mid x = xv\} \text{ and } \mathcal{R}_{ar}(1, v) = \{x \mid x = -xv\};$$

$$\mathcal{R}_r(w) = \mathcal{R}_r(w, w) \text{ and } \mathcal{R}_{ar}(w) = \mathcal{R}_{ar}(w, w).$$

If $x \in \mathcal{R}_r(w)$ ($x \in \mathcal{R}_{ar}(w)$), then x is a reflexive (anti-reflexive) element according to w .

Consider the following system of equations

$$ax = b, \quad xc = d. \tag{2}$$

We are interested in finding generalized reflexive solutions $x \in \mathcal{R}_r(w, v)$, and generalized anti-reflexive solutions $x \in \mathcal{R}_{ar}(w, v)$ of (2), for given $a, b, c, d \in \mathcal{R}$.

We use, as a main tool, the unique decomposition of every element in \mathcal{R} as a sum of generalized reflexive and generalized anti-reflexive element according to prescribed generalized reflection elements w and v .

This article is organized as follows. In Section 2 we study special properties of generalized reflexive and generalized anti-reflexive elements in $\mathcal{R}_r(w, v)$. Some lemmas are proved, which are used to prove the main results later on. In Section 3 we prove the necessary and sufficient conditions for the existence of generalized reflexive (w, v) and generalized anti-reflexive (w, v) solution of the system (2). Also, we get results for the generalized reflexive and generalized anti-reflexive (w, v) solutions of the equations $ax = b$ and $xc = d$ respectively.

2. Generalized Reflexive and Anti-Reflexive Elements

The motivation for this paper follows from [8]. Also, in [5] a reflexive solution of the equations from the system (2) in the setting of a ring is given.

In the rest of the paper let $w, v \in \mathcal{R}$ be generalized reflection elements such that $p = \frac{1}{2}(w + 1)$ and $q = \frac{1}{2}(v + 1)$ are projections in \mathcal{R} . First we study some special properties of elements in $\mathcal{R}_r(w, v)$ and $\mathcal{R}_{ar}(w, v)$.

Lemma 2.1. *For every $x \in \mathcal{R}$ there exist unique elements $x_1 \in \mathcal{R}_r(w, v)$ and $x_2 \in \mathcal{R}_{ar}(w, v)$ such that*

$$x = x_1 + x_2,$$

so we can write

$$\mathcal{R} = \mathcal{R}_r(w, v) \oplus \mathcal{R}_{ar}(w, v).$$

Moreover,

$$\begin{aligned} x_1 &= x - px - xq + 2pxq = pxq + (1 - p)x(1 - q), \\ x_2 &= px + xq - 2pxq = (1 - p)xq + px(1 - q). \end{aligned} \tag{3}$$

Proof. Existence: First of all, note that 0 is the only one generalized reflexive and generalized anti-reflexive element in \mathcal{R} , that is $\mathcal{R}_r(w, v) \cap \mathcal{R}_{ar}(w, v) = \{0\}$ (recall that $2 \in \mathcal{R}^{-1}$). Now, for any $x \in \mathcal{R}$ let $x_1 = \frac{1}{2}(x + wxv)$ and $x_2 = \frac{1}{2}(x - wxv)$. Then, it is obvious that $x = x_1 + x_2$, $x_1 \in \mathcal{R}_r(w, v)$ and $x_2 \in \mathcal{R}_{ar}(w, v)$.

Uniqueness: If $x = x_1 + x_2 = x'_1 + x'_2$, for $x_1, x'_1 \in \mathcal{R}_r(w, v)$ and $x_2, x'_2 \in \mathcal{R}_{ar}(w, v)$, then

$$x_1 - x'_1 = -(x_2 - x'_2) \in \mathcal{R}_r(w, v) \cap \mathcal{R}_{ar}(w, v) = \{0\}.$$

Thus, $x_1 = x'_1$ and $x_2 = x'_2$.

We have just proved that for given generalized reflection elements w and v , every element in \mathcal{R} can be represented as the unique sum of generalized reflexive and generalized anti-reflexive element in \mathcal{R} .

Also, $x_1 = \frac{1}{2}(x + wxv) = \frac{1}{2}(x + (2p - 1)x(2q - 1)) = x - px - xq + 2pxq = pxq + (1 - p)x(1 - q)$. Then we have $x_2 = x - x_1 = px + xq - 2pxq = (1 - p)xq + px(1 - q)$. \square

Now we characterize generalized reflexive and generalized anti-reflexive elements in rings with involution.

Lemma 2.2. *Let $x \in \mathcal{R}$. Then the following statements hold:*

1. $x \in \mathcal{R}_r(w, v)$ if and only if $px(1 - q) = 0 = (1 - p)xq$, or equivalently

$$x = pxq + (1 - p)x(1 - q);$$

2. $x \in \mathcal{R}_{ar}(w, v)$ if and only if $pxq = 0 = (1 - p)x(1 - q)$, or equivalently

$$x = px(1 - q) + (1 - p)xq.$$

Proof. The results follow from Lemma 2.1 taking into consideration that $x = pxq + px(1 - q) + (1 - p)xq + (1 - p)x(1 - q)$. \square

Lemma 2.3. *Let $x \in \mathcal{R}$, then the following statements hold:*

1. *If $x = x_1 + x_2 \in \mathcal{R}_r(w, 1) \oplus \mathcal{R}_{ar}(w, 1)$, then: $x_2^*x_1 = 0$, $x_1 = px$ and $x_2 = (1 - p)x$.*
2. *If $x = x_1 + x_2 \in \mathcal{R}_r(1, v) \oplus \mathcal{R}_{ar}(1, v)$, then: $x_1x_2^* = 0$, $x_1 = xq$ and $x_2 = x(1 - q)$.*
3. $\mathcal{R}_r(1, v) \cdot \mathcal{R}_{ar}(v, 1) = 0$.
4. $\mathcal{R}_{ar}(1, w) \cdot \mathcal{R}_r(w, 1) = 0$.

Proof. For $x \in \mathcal{R}$ we have:

1. $x = x_1 + x_2$, where $x_1 \in \mathcal{R}_r(w, 1)$ and $x_2 \in \mathcal{R}_{ar}(w, 1)$ because of Lemma 2.1. Then $x_2^*x_1 = (-wx_2)^*wx_1 = -x_2^*x_1 = 0$. Also, $x_1 = \frac{1}{2}(x + wx) = \frac{1}{2}(x + (2p - 1)x) = px$. The rest follows from Lemma 2.2.
2. Analogously.
3. Let $x \in \mathcal{R}_r(1, v)$ and $y \in \mathcal{R}_{ar}(v, 1)$. Then $x = xv$ and $y = -vy$, so $xy = xv(-v)y = -xy = 0$.
4. Analogously.

□

Let $a, b, c, d \in \mathcal{R}$, and let us decompose these elements as follows:

$$\begin{aligned}
 a &= a_1 + a_2 \in \mathcal{R}_r(1, w) \oplus \mathcal{R}_{ar}(1, w); \\
 b &= b_1 + b_2 \in \mathcal{R}_r(1, v) \oplus \mathcal{R}_{ar}(1, v); \\
 c &= c_1 + c_2 \in \mathcal{R}_r(v, 1) \oplus \mathcal{R}_{ar}(v, 1); \\
 d &= d_1 + d_2 \in \mathcal{R}_r(w, 1) \oplus \mathcal{R}_{ar}(w, 1).
 \end{aligned}
 \tag{4}$$

Lemma 2.4. *Let w, v_1 and v be generalized reflection elements in \mathcal{R} , then the following statements hold:*

1. *If $m \in \mathcal{R}_r(w, v_1)$ and $n \in \mathcal{R}_r(v_1, v)$, then $mn \in \mathcal{R}_r(w, v)$.*
2. *If $m \in \mathcal{R}_{ar}(w, v_1)$ and $n \in \mathcal{R}_{ar}(v_1, v)$, then $mn \in \mathcal{R}_{ar}(w, v)$.*
3. *If $m \in \mathcal{R}_{ar}(w, v_1)$ and $n \in \mathcal{R}_r(v_1, v)$, then $mn \in \mathcal{R}_{ar}(w, v)$.*
4. *If $m \in \mathcal{R}_r(w, v_1)$ and $n \in \mathcal{R}_{ar}(v_1, v)$, then $mn \in \mathcal{R}_{ar}(w, v)$.*

Proof. Simple computations lead to the results. □

Lemma 2.5. *Let a_1, a_2, c_1 and c_2 be as in the decomposition given by (4). Then the following statements hold:*

1. *The element a_1 is regular in \mathcal{R} if and only if there exists $\hat{a}_1 \in \mathcal{R}_r(w, 1) \cap a_1\{1\}$;*
2. *The element a_2 is regular in \mathcal{R} if and only if there exists $\hat{a}_2 \in \mathcal{R}_{ar}(w, 1) \cap a_2\{1\}$;*
3. *The element c_1 is regular in \mathcal{R} if and only if there exists $\hat{c}_1 \in \mathcal{R}_r(1, v) \cap c_1\{1\}$;*
4. *The element c_2 is regular in \mathcal{R} if and only if there exists $\hat{c}_2 \in \mathcal{R}_{ar}(1, v) \cap c_2\{1\}$.*

Proof. If a_1, a_2, c_1 and c_2 be regular elements then there exist a_1^-, a_2^-, c_1^- and c_2^- such that

$$a_1a_1^-a_1 = a_1, \quad a_2a_2^-a_2 = a_2, \quad c_1c_1^-c_1 = c_1 \quad \text{and} \quad c_2c_2^-c_2 = c_2.$$

Now, let's define

$$\begin{aligned}
 \hat{a}_1 &= \frac{1}{2}(a_1^- + wa_1^-) = pa_1^-, & \hat{a}_2 &= \frac{1}{2}(a_2^- - wa_2^-) = (1 - p)a_2^-, \\
 \hat{c}_1 &= \frac{1}{2}(c_1^- + c_1^-v) = c_1^-q, & \hat{c}_2 &= \frac{1}{2}(c_2^- - c_2^-v) = c_2^-(1 - q).
 \end{aligned}
 \tag{5}$$

Then we get:

1. $\hat{a}_1 \in a_1\{1\} \cap \mathcal{R}_r(w, 1)$;
2. $\hat{a}_2 \in a_2\{1\} \cap \mathcal{R}_{ar}(w, 1)$;
3. $\hat{c}_1 \in c_1\{1\} \cap \mathcal{R}_r(1, v)$;

4. $\hat{c}_2 \in c_2\{1\} \cap \mathcal{R}_{ar}(1, v)$.

The opposite direction is trivial. \square

If $\mathcal{R}^{m \times n}$ is the set of $m \times n$ matrices with entries in \mathcal{R} , then regularity of elements in $\mathcal{R}^{m \times n}$ can be defined in the same way. Namely, $A \in \mathcal{R}^{m \times n}$ is regular if there exists a matrix $A^- \in \mathcal{R}^{n \times m}$ such that $AA^-A = A$. In a ring with involution $A^* \in \mathcal{R}^{n \times m}$ denotes the involute transpose of A .

Theorem 2.6. *Let a_1, a_2, c_1 and c_2 be as in the decomposition given by (4). Then the following statements hold:*

1. $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \{1\} = \{[\hat{a}_1 \quad \hat{a}_2] \mid \hat{a}_1, \hat{a}_2 \text{ are given by (5)}\}$.
2. $[c_1 \quad c_2] \{1\} = \left\{ \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \end{bmatrix} \mid \hat{c}_1, \hat{c}_2 \text{ are given by (5)} \right\}$.

Proof. Let a_1, a_2, c_1 and c_2 be given by (4).

1. Let $m \in a\{1\}$, then there exist unique m_1 and m_2 such that $m = m_1 + m_2 \in \mathcal{R}_r(w, 1) \oplus \mathcal{R}_{ar}(w, 1)$ and

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cdot [m_1 \quad m_2] \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

holds. That is,

$$\begin{bmatrix} a_1 m_1 a_1 \\ a_2 m_2 a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix},$$

which means that m_1 and m_2 are inner inverses of a_1 and a_2 respectively.

Now, if \hat{a}_1, \hat{a}_2 are as in (5), using Lemma 2.3 we get

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cdot [\hat{a}_1 \quad \hat{a}_2] \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

2. Analogously.

\square

Lemma 2.7. *Let $e \in \mathcal{R}_r(w)$, $g \in \mathcal{R}_r(v)$ and $f \in \mathcal{R}$. Then, for $m = efg$ the following statements hold:*

1. If $f \in \mathcal{R}_r(w, v)$ then $m \in \mathcal{R}_r(w, v)$.
2. If $f \in \mathcal{R}_{ar}(w, v)$ then $m \in \mathcal{R}_{ar}(w, v)$.
3. If $f = f_1 + f_2 \in \mathcal{R}_r(w, v) \oplus \mathcal{R}_{ar}(w, v)$ then $m \in \mathcal{R}_r(w, v)$ if and only if $ef_2g = 0$, and $m = ef_1g$.

Proof. 1. $wmv = w(efg)v = w(wev)(wfv)(v) = efg = m$. So, $m \in \mathcal{R}_r(w, v)$.

2. Analogously.

3. $m = efg = ef_1g + ef_2g \in \mathcal{R}_r(w, v) \oplus \mathcal{R}_{ar}(w, v)$ and using (1) and (2) it is obvious that $m \in \mathcal{R}_r(w, v)$ if and only if $ef_2g = 0$.

\square

3. Equations in Rings

Let a, b, c and d be decomposed as in (4). Denote

$$\tilde{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \tilde{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \tilde{c} = [c_1 \ c_2] \text{ and } \tilde{d} = [d_1 \ d_2]. \tag{6}$$

Lemma 3.1. *Let $\tilde{a}, \tilde{b}, \tilde{c}$ and \tilde{d} be as in (6). Then the system given by (2) is equivalent to*

$$\begin{cases} \tilde{a}x = \tilde{b} \\ x\tilde{c} = \tilde{d} \end{cases} \tag{7}$$

under the condition that $x \in \mathcal{R}_r(w, v)$.

Proof. According to (4) and using the unique decomposition of \mathcal{R} together with Lemma 2.4, we have

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{8}$$

and

$$x [c_1 \ c_2] = [d_1 \ d_2]. \tag{9}$$

So, we get the equivalence between (2) and the system of equations (8) and (9). \square

The following result is actually Theorem 3.11 in [5], and we give the proof for completeness.

Theorem 3.2. *Let $a, b, c, d \in \mathcal{R}$ be such that a and c are regular elements. Then the following statements are equivalent:*

1. *There exist a solution $x \in \mathcal{R}$ of the system of equations (2);*
2. *$bc = ad$, $b = aa^-b$, and $d = dc^-c$.*

Moreover, if (1) or (2) is satisfied, then any solution of (2) can be expressed as

$$x = a^-b + (1 - a^-a)dc^- + (1 - a^-a)f(1 - cc^-), \quad f \in \mathcal{R} \tag{10}$$

for a^-, c^- inner inverses of a and c respectively.

Proof. (1) \Rightarrow (2) : Let x be a solution of the system (2), then $bc = axc = ad$, and for $a^- \in a\{1\}$ and $c^- \in c\{1\}$ we have $b = ax = aa^-ax = aa^-b$ and $d = xc = xcc^-c = dc^-c$.

(2) \Rightarrow (1) : It is easy to see that any element in \mathcal{R} of the form (10) is a solution of the system of equations (2).

On the other hand, let $x \in \mathcal{R}$ be a solution of (2). Then

$$\begin{aligned} x &= a^-b + (x - a^-b) \\ &= a^-b + (x - a^-ax) \\ &= a^-b + (1 - a^-a)x \\ &= a^-b + (1 - a^-a)(x - dc^- + dc^-) \\ &= a^-b + (1 - a^-a)dc^- + (1 - a^-a)(x - dc^-) \\ &= a^-b + (1 - a^-a)dc^- + (1 - a^-a)(x - xcc^-) \\ &= a^-b + (1 - a^-a)dc^- + (1 - a^-a)x(1 - cc^-). \end{aligned}$$

So, we get the result. \square

Now, we prove the main result of this paper.

Theorem 3.3. Let $a, b, c, d \in \mathcal{R}$ and let (4) hold such that a_1, a_2, c_1 and c_2 are regular elements. Then the following statements are equivalent:

1. There exists a solution $x \in \mathcal{R}_r(w, v)$ of the system of equations (2);
2. $b_1c_1 = a_1d_1, b_1 = a_1\hat{a}_1b_1, d_1 = d_1\hat{c}_1c_1, b_2c_2 = a_2d_2, b_2 = a_2\hat{a}_2b_2,$ and $d_2 = d_2\hat{c}_2c_2.$

Moreover, if (1) or (2) is satisfied, then any generalized reflexive solution of (2) can be expressed as

$$x = x_0 + efg, \quad \text{for } f \in \mathcal{R}_r(w, v), \tag{11}$$

where

$$x_0 = \hat{a}_1b_1 + (1 - \hat{a}_1a_1)d_1\hat{c}_1 + \hat{a}_2b_2 + (1 - \hat{a}_2a_2)d_2\hat{c}_2, \tag{12}$$

$$e = 1 - \hat{a}_1a_1 - \hat{a}_2a_2 \text{ and } g = 1 - c_1\hat{c}_1 - c_2\hat{c}_2, \tag{13}$$

$$\text{for } \hat{a}_1, \hat{a}_2, \hat{c}_1, \hat{c}_2 \text{ given by (5)} \tag{13}$$

Proof. (1) \Rightarrow (2) : Suppose that system (2) has a solution in $\mathcal{R}_r(w, v)$. By Lemma 3.1, it is equivalent with the fact that the system (7) has a reflexive solution $x \in \mathcal{R}_r(w, v) \subseteq \mathcal{R}$. Because of Theorem 3.2 we have

$$\tilde{b}\tilde{c} = \tilde{a}\tilde{d}, \quad \tilde{b} = \tilde{a}\tilde{a}^-\tilde{b} \text{ and } \tilde{d} = \tilde{d}\tilde{c}^-\tilde{c}. \tag{14}$$

Now, because of (4), (6), Lemma 2.3, and using (14) we obtain

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} [c_1, c_2] = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} [d_1, d_2],$$

that is

$$\begin{bmatrix} b_1c_1 & 0 \\ 0 & b_2c_2 \end{bmatrix} = \begin{bmatrix} a_1d_1 & 0 \\ 0 & a_2d_2 \end{bmatrix}.$$

So,

$$b_1c_1 = a_1d_1 \text{ and } b_2c_2 = a_2d_2.$$

Again, because of Lemma 2.3 and (14), we have

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} [\hat{a}_1\hat{a}_2] \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

or

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1\hat{a}_1 & 0 \\ 0 & a_2\hat{a}_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1\hat{a}_1b_1 \\ a_2\hat{a}_2b_2 \end{bmatrix}$$

that is

$$b_1 = a_1\hat{a}_1b_1 \text{ and } b_2 = a_2\hat{a}_2b_2.$$

In the same way, using the third equality in (14) one can obtain the equations

$$d_1 = d_1\hat{c}_1c_1 \text{ and } d_2 = d_2\hat{c}_2c_2.$$

(2) \Rightarrow (1) : Supposing (2), and using Theorem 3.2 we prove the solvability of the system (7) in \mathcal{R} . Moreover, any solution has the form

$$x = x_0 + (1 - \tilde{a}^-\tilde{a})f(1 - \tilde{c}\tilde{c}^-), \tag{15}$$

where

$$x_0 = \tilde{a}^- \tilde{b} + (1 - \tilde{a}^- \tilde{a}) \tilde{d} \tilde{c}^- \in \mathcal{R}_r(w, v), \text{ and } f \in \mathcal{R}_r(w, v).$$

Then we have

$$\begin{aligned} x_0 &= [\hat{a}_1 \quad \hat{a}_2] \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \left(1 - [\hat{a}_1 \quad \hat{a}_2] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) [d_1 \quad d_2] \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \end{bmatrix} \\ &= \hat{a}_1 b_1 + (1 - \hat{a}_1 a_1) d_1 \hat{c}_1 + \hat{a}_2 b_2 + (1 - \hat{a}_2 a_2) d_2 \hat{c}_2, \end{aligned}$$

and $x_0 \in \mathcal{R}_r(w, v)$.

Now, from (14) it is easy to check that x_0 is a particular solution of (7) in $\mathcal{R}_r(w, v)$. From Lemma 3.1, x_0 is also a solution of (2) in $\mathcal{R}_r(w, v)$.

Further on, we will prove that (11) is a general solution if (1) or (2) is satisfied. We have

$$e = 1 - \tilde{a}^- \tilde{a} = 1 - \hat{a}_1 a_1 - \hat{a}_2 a_2 \in \mathcal{R}_r(w)$$

and

$$g = 1 - \tilde{c} \tilde{c}^- = 1 - c_1 \hat{c}_1 - c_2 \hat{c}_2 \in \mathcal{R}_r(v)$$

According to Lemma 2.7, $x = x_0 + efg \in \mathcal{R}_r(w, v)$ for $f \in \mathcal{R}_r(w, v)$. Hence, because of Lemma 3.1, a general solution can be expressed as (11). \square

Now we look for a generalized (w, v) -anti-reflexive solution of the system (2). Let us denote

$$\tilde{b} = \begin{bmatrix} b_2 \\ b_1 \end{bmatrix}, \quad \tilde{d} = [d_2 \quad d_1] \tag{16}$$

Lemma 3.4. *Let $a, b, c, d \in \mathcal{R}$ be decomposed as in (4) and \tilde{a}, \tilde{c} be as in (6). If \tilde{b} and \tilde{d} are given by (16), then the system (2) is equivalent with*

$$\begin{cases} \tilde{a}x = \tilde{b} \\ x\tilde{c} = \tilde{d} \end{cases} \tag{17}$$

under the condition that $x \in \mathcal{R}_{ar}(w, v)$.

Proof. According to (4) and the unique decomposition of \mathcal{R} , using Lemma 2.4 we get the equivalence. \square

Theorem 3.5. *Let $a, b, c, d \in \mathcal{R}$ and let (4) hold such that a_1, a_2, c_1, c_2 are regular elements. Then the following statements are equivalent:*

1. *There exists a solution $x \in \mathcal{R}_{ar}(w, v)$ of the system (2).*
2. *$b_2 c_2 = a_1 d_1$, $b_2 = a_1 \hat{a}_1 b_2$, $d_2 = d_2 \hat{c}_1 c_1$, $b_1 c_1 = a_2 d_2$, $b_1 = a_2 \hat{a}_2 b_1$ and $d_1 = d_1 \hat{c}_2 c_2$.*

Moreover, if (1) or (2) is satisfied, then any generalized anti-reflexive solution of (2) can be expressed as

$$x = x_0 + efg, \text{ for } f \in \mathcal{R}_{ar}(w, v), \tag{18}$$

where

$$x_0 = \hat{a}_1 b_2 + (1 - \hat{a}_1 a_1) d_1 \hat{c}_2 + \hat{a}_2 b_1 + (1 - \hat{a}_2 a_2) d_2 \hat{c}_1,$$

$$e = 1 - \hat{a}_1 a_1 - \hat{a}_2 a_2 \text{ and } g = 1 - c_1 \hat{c}_1 - c_2 \hat{c}_2 \text{ for } \hat{a}_1, \hat{a}_2, \hat{c}_1, \hat{c}_2 \text{ given by (5).}$$

Proof. Using the same technic as in Theorem 3.3 we get the result. \square

The linear matrix equations

$$ax = b \tag{19}$$

and

$$xc = d, \tag{20}$$

have been considered by many authors, and some specific solutions are investigated. Here we want to find the generalized (w, v) -reflexive and anti-reflexive solution.

As special cases, we get the general solutions of the equations (19) and (20).

Theorem 3.6. *Let a, b be decomposed as in (4), such that a_1, a_2 are regular elements. The equation (19) has a generalized (w, v) -reflexive solution if and only if $a_1\hat{a}_1b_1 = b_1$ and $a_2\hat{a}_2b_2 = b_2$.*

Then any generalized reflexive solution x is given by

$$x = \hat{a}_1b_1 + \hat{a}_2b_2 + (1 - \hat{a}_1a_1 - \hat{a}_2a_2)u, \quad u \in \mathcal{R}_r(w, v) \text{ and } \hat{a}_1, \hat{a}_2 \text{ are given by (5).}$$

Proof. Using decomposition (4) for a and b , since $a_1\hat{a}_1b_1 = b_1$ and $a_2\hat{a}_2b_2 = b_2$ and for $x_0 = \hat{a}_1b_1 + \hat{a}_2b_2$, we have that $ax_0 = (a_1 + a_2)(\hat{a}_1b_1 + \hat{a}_2b_2) = b_1 + b_2 = b$. That is, x_0 is a particular (w, v) -generalized reflexive solution of (19).

Now, let x_0 be a generalized (w, v) -reflexive solution of the equation (19). That is $a_1x_0 + a_2x_0 = b_1 + b_2$, and because of Lemma 2.1, Lemma 2.3 and Lemma 2.4

$$b_1 = a_1x_0 = a_1\hat{a}_1a_1x_0 = a_1\hat{a}_1b_1 \text{ and } b_2 = a_2x_0 = a_2\hat{a}_2a_2x_0 = a_2\hat{a}_2b_2.$$

It is easy to check that

$$x = \hat{a}_1b_1 + \hat{a}_2b_2 + (1 - \hat{a}_1a_1 - \hat{a}_2a_2)u, \quad u \in \mathcal{R}_r(w, v)$$

is a solution of the equation (19). On the other hand, let $x \in \mathcal{R}$ be a solution of (19). Then

$$\begin{aligned} x &= (\hat{a}_1b_1 + \hat{a}_2b_2) + (x - \hat{a}_1b_1 - \hat{a}_2b_2) \\ &= x_0 + x - \hat{a}_1a_1x - \hat{a}_2a_2x \\ &= x_0 + (1 - \hat{a}_1a_1 - \hat{a}_2a_2)x, \end{aligned}$$

□

The general solution in the preceding theorem under the unique correspondence

$$x \leftrightarrow \begin{bmatrix} pxq & px(1-q) \\ (1-p)xq & (1-p)x(1-q) \end{bmatrix}$$

can be written in the form

$$x \leftrightarrow \begin{bmatrix} p(a_1^-b_1 + (1 - a_1^-a_1)u_1)q & 0 \\ 0 & (1-p)(a_2^-b_2 + (1 - a_2^-a_2)u_2)(1-q) \end{bmatrix} \tag{21}$$

Theorem 3.7. *Let c, d be decomposed as in (4), such that c_1, c_2 are regular elements. Then the equation (20) has a generalized (w, v) -reflexive solution if and only if $d_1 = d_1\hat{c}_1c_1$ and $d_2 = d_2\hat{c}_2c_2$. Then the generalized reflexive solution $x \in \mathcal{R}_r(w, v)$ is given by*

$$x = d_1\hat{c}_1 + d_2\hat{c}_2 + u(1 - c_1\hat{c}_1 - c_2\hat{c}_2), \text{ for } u \in \mathcal{R}_r(w, v).$$

Proof. Taking involution and using Theorem 3.6 we get the result. □

Theorem 3.8. Let a, b be decomposed as in (4), such that a_1, a_2 are regular elements. The equation (19) has a generalized (w, v) -anti-reflexive solution if and only if $a_1 \hat{a}_1 b_2 = b_2$ and $a_2 \hat{a}_2 b_1 = b_1$. Then any generalized (w, v) -anti-reflexive solution x is given by

$$x = \hat{a}_1 b_2 + \hat{a}_2 b_1 + (1 - \hat{a}_1 a_1 - \hat{a}_2 a_2)u \text{ for } v \in \mathcal{R}_{ar}(w, v).$$

The matrix form of the generalized anti-reflexive solution in the preceding theorem is given by

$$x = \begin{bmatrix} 0 & pa_1^- b_2 + p(1 - a_2^- a_2)u_1(1 - q) \\ (1 - p)a_2^- b_1 + (1 - p)(1 - a_2^- a_2)u_2q & 0 \end{bmatrix} \quad (22)$$

Analogously we can get the generalized (w, v) anti-reflexive solution for the equation $xc = d$.

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