Filomat 29:8 (2015), 1857–1867
DOI 10.2298/FIL1508857X

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Properties of Certain Subclasses of Multivalent Analytic Functions

Yi-Hui Xu[a], Jin-Lin Liu[b]

[a]*Department of Mathematics, Suqian College, Suqian 223800, P.R. China*
[b]*Department of Mathematics, Yangzhou University, Yangzhou 225002, P.R. China*
Dedicated to Professor Miodrag Mateljević on the occasion of his 65th birthday

Abstract. In the present paper the authors introduce two new subclasses $R_{p,k}(\lambda, A, B)$ and $T_{p,k}(\lambda, A, B)$ of multivalent analytic functions. Some properties such as distortion inequalities, inclusion relation and partial sums are given.

1. Introduction

Throughout this paper, we assume that

$$N = \{1, 2, 3, \cdots\}, k \in N \setminus \{1\}, -1 \leq B < 0, B < A \leq 1 \text{ and } 0 \leq \lambda \leq 1. \tag{1.1}$$

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=2p}^{\infty} a_n z^n \quad (p \in N), \tag{1.2}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. For functions f and g analytic in U, we say that f is subordinate to g in U and write $f(z) \prec g(z)$ $(z \in U)$, if there exists a Schwarz function $w(z)$ in U such that

$$|w(z)| \leq |z| \quad \text{and} \quad f(z) = g(w(z)) \quad (z \in U).$$

Recently, several authors obtained many important properties and characteristics of multivalent analytic functions in $A(p)$ (see, e.g., [1] to [6] and [13] to [18]; see also the recent works [7], [9], [10] and [12]). Inspired by these works, the authors will define two new subclasses of $A(p)$ in the present paper. Some properties of these subclasses such as distortion inequalities, inclusion relation and partial sums are derived.

We first define the following two subclasses of $A(p)$.

Definition 1. A function $f \in A(p)$ defined by (1.2) is said to be in the class $R_{p,k}(\lambda, A, B)$ if and only if it satisfies the coefficient inequality

$$\sum_{n=2p}^{\infty} [n(1 - B) - p(1 - A)(1 - \lambda + \lambda\delta_{n,p,k})]|a_n| \leq p(A - B). \tag{1.3}$$

2010 *Mathematics Subject Classification*. Primary 30C45.
Keywords. Multivalent analytic function; subordination; distortion; partial sums.
Received: 06 November 2013; Accepted: 12 April 2014
Communicated by Miodrag Mateljević
Research supported by the National Natural Science Foundation of China and Natural Science Foundation of Jiangsu Province
Email addresses: yuanziqixu@126.com (Yi-Hui Xu), jlliu@yzu.edu.cn, corresponding author (Jin-Lin Liu)

Definition 2. A function $f \in A(p)$ defined by (1.2) is said to be in the class $T_{p,k}(\lambda, A, B)$ if and only if it satisfies the coefficient inequality

$$\sum_{n=2p}^{\infty} n[n(1-B) - p(1-A)(1-\lambda+\lambda\delta_{n,p,k})]|a_n| \leq p^2(A-B). \tag{1.4}$$

It is obvious that

$$f(z) \in T_{p,k}(\lambda, A, B) \text{ if and only if } \frac{zf'(z)}{p} \in R_{p,k}(\lambda, A, B).$$

If we write

$$\alpha_n = \frac{n(1-B) - p(1-A)(1-\lambda+\lambda\delta_{n,p,k})}{p(A-B)} \text{ and } \beta_n = \frac{n}{p}\alpha_n \quad (n \geq 2p), \tag{1.5}$$

then it is easy to verify that

$$\frac{\partial \beta_n}{\partial \lambda} = \frac{n}{p}\frac{\partial \alpha_n}{\partial \lambda} \geq 0,$$

$$\frac{\partial \beta_n}{\partial A} = \frac{n}{p}\frac{\partial \alpha_n}{\partial A} < 0$$

and

$$\frac{\partial \beta_n}{\partial B} = \frac{n}{p}\frac{\partial \alpha_n}{\partial B} \geq 0.$$

Thus we obtain the following inclusion relations. If

$$0 \leq \lambda_1 \leq \lambda \leq 1, -1 \leq B_1 \leq B < 0, B < A \leq 1 \text{ and } A \leq A_1 \leq 1,$$

then

$$T_{p,k}(\lambda, A, B) \subset R_{p,k}(\lambda, A, B) \subseteq R_{p,k}(\lambda_1, A_1, B_1) \subseteq R_{p,k}(0, 1, -1) \tag{1.6}$$

and

$$T_{p,k}(\lambda, A, B) \subseteq T_{p,k}(\lambda_1, A_1, B_1) \subseteq T_{p,k}(0, 1, -1).$$

2. Main Results

The following lemma will be required in our investigation.

Lemma. Let $f \in A(p)$ defined by (1.2) satisfy the condition (1.3). Then

$$\frac{zf'(z)}{(1-\lambda)f(z)+\lambda f_{p,k}(z)} \prec p\frac{1+Az}{1+Bz} \quad (z \in U) \tag{2.1}$$

where

$$f_{p,k}(z) = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_k^{-jp} f(\varepsilon_k^j z), \quad \varepsilon_k = \exp\left(\frac{2\pi i}{k}\right) \tag{2.2}$$

and

$$\delta_{n,p,k} = \begin{cases} 0 & \left(\frac{n-p}{k} \notin N\right), \\ 1 & \left(\frac{n-p}{k} \in N\right). \end{cases} \tag{2.3}$$

Proof. For $f \in A(p)$ defined by (1.2), the function $f_{p,k}(z)$ in (2.2) can be expressed as

$$f_{p,k}(z) = z^p + \sum_{n=2p}^{\infty} \delta_{n,p,k} a_n z^n \tag{2.4}$$

with

$$\delta_{n,p,k} = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_k^{j(n-p)} = \begin{cases} 0 & \left(\frac{n-p}{k} \notin N\right), \\ 1 & \left(\frac{n-p}{k} \in N\right). \end{cases}$$

In view of (1.1) and (2.3), we see that

$$pA(1-\lambda+\lambda\delta_{n,p,k}) - nB \geq -B[n - p(1-\lambda+\lambda\delta_{n,p,k})] \geq 0 \quad (n \geq 2p). \tag{2.5}$$

Suppose that the inequality (1.3) holds. Then from (2.4) and (2.5) we deduce that

$$\begin{aligned}\left|\frac{\frac{zf'(z)}{(1-\lambda)f(z)+\lambda f_{p,k}(z)} - p}{pA - \frac{Bzf'(z)}{(1-\lambda)f(z)+\lambda f_{p,k}(z)}}\right| &= \left|\frac{\sum_{n=2p}^{\infty}[n-p(1-\lambda+\lambda\delta_{n,p,k})]a_n z^{n-p}}{p(A-B)+\sum_{n=2p}^{\infty}[pA(1-\lambda+\lambda\delta_{n,p,k})-nB]a_n z^{n-p}}\right| \\ &\leq \frac{\sum_{n=2p}^{\infty}[n-p(1-\lambda+\lambda\delta_{n,p,k})]|a_n|}{p(A-B)-\sum_{n=2p}^{\infty}[pA(1-\lambda+\lambda\delta_{n,p,k})-nB]|a_n|} \\ &\leq 1 \quad (|z|=1).\end{aligned}$$

Hence, by the maximum modulus theorem, we arrive at (2.1).

Remark 1. By Lemma and (1.6), one can see that each function in the classes $R_{p,k}(\lambda, A, B)$ and $T_{p,k}(\lambda, A, B)$ is multivalent starlike.

Theorem 1. Let $\frac{p}{k} \notin N$ and suppose that either

(a) $1 - B \geq p(1-A)$ and $0 \leq \lambda \leq 1$, or

(b) $1 - B < p(1-A)$ and $0 \leq \lambda \leq \frac{1-B}{p(1-A)}$.

(i) If $f \in R_{p,k}(\lambda, A, B)$, then for $z \in U$,

$$|z|^p - \frac{A-B}{2(1-B)-(1-A)(1-\lambda)}|z|^{2p} \leq |f(z)| \leq |z|^p + \frac{A-B}{2(1-B)-(1-A)(1-\lambda)}|z|^{2p}. \tag{2.6}$$

The bounds in (2.6) are best possible for the function f defined by

$$f(z) = z^p + \frac{A-B}{2(1-B)-(1-A)(1-\lambda)}z^{2p}. \tag{2.7}$$

(ii) If $f \in T_{p,k}(\lambda, A, B)$, then for $z \in U$,

$$p|z|^{p-1}\left(1 - \frac{A-B}{2(1-B)-(1-A)(1-\lambda)}|z|^p\right) \le |f'(z)| \le p|z|^{p-1}\left(1 + \frac{A-B}{2(1-B)-(1-A)(1-\lambda)}|z|^p\right). \tag{2.8}$$

The bounds in (2.8) are best possible for the function f defined by

$$f(z) = z^p + \frac{A-B}{4(1-B)-2(1-A)(1-\lambda)}z^{2p}. \tag{2.9}$$

Proof. Let $\frac{p}{k} \notin N$. For $n \ge 2p$ $(n \in N)$ and $\frac{n-p}{k} \notin N$, we have $\delta_{n,p,k} = \delta_{2p,p,k} = 0$, and so

$$\frac{n(1-B)-p(1-A)(1-\lambda+\lambda\delta_{n,p,k})}{p(A-B)} \ge \frac{2(1-B)-(1-A)(1-\lambda)}{A-B}. \tag{2.10}$$

For $n \ge 2p$ $(n \in N)$ and $\frac{n-p}{k} \in N$, we have $\delta_{n,p,k} = 1$ and

$$\frac{n(1-B)-p(1-A)(1-\lambda+\lambda\delta_{n,p,k})}{p(A-B)} \ge \frac{(2p+1)(1-B)-p(1-A)}{p(A-B)}. \tag{2.11}$$

If either (a) or (b) is satisfied, then

$$\frac{(2p+1)(1-B)-p(1-A)}{p(A-B)} \ge \frac{2(1-B)-(1-A)(1-\lambda)}{A-B} > 1. \tag{2.12}$$

(i) If

$$f(z) = z^p + \sum_{n=2p}^{\infty} a_n z^n \in R_{p,k}(\lambda, A, B),$$

then it follows from (2.10) to (2.12) that

$$\frac{2(1-B)-(1-A)(1-\lambda)}{A-B}\sum_{n=2p}^{\infty}|a_n| \le 1.$$

Hence we have

$$\begin{aligned} |f(z)| &\le |z|^p + |z|^{2p}\sum_{n=2p}^{\infty}|a_n| \\ &\le |z|^p + \frac{A-B}{2(1-B)-(1-A)(1-\lambda)}|z|^{2p} \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\ge |z|^p - |z|^{2p}\sum_{n=2p}^{\infty}|a_n| \\ &\ge |z|^p - \frac{A-B}{2(1-B)-(1-A)(1-\lambda)}|z|^{2p} > 0 \end{aligned}$$

for $z \in U$.

(ii) If

$$f(z) = z^p + \sum_{n=2p}^{\infty} a_n z^n \in T_{p,k}(\lambda, A, B),$$

then (2.10) to (2.12) yield

$$\frac{2(1-B)-(1-A)(1-\lambda)}{p(A-B)} \sum_{n=2p}^{\infty} n|a_n| \leq 1.$$

This leads to (2.8). Now the proof of the theorem is complete.

Theorem 2. Let

$$\frac{p}{k} \notin N, 1-B < p(1-A) \text{ and } \frac{1-B}{p(1-A)} < \lambda \leq 1. \tag{2.13}$$

(i) If $f(z) = z^p + a_{2p}z^{2p} + \cdots \in R_{p,k}(\lambda, A, B)$, then for $z \in U$,

$$|f(z)| \leq |z|^p + |a_{2p}||z|^{2p} + \frac{p(A-B) - p[2(1-B)-(1-A)(1-\lambda)]|a_{2p}|}{(2p+1)(1-B)-p(1-A)} |z|^{2p+1} \tag{2.14}$$

and

$$|f(z)| \geq |z|^p - |a_{2p}||z|^{2p} - \frac{p(A-B) - p[2(1-B)-(1-A)(1-\lambda)]|a_{2p}|}{(2p+1)(1-B)-p(1-A)} |z|^{2p+1}. \tag{2.15}$$

Equalities in (2.14) and (2.15) are attained, for example, by

$$f(z) = z^p + \frac{p(A-B)}{(2p+1)(1-B)-p(1-A)} z^{2p+1}.$$

(ii) If $f(z) = z^p + a_{2p}z^{2p} + \cdots \in T_{p,k}(\lambda, A, B)$, then for $z \in U$,

$$|f'(z)| \leq p|z|^{p-1} + 2p|a_{2p}||z|^{2p-1} + \frac{p^2[(A-B) - 2(2(1-B)-(1-A)(1-\lambda))|a_{2p}|]}{(2p+1)(1-B)-p(1-A)} |z|^{2p} \tag{2.16}$$

and

$$|f'(z)| \geq p|z|^{p-1} - 2p|a_{2p}||z|^{2p-1} - \frac{p^2[(A-B) - 2(2(1-B)-(1-A)(1-\lambda))|a_{2p}|]}{(2p+1)(1-B)-p(1-A)} |z|^{2p}. \tag{2.17}$$

Equalities in (2.16) and (2.17) are attained, for example, by

$$f(z) = z^p + \frac{p^2(A-B)}{(2p+1)[(2p+1)(1-B)-p(1-A)]} z^{2p+1}.$$

Proof. Note that (2.13) implies that

$$\frac{(2p+1)(1-B)-p(1-A)}{p(A-B)} < \frac{2(1-B)-(1-A)(1-\lambda)}{A-B}. \tag{2.18}$$

(i) For $f(z) = z^p + a_{2p}z^{2p} + \cdots \in R_{p,k}(\lambda, A, B)$, it follows from (2.10), (2.11) and (2.18) that

$$\frac{2(1-B)-(1-A)(1-\lambda)}{A-B}|a_{2p}| + \frac{(2p+1)(1-B)-p(1-A)}{p(A-B)} \sum_{n=2p+1}^{\infty} |a_n| \leq 1.$$

From this we can get (2.14) and (2.15).

(ii) For $f(z) = z^p + a_{2p}z^{2p} + \cdots \in T_{p,k}(\lambda, A, B)$, from (2.10), (2.11) and (2.18) we deduce

$$\frac{2[2(1-B)-(1-A)(1-\lambda)]}{A-B}|a_{2p}| + \frac{(2p+1)(1-B)-p(1-A)}{p^2(A-B)}\sum_{n=2p+1}^{\infty} n|a_n| \le 1.$$

Hence we have (2.16) and (2.17). The proof of the theorem is complete.

Theorem 3. Let $\frac{p}{k} \in N$.

(i) If $f \in R_{p,k}(\lambda, A, B)$, then for $z \in U$,

$$|z|^p - \frac{A-B}{1+A-2B}|z|^{2p} \le |f(z)| \le |z|^p + \frac{A-B}{1+A-2B}|z|^{2p}. \tag{2.19}$$

The bounds in (2.19) are best possible for the function f defined by

$$f(z) = z^p + \frac{A-B}{1+A-2B}z^{2p}. \tag{2.20}$$

(ii) If $f \in T_{p,k}(\lambda, A, B)$, then for $z \in U$,

$$p|z|^{p-1}\left(1 - \frac{A-B}{1+A-2B}|z|^p\right) \le |f'(z)| \le p|z|^{p-1}\left(1 + \frac{A-B}{1+A-2B}|z|^p\right). \tag{2.21}$$

The bounds in (2.21) are best possible for the function f defined by

$$f(z) = z^p + \frac{A-B}{2(1+A-2B)}z^{2p}. \tag{2.22}$$

Proof. Let $\frac{p}{k} \in N$. For $n \ge 2p$ $(n \in N)$ and $\frac{n-p}{k} \in N$, we have $n = 2p + k(l-1)$ $(l \in N)$, $\delta_{n,p,k} = \delta_{2p,p,k} = 1$ and

$$\frac{n(1-B) - p(1-A)(1-\lambda+\lambda\delta_{n,p,k})}{p(A-B)} \ge \frac{1+A-2B}{A-B}. \tag{2.23}$$

For $n \ge 2p$ $(n \in N)$ and $\frac{n-p}{k} \notin N$, we have $\delta_{n,p,k} = \delta_{2p+1,p,k} = 0$, and so

$$\begin{aligned}\frac{n(1-B) - p(1-A)(1-\lambda+\lambda\delta_{n,p,k})}{p(A-B)} &\ge \frac{(2p+1)(1-B) - p(1-A)(1-\lambda)}{p(A-B)}\\ &\ge \frac{1+A-2B}{A-B}.\end{aligned} \tag{2.24}$$

(i) If

$$f(z) = z^p + \sum_{n=2p}^{\infty} a_n z^n \in R_{p,k}(\lambda, A, B),$$

then it follows from (2.23) and (2.24) that

$$\frac{1+A-2B}{A-B}\sum_{n=2p}^{\infty} |a_n| \le 1. \tag{2.25}$$

Hence we have

$$\begin{aligned}|f(z)| &\le |z|^p + |z|^{2p}\sum_{n=2p}^{\infty}|a_n|\\ &\le |z|^p + \frac{A-B}{1+A-2B}|z|^{2p}\end{aligned}$$

and

$$\begin{aligned}|f(z)| &\geq |z|^p - |z|^{2p}\sum_{n=2p}^{\infty}|a_n| \\ &\geq |z|^p - \frac{A-B}{1+A-2B}|z|^{2p}\end{aligned}$$

for $z \in U$.

(ii) If

$$f(z) = z^p + \sum_{n=2p}^{\infty} a_n z^n \in T_{p,k}(\lambda, A, B),$$

then (2.23) and (2.24) yield

$$\frac{1+A-2B}{p(A-B)}\sum_{n=2p}^{\infty} n|a_n| \leq 1.$$

This leads to (2.21). The proof is completed.

Next, we generalize the inclusion relation $T_{p,k}(\lambda, A, B) \subset R_{p,k}(\lambda, A, B)$ which is mentioned in (1.6).

Theorem 4. If $-1 \leq D < 0$, then

$$T_{p,k}(\lambda, A, B) \subset R_{p,k}(\lambda, C(D), D), \tag{2.26}$$

where

$$C(D) = D + \frac{(1-D)(A-B)}{2(1-B)+(A-B)}.$$

Proof. Since $B < A \leq 1$ and $-1 \leq D < 0$, we see that

$$D < C(D) < 1.$$

Let $f \in T_{p,k}(\lambda, A, B)$. In order to prove (2.26), we need only to find the smallest C ($D < C \leq 1$) such that

$$\frac{n(1-D)-p(1-C)(1-\lambda+\lambda\delta_{n,p,k})}{p(C-D)} \leq \frac{n[n(1-B)-p(1-A)(1-\lambda+\lambda\delta_{n,p,k})]}{p^2(A-B)} \tag{2.27}$$

for all $n \geq 2p$, that is , that

$$\frac{(1-D)[n-p(1-\lambda+\lambda\delta_{n,p,k})]}{p(C-D)} + (1-\lambda+\lambda\delta_{n,p,k}) \leq \frac{n}{p}\left\{\frac{(1-B)[n-p(1-\lambda+\lambda\delta_{n,p,k})]}{p(A-B)} + (1-\lambda+\lambda\delta_{n,p,k})\right\}. \tag{2.28}$$

For $n \geq 2p$ and $\frac{n-p}{k} \notin N$, (2.28) is equivalent to

$$C \geq D + \frac{1-D}{\frac{(n-p)(1-\lambda)}{n-p(1-\lambda)} + \frac{n(1-B)}{p(A-B)}} = \varphi(\lambda, n) \quad \text{(say)}. \tag{2.29}$$

Noting that (1.1), a simple calculation shows that $\frac{\partial\varphi(\lambda,x)}{\partial x} < 0$ for all real $x \geq 2p$ and $0 \leq \lambda \leq 1$, and so the function $\varphi(\lambda, n)$ is decreasing in n ($n \geq 2p$). Therefore

$$\varphi(\lambda, n) \leq \begin{cases} \varphi(\lambda, 2p+1) & \left(\frac{p}{k} \in N\right), \\ \varphi(\lambda, 2p) & \left(\frac{p}{k} \notin N\right). \end{cases} \tag{2.30}$$

For $n \geq 2p$ and $\frac{n-p}{k} \in N$, (2.28) becomes

$$C \geq D + \frac{p(1-D)(A-B)}{n(1-B)+p(A-B)} = \varphi(0, n) \tag{2.31}$$

and

$$\varphi(0,n) \le \begin{cases} \varphi(0,2p) & \left(\frac{p}{k} \in N\right), \\ \varphi\left(0, k\left(\left[\frac{p}{k}\right]+1\right)+p\right) & \left(\frac{p}{k} \notin N\right). \end{cases} \tag{2.32}$$

Consequently, by taking

$$C(D) = \varphi(0,2p) = D + \frac{(1-D)(A-B)}{2(1-B)+(A-B)}, \tag{2.33}$$

it follows from (2.27) to (2.33) that $f \in R_{p,k}(\lambda, C(D), D)$. Thus the proof is complete.

Now we derive certain results of the partial sums of functions in the classes $R_{p,k}(\lambda, A, B)$ and $T_{p,k}(\lambda, A, B)$. Let $f \in A(p)$ be given by (1.2) and define the partial sums $s_1(z)$ and $s_m(z)$ by

$$s_1(z) = z^p \text{ and } s_m(z) = z^p + \sum_{n=2p}^{2p+m-2} a_n z^n \quad (m \in N \setminus \{1\}). \tag{2.34}$$

For simplicity we use the notation α_n $(n \ge 2p)$ defined by (1.5).

Theorem 5. Let $f \in R_{p,k}(\lambda, A, B)$ and let either
(a) $1 - B \ge p(1-A)$ and $0 \le \lambda \le 1$, or
(b) $1 - B < p(1-A)$ and $0 \le \lambda \le \frac{1-B}{p(1-A)}$.
Then for $m \in N$, we have

$$\mathrm{Re}\left(\frac{f(z)}{s_m(z)}\right) > 1 - \frac{1}{\alpha_{2p+m-1}} \quad (z \in U) \tag{2.35}$$

and

$$\mathrm{Re}\left(\frac{s_m(z)}{f(z)}\right) > \frac{\alpha_{2p+m-1}}{1+\alpha_{2p+m-1}} \quad (z \in U). \tag{2.36}$$

The bounds in (2.35) and (2.36) are sharp for each m.

Proof. If either (a) or (b) is satisfied, then for $n \ge 2p$,

$$\alpha_n = \frac{n(1-B) - p(1-A)(1-\lambda+\lambda\delta_{n,p,k})}{p(A-B)} \ge \frac{1-B}{A-B} \ge 1 \tag{2.37}$$

and

$$\begin{aligned} \alpha_{n+1} &= \alpha_n + \frac{1-B-p\lambda(1-A)(\delta_{n+1,p,k}-\delta_{n,p,k})}{p(A-B)} \\ &\ge \alpha_n + \frac{1-B-p\lambda(1-A)}{p(A-B)} \\ &\ge \alpha_n. \end{aligned} \tag{2.38}$$

Let $f \in R_{p,k}(\lambda, A, B)$. Then it follows from (2.37) and (2.38) that

$$\sum_{n=2p}^{2p+m-2} |a_n| + \alpha_{2p+m-1} \cdot \sum_{n=2p+m-1}^{\infty} |a_n| \le \sum_{n=2p}^{\infty} \alpha_n |a_n| \le 1 \quad (m \in N \setminus \{1\}). \tag{2.39}$$

If we put

$$p_1(z) = 1 + \alpha_{2p+m-1}\left(\frac{f(z)}{s_m(z)} - 1\right)$$

for $z \in U$ and $m \in N \setminus \{1\}$, then $p_1(0) = 1$ and we deduce from (2.39) that

$$\begin{aligned}\left|\frac{p_1(z)-1}{p_1(z)+1}\right| &= \left|\frac{\alpha_{2p+m-1}\sum_{n=2p+m-1}^{\infty} a_n z^{n-p}}{2\left(1+\sum_{n=2p}^{2p+m-2} a_n z^{n-p}\right)+\alpha_{2p+m-1}\sum_{n=2p+m-1}^{\infty} a_n z^{n-p}}\right| \\ &\leq \frac{\alpha_{2p+m-1}\sum_{n=2p+m-1}^{\infty}|a_n|}{2-2\sum_{n=2p}^{2p+m-2}|a_n|-\alpha_{2p+m-1}\sum_{n=2p+m-1}^{\infty}|a_n|} \\ &\leq 1 \quad (z\in U;\ m\in N\setminus\{1\}).\end{aligned}$$

This implies that $\mathrm{Re}(p_1(z)) > 0$ for $z \in U$, and so (2.35) holds for $m \in N \setminus \{1\}$.

Similarly, by setting

$$p_2(z) = (1+\alpha_{2p+m-1})\frac{s_m(z)}{f(z)} - \alpha_{2p+m-1},$$

it follows from (2.39) that

$$\begin{aligned}\left|\frac{p_2(z)-1}{p_2(z)+1}\right| &= \left|\frac{-(1+\alpha_{2p+m-1})\sum_{n=2p+m-1}^{\infty} a_n z^{n-p}}{2\left(1+\sum_{n=2p}^{2p+m-2} a_n z^{n-p}\right)+(1-\alpha_{2p+m-1})\sum_{n=2p+m-1}^{\infty} a_n z^{n-p}}\right| \\ &\leq \frac{(1+\alpha_{2p+m-1})\sum_{n=2p+m-1}^{\infty}|a_n|}{2-2\sum_{n=2p}^{2p+m-2}|a_n|-(\alpha_{2p+m-1}-1)\sum_{n=2p+m-1}^{\infty}|a_n|} \\ &\leq 1 \quad (z\in U;\ m\in N\setminus\{1\}).\end{aligned}$$

Hence we have (2.36) for $m \in N \setminus \{1\}$.

For $m = 1$, replacing (2.39) by

$$\alpha_{2p}\sum_{n=2p}^{\infty}|a_n| \leq \sum_{n=2p}^{\infty}\alpha_n|a_n| \leq 1$$

and proceeding as the above, we see that (2.35) and (2.36) are also true.

Furthermore, taking the function f defined by

$$f(z) = z^p + \frac{z^{2p+m-1}}{\alpha_{2p+m-1}} \in R_{p,k}(\lambda, A, B),$$

we have $s_m(z) = z^p$,

$$\mathrm{Re}\left(\frac{f(z)}{s_m(z)}\right) \to 1 - \frac{1}{\alpha_{2p+m-1}} \quad \text{as } z \to \exp\left(\frac{\pi i}{p+m-1}\right)$$

and

$$\mathrm{Re}\left(\frac{s_m(z)}{f(z)}\right) \to \frac{\alpha_{2p+m-1}}{1+\alpha_{2p+m-1}} \quad \text{as } z \to 1.$$

Thus the proof of Theorem 5 is completed.

Corollary. Let the assumptions of Theorem 5 hold. Then, for $z \in U$, we have

$$\mathrm{Re}\left(\frac{f(z)}{z^p}\right) > \begin{cases} \frac{1-B}{1+A-2B} & \left(\frac{p}{k}\in N\right), \\ \frac{1-B+\lambda(1-A)}{2(1-B)-(1-\lambda)(1-A)} & \left(\frac{p}{k}\notin N\right). \end{cases}$$

and

$$\mathrm{Re}\left(\frac{z^p}{f(z)}\right) > \begin{cases} \frac{1+A-2B}{1-B+2(A-B)} & \left(\frac{p}{k}\in N\right), \\ \frac{2(1-B)-(1-\lambda)(1-A)}{1-B+\lambda(1-A)+2(A-B)} & \left(\frac{p}{k}\notin N\right). \end{cases}$$

The results are sharp.

Remark 2. Replacing $R_{p,k}(\lambda, A, B)$ by $T_{p,k}(\lambda, A, B)$, it follows from Theorem 5 that the inequalities (2.35) and (2.36) are true.

In Theorem 6 below we improve the bounds in (2.35) and (2.36) for $f \in T_{p,k}(\lambda, A, B)$.

Theorem 6. Let $f \in T_{p,k}(\lambda, A, B)$ and let either (a) or (b) in Theorem 5 be satisfied. Then for $m \in N$, we have

$$\operatorname{Re}\left(\frac{f(z)}{s_m(z)}\right) > 1 - \frac{p}{(2p+m-1)\alpha_{2p+m-1}} \quad (z \in U) \tag{2.40}$$

and

$$\operatorname{Re}\left(\frac{s_m(z)}{f(z)}\right) > \frac{(2p+m-1)\alpha_{2p+m-1}}{p+(2p+m-1)\alpha_{2p+m-1}} \quad (z \in U). \tag{2.41}$$

The bounds in (2.40) and (2.41) are sharp for the function f defined by

$$f(z) = z^p + \frac{pz^{2p+m-1}}{(2p+m-1)\alpha_{2p+m-1}} \in T_{p,k}(\lambda, A, B). \tag{2.42}$$

Proof. By virtue of the assumptions of Theorem 6, it follows from (2.37) and (2.38) that

$$\sum_{n=2p}^{2p+m-2} |a_n| + \frac{(2p+m-1)\alpha_{2p+m-1}}{p} \sum_{n=2p+m-1}^{\infty} |a_n| \le \sum_{n=2p}^{\infty} \frac{n}{p}\alpha_n |a_n| \le 1 \quad (m \in N \setminus \{1\}) \tag{2.43}$$

and

$$\alpha_{2p} \sum_{n=2p}^{\infty} |a_n| \le \sum_{n=2p}^{\infty} \frac{n}{p}\alpha_n |a_n| \le 1. \tag{2.44}$$

If we put

$$p_1(z) = 1 + \frac{(2p+m-1)\alpha_{2p+m-1}}{p}\left(\frac{f(z)}{s_m(z)} - 1\right)$$

and

$$p_2(z) = \left(1 + \frac{(2p+m-1)\alpha_{2p+m-1}}{p}\right)\frac{s_m(z)}{f(z)} - \frac{(2p+m-1)\alpha_{2p+m-1}}{p},$$

then (2.43) and (2.44) lead to $\operatorname{Re}(p_j(z)) > 0$ ($z \in U$; $m \in N$; $j = 1, 2$). Hence we have (2.40) and (2.41). Sharpness can be verified easily.

ACKNOWLEDGEMENT

The authors would like to express sincere thanks to the referee for careful reading and suggestions which helped us to improve the paper.

References

[1] M. K. Aouf, J. Dziok, Distortion and convolutional theorems for operators of generalized fractional calculus involving Wright function, J. Appl. Anal. 14 (2008) 183–192.
[2] Y.-L. Cang, J.-L. Liu, Certain subclasses of multivalent analytic functions, Abst. Appl. Anal. Volume 2013, Article ID 890404, 8 pages.
[3] J. Dziok, R. K. Raina, J. Sokol, Differential subordinations and α-convex functions, Acta Math. Scientia 33B (2013) 609–620.
[4] J.-L. Liu, The Noor integral and strongly starlike functions, J. Math. Anal. Appl. 261 (2001) 441–447.

[5] J.-L. Liu, H. M. Srivastava, A linear operator and associated families of meromorphically multivalent functions, J. Math. Anal. Appl. 259 (2001) 566–581.

[6] M.-S. Liu, S. Owa, N.-S. Song, Properties of certain transforms defined by convolution of analytic functions, Appl. Math. Comput. 219 (2013) 4702–4709.

[7] M. Obradovic, S. Ponnusamy, Radius of univalence of certain class of analytic functions, Filomat 27(6) (2013) 1085–1090.

[8] R. Pavatham, S. Radha, On α-starlike and α-close-to-convex functions with respect to n-symmetric points, Indian J. Pure Appl. Math. 16 (1986) 1114–1122.

[9] H. M. Srivastava, S. Bulut, M. Caglar, N. Yagmur, Coefficient estimates for a general subclass of analytic and bi-univalent functions, Filomat 27(5) (2013) 831–842.

[10] H. M. Srivastava, J. K. Prajapat, G. I. Oros, R. Sendrutiu, Geometric properties of a certain general family of integral operators, Filomat 28(4) (2014) 745–754.

[11] H.M. Srivastava, D.-G. Yang, N-E. Xu, Some subclasses of meromorphically multivalent functions associated with a linear operator, Appl. Math. Comput. 195 (2008) 11–23.

[12] H. Tang, G. Deng, J. Sokol, S. Li, Inclusion and argument properties for the Srivastava-Khairnar-More operator, Filomat 28(8) (2014) 1603–1618.

[13] H. Tang, H. M. Srivastava, S.-H. Li, L.-N. Ma, Third-order differential subordination and superordination results for meromorphically multivalent functions associated with the Liu-Srivastava operator, Abstr. Appl. Anal. 2014 (2014), Article ID 792175 1–11.

[14] Z.-G. Wang, C.-Y. Gao, S.-M. Yuan, On certain subclasses of close-to-convex and quasi-convex functions with respect to k-symmetric points, J. Math. Anal. Appl. 322 (2006) 97–106.

[15] Z.-G. Wang, H. M. Srivastava, S.-M. Yuan, Some basic properties of certain subclasses of meromorphically starlike functions, J. Inequal. Appl. 2014 (2014) Article ID 2014:29, 1–12.

[16] N.-E. Xu, D.-G. Yang, Some classes of analytic and multivalent functions involving a linear operator, Math. Comput. Modelling 49 (2009) 955–965.

[17] Y.-H. Xu, B.A. Frasin, J.-L. Liu, Certain sufficient conditions for starlikeness and convexity of meromorphically multivalent functions, Acta Math. Scientia 33B (2013) 1300–1304.

[18] S.-M. Yuan, Z.-M. Liu, Some properties of α-convex and α-quasiconvex functions with respect to n-symmetric points, Appl. Math. Comput. 188 (2007) 1142–1150.