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Stability of Essential Approximate Point Spectrum and Essential Defect Spectrum of Linear Operator

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Abstract. In the present paper, we use the notion of measure of noncompactness to give some results on Fredholm operators and we establish a fine description of the essential approximate point spectrum and the essential defect spectrum of a closed densely defined linear operator.

1. Introduction

Let X and Y be two infinite-dimensional Banach spaces. By an operator A from X to Y we mean a linear operator with domain $\mathcal{D}(A) \subset X$ and range $R(A) \subset Y$. We denote by $C(X, Y)$ (resp. $\mathcal{L}(X, Y)$) the set of all closed, densely defined linear operators (resp. the Banach algebra of all bounded linear operators) from X into Y and we denote by $\mathcal{K}(X, Y)$ the subspace of all compact operators from X into Y. We denote by $\sigma(A)$ and $\rho(A)$ respectively the spectrum and the resolvent set of A. The nullity, $\alpha(A)$, of A is defined as the dimension of $N(A)$ and the deficiency, $\beta(A)$, of A is defined as the codimension of $R(A)$ in Y. The set of upper semi-Fredholm operators is defined by

$$\Phi_+(X, Y) = \Big\{A \in C(X, Y) \text{ such that } \alpha(A) < \infty, \ \ R(A) \text{ is closed in } Y\Big\}.$$

and the set of lower semi-Fredholm operators is defined by

$$\Phi_-(X, Y) = \Big\{A \in C(X, Y) \text{ such that } \beta(A) < \infty, \ \ R(A) \text{ is closed in } Y\Big\}.$$

The set of Fredholm operators from X into Y is defined by

$$\Phi(X, Y) = \Phi_+(X, Y) \cap \Phi_-(X, Y).$$

The set of bounded upper (resp. lower) semi-Fredholm operator from X into Y is defined by

$$\Phi^b_+(X, Y) = \Phi_+(X, Y) \cap \mathcal{L}(X, Y) \quad \Big(\text{resp. } \Phi_-(X, Y) \cap \mathcal{L}(X, Y)\Big).$$

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We denote by $\Phi^b(X,Y) = \Phi(X,Y) \cap \mathcal{L}(X,Y)$ the set of bounded Fredholm operators from X into Y. If A is semi-Fredholm operator (either upper or lower) the index of A, is defined by $i(A) = \alpha(A) - \beta(A)$. It is clear that if $A \in \Phi(X,Y)$ then $-\infty < i(A) < \infty$. If $A \in \Phi_+(X,Y) \setminus \Phi(X,Y)$ then $i(A) = -\infty$ and if $A \in \Phi_-(X,Y) \setminus \Phi(X,Y)$ then $i(A) = +\infty$. If $X = Y$ then $\mathcal{L}(X,Y)$, $\mathcal{C}(X,Y)$, $\mathcal{K}(X,Y)$, $\Phi(X,Y)$, $\Phi_+(X,Y)$ and $\Phi_-(X,Y)$ are replaced by $\mathcal{L}(X)$, $\mathcal{C}(X)$, $\mathcal{K}(X)$, $\Phi(X)$, $\Phi_+(X)$ and $\Phi_-(X)$ respectively.

There are several, and in general, non-equivalent definitions of the essential spectrum of a closed operator on a Banach space. In this paper, we are concerned with the following essential spectra:

Definition 1.1. *Let $A \in \mathcal{C}(X)$. We define the essential spectrum of A, by*

$$\begin{array}{rcl} \sigma_{eg}(A) & = & \mathbb{C}\backslash\rho^+_{ess}(A), \\ \sigma_{ew}(A) & = & \mathbb{C}\backslash\rho^-_{ess}(A), \\ \sigma_{ess}(A) & = & \mathbb{C}\backslash\rho_{ess}(A), \\ \sigma_{eap}(A) & = & \bigcap\limits_{K\in\mathcal{K}(X)} \sigma_{ap}(A+K), \end{array}$$

and

$$\sigma_{e\delta}(A) = \bigcap_{K\in\mathcal{K}(X)} \sigma_\delta(A+K),$$

where

$$\begin{array}{rcl} \rho^+_{ess}(A) & = & \Big\{\lambda \in \mathbb{C} \ \text{ such that } \ \lambda - A \in \Phi_+(X)\Big\}, \\ \rho^-_{ess}(A) & = & \Big\{\lambda \in \mathbb{C} \ \text{ such that } \ \lambda - A \in \Phi_-(X)\Big\}, \\ \rho_{ess}(A) & = & \Big\{\lambda \in \mathbb{C} \ \text{ such that } \ \lambda - A \in \Phi(X)\Big\}, \\ \sigma_{ap}(A) & = & \Big\{\lambda \in \mathbb{C} \ \text{ such that } \inf\limits_{\|x\|=1, x\in\mathcal{D}(A)} \|(\lambda - A)x\| = 0\Big\}, \end{array}$$

and

$$\sigma_\delta(A) = \Big\{\lambda \in \mathbb{C} \text{ such that } \lambda - A \ \text{ is not surjective}\Big\}.$$

The subsets $\sigma_{eap}(.)$ was introduced by V. Rakočević in [11] and designates the essential approximate point spectrum, $\sigma_{e\delta}(.)$ is the essential defect spectrum and was introduced by C. Shmoeger in [15], $\sigma_{eg}(.)$ and $\sigma_{ew}(.)$ are the Gustafson and Weidmann essential spectra [4]. Note that, in general, we have

$$\sigma_{ess}(A) = \sigma_{eap}(A) \cup \sigma_{e\delta}(A), \ \sigma_{eg}(A) \subset \sigma_{eap}(A) \text{ and } \sigma_{ew}(A) \subset \sigma_{e\delta}(A).$$

It is well known that if a self-adjoint operator in a Hilbert space, there seems to be only one reasonable way to define the essential spectrum: the set of all points of the spectrum that are not isolated eigenvalues of finite algebraic multiplicity [8, 17, 18]. Note that all these sets are closed and if X is a Hilbert space and A is a self-adjoint operator on X, then all these sets coincide.

Definition 1.2. *Let $F \in \mathcal{L}(X,Y)$. F is called Fredholm perturbation if $A + F \in \Phi(X,Y)$ whenever $A \in \Phi(X,Y)$. F is called an upper (resp. lower) Fredholm perturbation if $A + F \in \Phi_+(X,Y)$ $\big($resp. $\Phi_-(X,Y)\big)$ whenever $A \in \Phi_+(X,Y)$ $\big($resp. $\Phi_-(X,Y)\big)$.*
The sets of Fredholm, upper semi-Fredholm and lower semi-Fredholm perturbations are denoted by $\mathcal{F}(X,Y)$, $\mathcal{F}_+(X,Y)$ and $\mathcal{F}_-(X,Y)$, respectively.

If $X = Y$ then $\mathcal{F}(X) := \mathcal{F}(X,X)$, $\mathcal{F}_+(X) := \mathcal{F}_+(X,X)$ and $\mathcal{F}_-(X) := \mathcal{F}_-(X,X)$.
We would like to mention that the study of the sets of Fredholm perturbations starts with the investigations of I. C. Gohberg, A. S. Markus and I. A. Feldman in [5]. In particular, it is shown that $\mathcal{F}(X)$ is closed two-sided ideal of $\mathcal{L}(X)$.

Definition 1.3. [3, Definition 1.2] *An operator $A \in \mathcal{L}(X)$ is said to be polynomially Fredholm perturbation if there exists a nonzero complex polynomial P such that $P(A)$ is a Fredholm perturbation. We denote by $\mathcal{PF}(X)$ the set of polynomially perturbation operators defined by*

$$\mathcal{PF}(X) : \;=\; \Big\{A \in \mathcal{L}(X) \textit{ such that there exists a nonzero complex polynomial } P(z) = \sum_{n=0}^{p} a_n z^n \textit{ satisfing } P(z) \in \mathcal{F}(X)\Big\}.$$

Recently, A. Dehici and N. Boussetila [3, Theorem 2.1] showed that if A is a Riesz operator on X then A is polynomially Fredholm perturbation if and only if A^n is a Fredholm perturbation for some $n \in \mathbb{N}$. Besides, they have proved the implication $A \in \mathcal{L}(X)$ is polynomially Fredholm perturbation.

Lemma 1.4. [6, Lemma 2.1] *Let $A \in \mathcal{C}(X, Y)$ and $F \in \mathcal{L}(X, Y)$. Then*

(i) If $A \in \Phi(X, Y)$ and $F \in \mathcal{F}(X, Y)$, then $A + F \in \Phi(X, Y)$ and $i(A + F) = i(A)$.

(ii) If $A \in \Phi_+(X, Y)$ and $F \in \mathcal{F}_+(X, Y)$, then $A + F \in \Phi_+(X, Y)$ and $i(A + F) = i(A)$.

(iii) If $A \in \Phi_-(X, Y)$ and $F \in \mathcal{F}_-(X, Y)$, then $A + F \in \Phi_-(X, Y)$ and $i(A + F) = i(A)$.

The following proposition gives a characterization of the essential approximate point spectrum and the essential defect spectrum by means of upper semi-Fredholm and lower semi-Fredholm operators respectively.

Proposition 1.5. [6, Proposition 3.1] *Let $A \in \mathcal{C}(X)$. Then:*

(i) $\lambda \notin \sigma_{eap}(A)$ if and only if $\lambda - A \in \Phi_+(X)$ and $i(\lambda - A) \leq 0$.

(ii) $\lambda \notin \sigma_{e\delta}(A)$ if and only if $\lambda - A \in \Phi_-(X)$ and $i(\lambda - A) \geq 0$.

(iii) If A is a bounded linear operator, then $\sigma_{e\delta}(A) = \sigma_{eap}(A^*)$, where A^* stands for the adjoint operator.

This is equivalent to say that

$$\sigma_{eap}(A) = \sigma_{eg}(A) \cup \Big\{\lambda \in \mathbb{C} \textit{ such that } i(A - \lambda) > 0\Big\},$$

and

$$\sigma_{e\delta}(A) = \sigma_{ew}(A) \cup \Big\{\lambda \in \mathbb{C} \textit{ such that } i(A - \lambda) < 0\Big\}.$$

If, in addition, $\rho_{ess}(A)$ is connected and $\rho(A) \neq \emptyset$, then

$$\sigma_{eg}(A) = \sigma_{eap}(A)$$

and

$$\sigma_{ew}(A) = \sigma_{e\delta}(A).$$

Let $A \in \mathcal{C}(X)$. It follows from the closedness of A that $\mathcal{D}(A)$ endowed with the graph norm $\|.\|_A$ is a Banach space denoted by X_A, where

$$\|x\|_A = \|x\| + \|Ax\|, \;\; x \in \mathcal{D}(A).$$

Clearly, for $x \in \mathcal{D}(A)$ we have $\|Ax\| \leq \|x\|_A$, so $A \in \mathcal{L}(X_A, X)$. If B be a linear operator with $\mathcal{D}(A) \subseteq \mathcal{D}(B)$, then B is said to A-defined. The restriction of B to $\mathcal{D}(A)$ will be denoted by $\hat{B}$. Let B be an arbitrary A-bounded operator, hence we can regard A and B as operators from X_A into X, they will be denoted by $\hat{A}$ and $\hat{B}$ respectively, these belong to $\mathcal{L}(X_A, X)$. Furthermore, we have the obvious relations

$$\begin{cases} \alpha(\hat{A}) = \alpha(A), \;\; \beta(\hat{B}) = \beta(B), \;\;\; R(\hat{A}) = R(A), \\ \alpha(\hat{A} + \hat{B}) = \alpha(A + B), \\ \beta(\hat{A} + \hat{B}) = \beta(A + B) \text{ and } R(\hat{A} + \hat{B}) = R(A + B). \end{cases} \quad (1)$$

The notion of measure of noncompactness turned out to be a useful tool in some problems of topology, functional analysis, and operator theory (see, [2, 8]). In order to recall this notion, consider, for X a Banach space, M_X the family of all nonempty and bounded subsets of X while N_X denotes its subfamily consisting of all relatively compact sets. Moreover, let us denote by $cvx(A)$ the convex hull of a set $A \subset X$. Let us recall the following definition.

Definition 1.6. *A mapping $\gamma : M_X \longrightarrow [0, +\infty[$ is said to be a measure of noncompactness in the space X if it satisfies the following conditions:*

(i) The family $ker(\gamma) := \Big\{D \in M_X \text{ such that } \gamma(D) = 0\Big\}$ is nonempty and $ker(\gamma) \subset N_X$,

for $A,\ B \in M_X$, we have the following:

(ii) If $A \subset B$, then $\gamma(A) \leq \gamma(B)$,

(iii) $\gamma(\bar{A}) = \gamma(A)$,

(iv) $\gamma(\overline{cvx(A)}) = \gamma(A)$,

(v) $\gamma(\lambda A + (1-\lambda)B) \leq \lambda\gamma(A) + (1-\lambda)\gamma(B)$, for all $\lambda \in [0, 1]$,

(vi) If $A \in \mathcal{L}(X)$, $\gamma(A) \leq \|A\|$.

Proposition 1.7. [1, Corollary 2.3] *Let X be a Banach space and $A \in \mathcal{L}(X)$.*
If $\gamma(A^n) < 1$, for some $n > 0$, then $(I - A)$ is a Fredholm operator with $i(I - A) = 0$.

We will denote the set of non negative integers by $\mathbb{N}$ and if $A \in \mathcal{L}(X)$, we define the ascent and the descent of A respectively by:

$$\mathrm{asc}(A) := \min\Big\{n \in \mathbb{N} \text{ such that } N(A^n) = N(A^{n+1})\Big\},$$

and

$$\mathrm{desc}(A) := \min\Big\{n \in \mathbb{N} \text{ such that } R(A^n) = R(A^{n+1})\Big\}.$$

If no such integer exists, we shall say that A has infinite ascent or infinite descent. In [16, Theorem 3.6], A. E. Taylor proved that, if ascent and descent are finite, then $\mathrm{asc}(A) = \mathrm{desc}(A)$.

Remark 1.8. *Let A be a bounded linear operator on a Banach space X. If $A \in \Phi(X)$. with $asc(A)$ and $desc(A)$ are finite. Then $i(A) = 0$.*

Indeed. Since $asc(A)$ and $desc(A)$ are finite. Using [16, Theorem 3.6] *there exists an integer k such that $asc(A) = desc(A) = k$, hence $N(A^k) = N(A^{n+k})$ and $R(A^k) = R(A^{n+k})$ for all $n \in \mathbb{N}$, therefore*

$$i(A^k) = i(A^{n+k}).$$

However, $A \in \Phi(X)$ then by [14, Theorem 5.7] *implies that $i(A^k) = ki(A) = i(A^{n+k}) = (n + k)i(A)$, for all $n \geq 0$. So, $i(A) = 0$.*

Two important classes of operators in Fredholm theory are given by the classes of semi-Fredholm operators which possess finite ascent or finite descent. We shall distinguish two classes of operators. The class of all upper semi-Browder operators on a Banach space X that is defined by:

$$\mathcal{B}_+(X) := \Big\{A \in \Phi_+(X) \text{ such that } \mathrm{asc}(A) < \infty\Big\},$$

and the class of all lower semi-Browder operators that is defined by:

$$\mathcal{B}_-(X) := \Big\{A \in \Phi_-(X) \text{ such that } \mathrm{desc}(A) < \infty\Big\}.$$

The class of all Browder operators (known in the literature also as Riesz-Schauder operators) is defined by:

$$\mathcal{B}(X) := \mathcal{B}_+(X) \cap \mathcal{B}_-(X).$$

Definition 1.9. *Let X and Y be two Banach spaces.*

(i) *An operator $A \in \mathcal{C}(X,Y)$ is said to have a left Fredholm inverse if there are maps $R_l \in \mathcal{L}(Y,X)$ and $F \in \mathcal{F}(X)$ such that $I_X + F$ extends R_lA. The operator R_l is called left Fredholm inverse of A.*

(ii) *An operator $A \in \mathcal{C}(X,Y)$ is said to have a right Fredholm inverse if there are maps $R_r \in \mathcal{L}(Y,X)$ such that $R_r(Y) \subset \mathcal{D}(A)$ and $AR_r - I_Y \in \mathcal{F}(X)$. The operator R_r is called right Fredholm inverse of A.*

The remainder of this work is to characterize the essential approximate point spectrum and the essential defect spectrum. In the first part of this paper we extend the analysis in [7] to bounded linear operator $\gamma(K^n) < 1$ where $\gamma(.)$ is a measure of noncompactness. More precisely, let $A,\ B \in \mathcal{L}(X)$, if $\lambda - A \in \Phi_+(X)$ (resp. $\Phi_-(X)$) and $A_{\lambda r}$ is a left (resp. $A_{\lambda l}$) right Fredholm inverse of $\lambda - A$, such that $\gamma([BA_{\lambda r}]^n) < 1$ (resp. $\gamma([A_{\lambda l}B] < 1)$ and $\|BA_{\lambda r}\| < 1$ (resp. $\|A_{\lambda l}B\| < 1$), then $\sigma_{eap}(A+B) = \sigma_{eap}(A)$ (resp. $\sigma_{e\delta}(A+B) = \sigma_{e\delta}(A)$). In the second part of this paper we derive also useful stability results for the essential approximate point spectrum and the essential defect spectrum. In fact, let A and B be two closed linear operator in $\Phi(X)$. Assume that there are $A_0,\ B_0 \in \mathcal{L}(X)$ and $F_1,\ F_2 \in \mathcal{PF}(X)$ such that $AA_0 = I - F_1$ and $BB_0 = I - F_2$. Then if $0 \in \rho^+_{ess}(A) \cap \rho^+_{ess}(B)$, $A_0 - B_0$ is upper (resp. lower) semi perturbation and $i(A) = i(B)$ then $\sigma_{eap}(A) = \sigma_{eap}(B)$ (resp. $\sigma_{e\delta}(A) = \sigma_{e\delta}(B)$).

2. Stability of Essential Spectra

The purpose of this this Section, we have also the following useful stability of essential spectra.

Theorem 2.1. *Let X be Banach space, A and B be two operators in $\mathcal{L}(X)$. Then*
(i) *Assume that for each $\lambda \in \rho^+_{ess}(A)$, there exists a left Fredholm inverse $A_{\lambda l}$ of $\lambda - A$ such that $\|BA_{\lambda l}\| < 1$, then*

$$\sigma_{eap}(A+B) = \sigma_{eap}(A).$$

(ii) *Assume that for each $\lambda \in \rho^-_{ess}(A)$, there exists a right Fredholm inverse $A_{\lambda r}$ of $\lambda - A$ such that $\|A_{\lambda r}B\| < 1$, then*

$$\sigma_{e\delta}(A+B) = \sigma_{e\delta}(A).$$

Proof. Let $\mathcal{P}_\gamma(X) = \Big\{A \in \mathcal{L}(X) \text{ such that } \gamma(A^n) < 1, \text{ for some } n > 0\Big\}$. Since $\|BA_{\lambda l}\| < 1$ (resp. $\|A_{\lambda r}B\| < 1$), then $\gamma(BA_{\lambda l}) < 1$ (resp. $\gamma(A_{\lambda r}B) < 1$), so $BA_{\lambda l} \in \mathcal{P}_\gamma(X)$ (resp. $A_{\lambda r}B \in \mathcal{P}_\gamma(X)$). Applying Proposition 1.7 we have

$$I - BA_{\lambda l} \in \Phi^b(X) \text{ and } i(I - BA_{\lambda l}) = 0, \tag{2}$$

and

$$I - A_{\lambda r}B \in \Phi^b(X) \text{ and } i(I - A_{\lambda r}B) = 0. \tag{3}$$

(i) Let $\lambda \notin \sigma_{eap}(A)$ then by Proposition 1.5 (i) we get

$$\lambda - A \in \Phi^b_+(X) \text{ and } i(\lambda - A) \leq 0.$$

As, $A_{\lambda l}$ is a left Fredholm inverse of $\lambda - A$, then there exists $F \in \mathcal{F}(X)$ such that

$$A_{\lambda l}(\lambda - A) = I - F \text{ on } X. \tag{4}$$

By, Eq. (4) the operator $\lambda - A - B$ can be written in the from

$$\lambda - A - B = \lambda - A - B\Big(A_{\lambda l}(\lambda - A) + F\Big) = (I - BA_{\lambda l})(\lambda - A) - BF. \tag{5}$$

According of the Eq. (2) we have $I - BA_{\lambda l} \in \Phi^b_+(X)$, using [9, Theorem 12] we have $(I - BA_{\lambda l})(\lambda - A) \in \Phi_+(X)$

and

$$\begin{aligned} i[(I - BA_{\lambda l})(\lambda - A)] &= i(I - BA_{\lambda l}) + i(\lambda - A) \\ &= i(\lambda - A) \leq 0. \end{aligned}$$

Using Eq. (5) and Lemma 1.4 (*ii*) , we get

$$\lambda - A - B \in \Phi_+^b(X) \text{ and } i(\lambda - A - B) = i(\lambda - A) \leq 0,$$

hence $\lambda \notin \sigma_{eap}(A + B)$. Conversely, let $\lambda \notin \sigma_{eap}(A + B)$ then by Proposition 1.5 (*i*) we have

$$\lambda - A - B \in \Phi_+^b(X) \text{ and } i(\lambda - A - B) \leq 0.$$

Since $\|BA_{\lambda l}\| < 1$, and by Eq. (5) the operator $\lambda - A$ can be written in the from

$$\lambda - A = (I - BA_{\lambda l})^{-1}\Big(\lambda - A - B - BF\Big). \tag{6}$$

Using Lemma 1.4 (*ii*), we get

$$\lambda - A - B - BF \in \Phi_+^b(X)$$

and

$$i(\lambda - A - B - BF) = i(\lambda - A - B) \leq 0,$$

since $I - BA_{\lambda l}$ is boundedly invertible, then by Eq. (6), we have

$$\lambda - A \in \Phi_+^b(X) \text{ and } i(\lambda - A) \leq 0.$$

This proves that $\lambda \notin \sigma_{eap}(A)$. We find

$$\sigma_{eap}(A + B) = \sigma_{eap}(A).$$

(*ii*) Let $\lambda \notin \sigma_{e\delta}(A)$ then according of Proposition 1.5 we get

$$\lambda - A \in \Phi_-^b(X) \text{ and } i(\lambda - A) \geq 0.$$

Since $A_{\lambda r}$ is a right Fredholm inverse of $\lambda - A$, then there exists $F \in \mathcal{F}(X)$ such that

$$(\lambda - A)A_{\lambda r} = I - F \text{ on } X. \tag{7}$$

By, Eq. (7) the operator $\lambda - A - B$ can be written in the from

$$\lambda - A - B = \lambda - A - \Big((\lambda - A)A_{\lambda r} + F\Big)B = (\lambda - A)(I - A_{\lambda r}B) - FB. \tag{8}$$

A similar proof as (*i*), it suffices to replace $\Phi_+^b(.)$, $\sigma_{eap}(.)$, Eq. (5) and Lemma 1.4 (*ii*) by $\Phi_-^b(.)$, $\sigma_{e\delta}(.)$, Eq. (8) and Lemma 1.4 (*iii*) respectively. Hence, we show that

$$\sigma_{e\delta}(A + B) \subset \sigma_{e\delta}(A).$$

Conversely, let $\lambda \notin \sigma_{e\delta}(A + B)$ then by Proposition 1.5 (*ii*)we have

$$\lambda - A - B \in \Phi_-^b(X) \text{ and } i(\lambda - A - B) \geq 0.$$

Since $\|A_{\lambda r}B\| < 1$, and by Eq. (8) the operator $\lambda - A$ can be written in the from

$$\lambda - A = (\lambda - A - B - BF)(I - A_{\lambda r}B)^{-1}. \tag{9}$$

According of the Lemma 1.4 (*iii*), we get

$$\lambda - A - B - FB \in \Phi_-^b(X)$$

and

$$i(\lambda - A - B - FB) = i(\lambda - A - B) \geq 0.$$

So, $I - A_{\lambda r}B$ is boundedly invertible, then by Eq. (9), we have $\lambda - A \in \Phi_-^b(X)$ and $i(\lambda - A) \geq 0$. This proves that $\lambda \notin \sigma_{e\delta}(A)$. We find

$$\sigma_{e\delta}(A + B) = \sigma_{e\delta}(A).$$ □

3. Invariance of Essential Spectra

The purpose of this this Section, we also the following useful stability result for the essential approximate point spectrum and the essential defect spectrum of a closed, densely defined linear operator on a Banach space X. we begin with the following useful result.

Theorem 3.1. *Let $A \in \mathcal{PF}(X)$ i.e., there exists a nonzero complex polynomial $P(z) = \sum_{i=0}^{n} a_i z^i$ satisfying $P(A) \in \mathcal{F}(X)$. Let $\lambda \in \mathbb{C}$ with $P(\lambda) \neq 0$ and set $B = \lambda - A$. Then, B is a Fredholm operator on X with finite ascent and descent.*

To prove Theorem 3.1 we will need the following lemma.

Lemma 3.2. *Let P be a complex polynomial and $\lambda \in \mathbb{C}$ such that $P(\lambda) \neq 0$. Then, for all $A \in \mathcal{L}(X)$ satisfying $P(A) \in \mathcal{F}(X)$, the operator $P(\lambda) - P(A)$ is a Fredholm operator on X with finite ascent and descent.*

Proof. Put $B = P(\lambda) - P(A) = P(\lambda)\left(I - \frac{P(A)}{P(\lambda)}\right) = P(\lambda)(I - F)$ where $F = \frac{P(A)}{P(\lambda)} \in \mathcal{F}(X)$. Let $C = I - F$, then $B = P(\lambda)C$. It is clear that $C + F \in \mathcal{B}(X)$ and, so, we can write $C = C + F - F$ with $C + F \in \mathcal{B}(X)$ and $F \in \mathcal{F}(X)$. On the other hand, we have $(C + F)F = F(C + F)$. By using [12, Theorem 1], we deduce that $C \in \mathcal{B}(X)$ and therefore $B \in \mathcal{B}(X)$. □

Proof of Theorem 3.1 Let $\lambda \in \mathbb{C}$ with $P(\lambda) \neq 0$. We have:

$$P(\lambda) - P(A) = \sum_{i=1}^{n} a_i(\lambda^i - A^i).$$

On the other hand, for any $i \in \left\{1, ..., n\right\}$, we have

$$\lambda^i - A^i = (\lambda - A)\sum_{j=0}^{k-1} \lambda^j A^{k-1-j}.$$

So,

$$P(\lambda) - P(A) = (\lambda - A)Q(A) = Q(A)(\lambda - A), \tag{10}$$

where

$$Q(A) = \sum_{i=1}^{n} a_i \sum_{j=0}^{k-1} \lambda^j A^{k-1-j}.$$

Let $p \in \mathbb{N}$, the Eq. (10) gives

$$\Big(P(\lambda) - P(A)\Big)^p = (\lambda - A)^p Q(A)^p = Q(A)^p (\lambda - A)^p.$$

Hence,

$$N\Big[(\lambda - A)^p\Big] \subset N\Big[\Big(P(\lambda) - P(A)\Big)^p\Big], \quad \forall p \in \mathbb{N}$$

and

$$R\Big[\Big(P(\lambda) - P(A)\Big)^p\Big] \subset R\Big[(\lambda - A)^p\Big], \quad \forall p \in \mathbb{N}.$$

This leads to

$$\bigcup_{p\in\mathbb{N}} N\Big[(\lambda - A)^p\Big] \subset \bigcup_{p\in\mathbb{N}} N\Big[\Big(P(\lambda) - P(A)\Big)^p\Big], \tag{11}$$

and

$$\bigcap_{p\in\mathbb{N}} R\Big[\Big(P(\lambda) - P(A)\Big)^p\Big] \subset \bigcap_{p\in\mathbb{N}} R\Big[(\lambda - A)^p\Big]. \tag{12}$$

On the other hand, it follows from Lemma 1.2 that $P(\lambda) - P(A)$ is a Fredholm operator on X with finite ascent and descent. So, by [16, Theorem 3.6], we have

$$\operatorname{asc}\Big(P(\lambda) - P(A)\Big) = \operatorname{desc}\Big(P(\lambda) - P(A)\Big).$$

Let p_0 this quantity, then

$$\dim \bigcup_{p\in\mathbb{N}} N\Big[\Big(P(\lambda) - P(A)\Big)^p\Big] = \dim N\Big[\Big(P(\lambda) - P(A)\Big)^{p_0}\Big] < \infty,$$

and

$$\operatorname{codim} \bigcap_{p\in\mathbb{N}} R\Big[\Big(P(\lambda) - P(A)\Big)^p\Big] = \operatorname{codim} R\Big[\Big(P(\lambda) - P(A)\Big)^{p_0}\Big] < \infty.$$

Using Eqs (11) and (12), we have

$$\dim \bigcup_{p\in\mathbb{N}} N\Big[(\lambda - A)^p\Big] < \infty,$$

and

$$\operatorname{codim} \bigcap_{p\in\mathbb{N}} R\Big[(\lambda - A)^p\Big] < \infty.$$

Therefore, asc $(\lambda - A) < \infty$ and desc $(\lambda - A) < \infty$. We have also, $\alpha(\lambda - A) < \infty$ and $\beta(\lambda - A) < \infty$.

Corollary 3.3. *Let $A \in \mathcal{L}(X)$. Let $F \in \mathcal{PF}(X)$, i.e., there exists a nonzero complex polynomial $P(z) = \sum_{r=0}^{p} a_r z^r$ satisfying $P(F) \in \mathcal{F}(X)$. Let $\lambda \in \mathbb{C}$ with $P(\lambda) \neq 0$. If their $A = \lambda - F$, then A is a Fredholm operator on X of index zero.*

Proof. This corollary immediately follows from Theorem 3.1 and Remark 1.8. □

The following theorem is the main result of this section.

Theorem 3.4. *Let X be a Banach space and let $A,\ B \in \Phi(X)$. Assume that there are $A_0,\ B_0 \in \mathcal{L}(X)$ and $F_1,\ F_2 \in \mathcal{PF}(X)$ such that*

$$AA_0 = I - F_1 \tag{13}$$

$$BB_0 = I - F_2. \tag{14}$$

(i) If $0 \in \rho^+_{ess}(A) \cap \rho^+_{ess}(B)$, $A_0 - B_0 \in \mathcal{F}_+(X)$ and $i(A) = i(B)$ then

$$\sigma_{eap}(A) = \sigma_{eap}(B).$$

(ii) If $0 \in \rho^+_{ess}(A) \cap \rho^+_{ess}(B)$, $A_0 - B_0 \in \mathcal{F}_-(X)$ and $i(A) = i(B)$ then

$$\sigma_{e\delta}(A) = \sigma_{e\delta}(B).$$

Proof. Let $\lambda \in \mathbb{C}$, Eqs (13) and (14) imply

$$(\lambda - A)A_0 - (\lambda - B)B_0 = F_1 - F_2 + \lambda(A_0 - B_0). \tag{15}$$

(i) Let $\lambda \notin \sigma_{eap}(B)$, then by Proposition 1.5 we have

$$(\lambda - B) \in \Phi_+(X) \ \text{ and } \ i(\lambda - B) \leq 0.$$

It is clearly that $B \in \mathcal{L}(X_B, X)$ where $X_B = (\mathcal{D}(B), \|.\|_B)$ is Banach space for the graph norm $\|.\|_B$. We can regard B as operator from X_B into X. This will be denoted by $\hat{B}$. Then

$$(\lambda - \hat{B}) \in \Phi_+(X_B, X) \ \text{ and } \ i(\lambda - \hat{B}) \leq 0.$$

Moreover, as $F_2 \in \mathcal{PF}(X)$, Eq. (14), Theorem 3.1 and [14, Theorem 2.7] imply that $B_0 \in \Phi^b(X, X_B)$ and consequently

$$(\lambda - \hat{B})B_0 \in \Phi^b_+(X_B, X).$$

Using Eq. (15) and Lemma 1.4, the operator $A_0 - B_0 \in \mathcal{F}_+(X)$ imply that $(\lambda - \hat{A})A_0 \in \Phi^b_+(X)$ and

$$i\Big((\lambda - \hat{A})A_0\Big) = i\Big((\lambda - \hat{B})B_0\Big). \tag{16}$$

A similar reasoning as before combining Eq. (13), Theorem 3.1 and [14, Theorem 2.6] shows that $A_0 \in \Phi^b(X, X_A)$ where $X_A = (\mathcal{D}(A), \|.\|_A)$. According of [14, Theorem 1.4] we can write

$$A_0T = I - F \ \text{ on } \ X_A \tag{17}$$

where

$$T \in \mathcal{L}(X_A, X) \ \text{ and } \ F \in \mathcal{F}(X_A),$$

by Eq. (17) we have

$$(\lambda - \hat{A})A_0T = (\lambda - \hat{A}) - (\lambda - \hat{A})F.$$

Since $T \in \Phi^b(X_A, X)$, according of [14, Theorem 6.6] we have

$$(\lambda - \hat{A})A_0T \in \Phi^b_+(X_A, X).$$

Using [14, Theorem 6.3] we prove that

$$(\lambda - \hat{A}) \in \Phi^b_+(X_A, X),$$

by Eq. (1) we have

$$(\lambda - A) \in \Phi_+(X_A, X). \tag{18}$$

As, $F_1,\ F_2 \in \mathcal{PF}(X)$, Eqs (13), (14) and [14, Theorem 2.3] give

$$i(A) + i(A_0) = i(I - F_1) = 0 \ \text{ and } \ i(B) + i(B_0) = i(I - F_1) = 0,$$

since $i(A) = i(B)$ then $i(A_0) = i(B_0)$. Using Eq. (16) we can write

$$i(\lambda - A) + i(A_0) = i(\lambda - B) + i(B_0).$$

Therefore

$$i(\lambda - A) \leq 0. \tag{19}$$

Using Eqs (18) and (19), we get $\lambda \notin \sigma_{eap}(A)$. Hence we prove that

$$\sigma_{eap}(A) \subset \sigma_{eap}(B).$$

The opposite inclusion follows from symmetric and we obtain

$$\sigma_{eap}(A) = \sigma_{eap}(B).$$

(*ii*) The proof of (*ii*) may be checked in a similar way to that in (*i*). It suffices to replace $\sigma_{eap}(.)$, $\Phi_+(.)$, $i(.) \leq 0$, [14, Theorem 6.6], [14, Theorem 6.3] and Proposition 1.5 (*i*) by $\sigma_{e\delta}(.)$, $\Phi_-(.)$, $i(.) \geq 0$, [9, Theorem 5], [14, Theorem 6.7] and Proposition 1.5 (*ii*) respectively. $\square$

4. Characterization of Essential Spectra

In this Section, we discuss the essential approximate point spectrum and the essential defect spectrum by means of the measure of noncompactness. Let $A \in \mathcal{C}(X)$, with a non-empty resolvent set, we will give a refinement on the definition of the essential approximate point spectrum and the essential defect spectrum of A respectively, by

$$\sigma_+(A) := \bigcap_{K \in \mathcal{H}_A^n(X)} \sigma_{ap}(A + K),$$

and

$$\sigma_-(A) := \bigcap_{K \in \mathcal{H}_A^n(X)} \sigma_{\delta}(A + K),$$

where $\mathcal{H}_A^n(X) = \Big\{K \in \mathcal{C}(X) \text{ such that } \gamma([K(\lambda - A - K)^{-1}]^n) < 1,\ \ \forall \lambda \in \rho(A + K)\Big\}$.

Theorem 4.1. *Let $A \in \mathcal{C}(X)$ with a non-empty resolvent set. Then*

$$(i)\ \ \sigma_{eap}(A) := \sigma_+(A),$$

$$(ii)\ \sigma_{e\delta}(A) := \sigma_-(A).$$

Proof. (i) Since $\mathcal{K}(X) \subset \mathcal{H}_A^n(X)$, we infer that $\sigma_+(A) \subset \sigma_{eap}(A)$. Conversely, let $\lambda \notin \sigma_+(A)$ then there exists $K \in \mathcal{H}_A^n(X)$ such that

$$\inf_{\|x\|=1,\ x\in\mathcal{D}(A)} \|(\lambda - A - K)x\| > 0.$$

The use of [13, Theorem 5.1] makes us conclude that $\lambda - A - K \in \Phi_+(X)$. So, $\lambda \in \rho(A+K)$ and $\gamma([(\lambda - A - K)^{-1}K]^n) < 1$. Hence applying Proposition 1.5, we get

$$\Big[I + K(\lambda - A - K)^{-1}\Big] \in \Phi(X) \text{ and } i\Big[I + K(\lambda - A - K)^{-1}\Big] = 0.$$

Then

$$\Big[I + K(\lambda - A - K)^{-1}\Big] \in \Phi_+(X) \text{ and } i\Big[I + K(\lambda - A - K)^{-1}\Big] \leq 0.$$

So,

$$\Big[I + \hat{K}(\lambda - \hat{A} - \hat{K})^{-1}\Big] \in \Phi_+(X) \text{ and } i\Big[I + \hat{K}(\lambda - \hat{A} - \hat{K})^{-1}\Big] \leq 0. \tag{20}$$

Thus writing $\lambda - \hat{A}$ in the form

$$\lambda - \hat{A} = \Big[I + \hat{K}(\lambda - \hat{A} - \hat{K})^{-1}\Big](\lambda - \hat{A} - \hat{K}). \tag{21}$$

Using Eqs (20) and (21) together with [10, Theorem 5] and [10, Theorem 12] we get

$$\lambda - \hat{A} \in \Phi_+^b(X) \text{ and } i(\lambda - \hat{A}) \leq 0.$$

Now, using Eq. (1) we infer that

$$\lambda - A \in \Phi_+(X) \text{ and } i(\lambda - A) \leq 0.$$

Finally, by Proposition 1.5 (i) we get $\lambda \notin \sigma_{eap}(A)$, this proves the assertion (i).

(ii) Since $\mathcal{K}(X) \subset \mathcal{H}_A^n(X)$, then $\sigma_-(A) \subset \sigma_{e\delta}(A)$. It remains to show that $\sigma_{e\delta}(A) \subset \sigma_-(A)$. To do this consider $\lambda \notin \sigma_-(A)$, then there exists $K \in \mathcal{H}_A^n(X)$ such that $\lambda \notin \sigma_{e\delta}(A+K)$. Thus $\lambda - A - K$ is surjective, then

$$\lambda - A - K \in \Phi_-(X)$$

and

$$i(\lambda - A - K) = \alpha(\lambda - A - K) \geq 0.$$

Hence, by [14, Theorem 5.30] and Eq. (21) we deduce that

$$\lambda - \hat{A} \in \Phi_-(X)$$

and

$$i(\lambda - \hat{A}) = i(\lambda - \hat{A} - \hat{K}) \geq 0.$$

Using, Eq. (1) we get

$$\lambda - A \in \Phi_-(X) \text{ and } i(\lambda - A) \geq 0.$$

Finally, by Proposition 1.5 (ii) we get $\lambda \notin \sigma_{e\delta}(A)$. $\square$

Remark 4.2. *It follows, immediately, from* Theorem 4.1 *that*

$$\sigma_{eap}(A+K) = \sigma_{eap}(A) \ \text{ and } \sigma_{e\delta}(A+K) = \sigma_{e\delta}(A)$$

for all $K \in \mathcal{H}_A^n(X)$.

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