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Coefficient Estimates for Certain Subfamilies of Close-to-convex Functions of Complex Order

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Abstract. Motivated from the recent work of Srivastava et al. (H.M. Srivastava, O. Altıntaş, S. K. Serenbay, Coefficient bounds for certain subclasses of starlike functions of complex order, Appl. Math. Lett. 24(2011)1359-1363.), we aim to determine the coefficient estimates for functions in certain subclasses of close-to-convex and related functions of complex order, which are here defined by means of Cauchy-Euler type non-homogeneous differential equation. Several interesting consequences of our results are also observed.

1. Introduction

We denote by $\mathcal{A}$ the class of functions $f(z)$ which are analytic in the open unit disc $E = \{z : |z| < 1\}$ and of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

Let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent. Also let $\mathcal{S}^*_\gamma$, C_γ, $\mathcal{K}_\gamma$ and $Q_\gamma$be the subclasses of $\mathcal{A}$ consisting of all functions which are starlike, convex, close-to-convex and quasi convex of complex order γ $(\gamma \neq 0)$ respectively, for details see [3–5]. We note that for $0 < \gamma \leq 1$, these classes coincide with the well known classes of starlike, convex and close-to-convex of order $1-\gamma$. Recently Altintaş et al.[1] considered the following class of functions denoted by $\mathcal{SC}(\gamma, \lambda, \beta)$ and defined as:

$$\mathcal{SC}(\gamma, \lambda, \beta) = \left\{ f(z) \in \mathcal{A} : Re\left[1 + \frac{1}{\gamma}\left(\frac{z[(1-\lambda)f(z) + \lambda z f'(z)]'}{(1-\lambda)f(z) + \lambda z f'(z)} - 1\right)\right] > \beta,\ z \in E \right\}, \tag{1.1}$$

where $0 \leq \beta < 1, 0 \leq \lambda \leq 1,\ \gamma \in \mathbb{C} - \{0\}$. Note that $\mathcal{SC}(\gamma, 0, 0) = \mathcal{S}^*_\gamma$, $\mathcal{SC}(\gamma, 1, 0) = C_\gamma$. Throughout the entire paper onward we assume the restrictions $0 \leq \beta < 1, 0 \leq \lambda \leq 1, \gamma \in \mathbb{C} - \{0\}$ unless otherwise mentioned.

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Now we denote $\mathcal{KQ}(\gamma,\lambda,\beta)$ be the class of functions $f(z)\in\mathcal{A}$ if there exists a function $g(z)\in\mathcal{SC}(1,\lambda,\beta)$ such that

$$\Re\left[1+\frac{1}{\gamma}\left(\frac{z[(1-\lambda)f(z)+\lambda zf'(z)]'}{(1-\lambda)g(z)+\lambda zg'(z)}-1\right)\right]>\beta,\ z\in E. \tag{1.2}$$

As special choices we have the following relationships

$$\mathcal{KQ}(\gamma,0,0)=\mathcal{K}_\gamma,\ \ \mathcal{KQ}(\gamma,1,0)=\mathcal{Q}_\gamma.$$

Motivated from the recent work of Altintaş et al. [2] and Srivastava et al. [7] the main purpose of our investigation is to derive coefficient estimates of a subfamily $\mathcal{BK}(\gamma,\lambda,\beta;\mu)$ of $\mathcal{A}$, which consists of functions $f(z)$ in $\mathcal{A}$ satisfying the following Cauchy Euler type non homogenous differential equation

$$z^2\frac{d^2w}{dz^2}+2(1+\mu)z\frac{dw}{dz}+\mu(1+\mu)w=(1+\mu)(2+\mu)h(z), \tag{1.3}$$

where $w=f(z)$, $h(z)\in\mathcal{KQ}(\gamma,\lambda,\beta)$, $\mu\in\mathbb{R}-(-\infty,-1]$, for details we refer to [2, 6–9].

The following result which is due to Altintaş et al. [2] is essential in deriving our main results.

Lemma 1 [2].

Let $f(z)\in\mathcal{SC}(\gamma,\lambda,\beta)$ *and be of the form* (1.1). *Then*

$$|a_n|\le\frac{\prod_{j=0}^{n-2}\left[j+2\left|\gamma\right|(1-\beta)\right]}{(n-1)![1+\lambda(n-1)]},\ n\in N^*=\{2,3,4,\ldots\}.$$

2. Coefficient Estimates for Functions in the Class $\mathcal{KQ}(\gamma,\lambda,\beta)$

We first establish the below result for the functions in the class $\mathcal{KQ}(\gamma,\lambda,\beta)$.

Theorem 1. *Let* $f(z)\in\mathcal{KQ}(\gamma,\lambda,\beta)$ *and be defined by* (1.1). *Then*

$$|a_n|\le\frac{\prod_{j=0}^{n-2}[j+2(1-\beta)]}{n![1+\lambda(n-1)]}+\frac{2\left|\gamma\right|(1-\beta)}{n[1+\lambda(n-1)]}\sum_{k=1}^{n-1}\frac{\prod_{j=0}^{n-k-2}[j+2(1-\beta)]}{(n-k-1)!},\ n\in N^*=\{2,3,4,\ldots\}. \tag{2.1}$$

Proof. Since $f(z)\in\mathcal{KQ}(\gamma,\lambda,\beta)$, then there exists $g(z)=z+\sum_{n=2}^\infty b_nz^n$ belonging to the class $\mathcal{SC}(1,\lambda,\beta)$ such that

$$\Re\left[1+\frac{1}{\gamma}\left(\frac{zF'(z)}{G(z)}-1\right)\right]>\beta,\ \text{for } z\in E,$$

where $F(z)=z+\sum_{n=2}^\infty A_nz^n$ and $G(z)=z+\sum_{n=2}^\infty B_nz^n$, with

$$A_n=[1+\lambda(n-1)]a_n,\ B_n=[1+\lambda(n-1)]b_n,\quad n\ge2. \tag{2.2}$$

Let

$$\frac{1+\frac{1}{\gamma}\left(\frac{zF'(z)}{G(z)}-1\right)-\beta}{1-\beta}=q(z)=1+\sum_{n=1}^\infty c_nz^n,\ \text{for } z\in E. \tag{2.3}$$

Since $\Re q(z)>0$, $z\in E$, we find that

$$|c_n|\le2,\ n\in\mathbb{N}. \tag{2.4}$$

Then from (2.3), we obtain

$$nA_n=B_n+\gamma(1-\beta)\left[c_{n-1}+\sum_{k=1}^{n-2}c_kB_{n-k}\right],\ n\ge2\,,$$

Now using Lemma 1 together with (2.2) and (2.4), we have

$$|A_n| \leq \frac{\prod_{j=0}^{n-2}[j+2(1-\beta)]}{n!} + \frac{2|\gamma|(1-\beta)}{n}\sum_{k=1}^{n-1}\frac{\prod_{j=0}^{n-k-2}[j+2(1-\beta)]}{(n-k-1)!},$$

and hence from the relation between $F(z)$ and $f(z)$ as in (2.2), we obtain the desired result. ■

By assigning different specific values to the involved parameters $\beta,\ \gamma,\ \lambda$ in Theorem 1, we deduce the following interesting results.

Corollary 1. *Let* $f(z) \in \mathcal{KQ}(\gamma, \lambda, 0)$ *and be defined by* (1.1). *Then*

$$|a_n| \leq \frac{1}{[1+\lambda(n-1)]}\left[1+(n-1)|\gamma|\right],\ n \in N^* = \{2,3,4,\ldots\}.$$

Corollary 2 [3]. *Let* $f(z) \in \mathcal{KQ}(\gamma, 0, 0) = \mathcal{K}(\gamma)$ *and be defined by* (1.1). *Then*

$$|a_n| \leq 1+(n-1)|\gamma|,\ n \in N^* = \{2,3,4,\ldots\}.$$

Corollary 3 [4]. *Let* $f(z) \in \mathcal{KQ}(\gamma, 1, 0) = \mathcal{Q}(\gamma)$ *and be defined by* (1.1). *Then for* $n \in N^* = \{2,3,4,\ldots\}$.

$$|a_n| \leq \frac{1+(n-1)|\gamma|}{n},\ n \in N^* = \{2,3,4,\ldots\}.$$

For $\gamma = 1$ in Corollary 2 and Corollary 3, we obtain the well-known coefficient estimates for close-to-convex and quasi convex functions.

Corollary 4. *Let* $f(z) \in \mathcal{KQ}(1-\alpha, \lambda, \beta)$ *and be defined by* (1.1). *Then for* $n \in N^* = \{2,3,4,\ldots\}$

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}[j+2(1-\beta)]}{n![1+\lambda(n-1)]} + \frac{2(1-\alpha)(1-\beta)}{n[1+\lambda(n-1)]}\sum_{k=1}^{n-1}\frac{\prod_{j=0}^{n-k-2}[j+2(1-\beta)]}{(n-k-1)!}.$$

Corollary 5. *Let* $f(z) \in \mathcal{KQ}(1-\alpha, 0, 0) = \mathcal{K}(1-\alpha)$ *and be defined by* (1.1). *Then*

$$|a_n| \leq n(1-\alpha)+\alpha,\ \ n \in N^* = \{2,3,4,\ldots\}.$$

Corollary 6. *Let* $f(z) \in \mathcal{KQ}(1-\alpha, 1, 0) = \mathcal{Q}(1-\alpha)$ *and be defined by* (1.1). *Then*

$$|a_n| \leq 1-\alpha+\frac{\alpha}{n},\ n \in N^* = \{2,3,4,\ldots\}.$$

3. Coefficient Estimates of the Class $\mathcal{BK}(\gamma, \lambda, \beta; \mu)$

The theorem below is our main coefficient estimates for functions in the class $\mathcal{BK}(\gamma, \lambda, \beta; \mu)$.

Theorem 2. *Let* $f(z) \in \mathcal{BK}(\gamma, \lambda, \beta)$ *and be defined by* (1.1). *Then for* $n \in N^* = \{2,3,4,\ldots\}$

$$|a_n| \leq \frac{(1+\mu)(2+\mu)}{(n+1+\mu)(n+\mu)}\left[\frac{\prod_{j=0}^{n-2}[j+2(1-\beta)]}{n![1+\lambda(n-1)]} + \frac{2|\gamma|(1-\beta)}{n[1+\lambda(n-1)]}\sum_{k=1}^{n-1}\frac{\prod_{j=0}^{n-k-2}[j+2(1-\beta)]}{(n-k-1)!}\right]. \tag{3.1}$$

Proof. Since $f(z) \in \mathcal{BK}(\gamma, \lambda, \beta; \mu)$, then there exist $h(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{KQ}(\gamma, \lambda, \beta)$, such that (1.3) holds true. Thus it follows that

$$a_n = \frac{(1+\mu)(2+\mu)}{(n+1+\mu)(n+\mu)}b_n,\quad n \in N^*,\ \mu \in \mathbb{R}-(-\infty,-1].$$

Hence, by using Theorem 1, we immediately obtain the desired inequality (3.1). ■

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