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An Extension of Darbo Fixed Point Theorem and its Applications to Coupled Fixed Point and Integral Equations

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Abstract. In this paper, an extension of Darbo fixed point theorem is introduced. By applying our extension, we obtain a coupled fixed point theorem and a solution for an integral equation. The proofs of our results are based on the technique of measure of noncompactness.

1. Introduction

Recently many papers have been appeared about the notion of measure of noncompactness and its applications where it was initiated by Kuratowski [12]. Darbo [11] applied the technique of measure of noncompactness to the fixed point theory and extended Schauder fixed point theory to noncompact operators. Some authors have used of Darbo fixed point theory to construct applications of measure of noncompactness, see for example [1, 2, 8]. In this paper, first some essential concepts and results are recalled. In the second section, an extension of Darbo fixed point theorem and some related results are presented, while in the third section, we apply our extension to obtain a coupled fixed point. Finally, by applying our extension, a solution of an integral equation is obtained.
Now some notions, definitions and results which will be used in sequel are recalled. Let $(E, \|.\|)$ be a Banach space with the zero element θ. Denote by $B(\theta, r)$ the closed ball in E centered at θ with radius r. Assume B_r denote $B(\theta, r)$. If X be a subset of E, the closure and the closed convex hull of X will be denoted by $\overline{X}$ and Conv(X), respectively. Moreover, we use the symbols M_E and N_E to indicate the family of all nonempty and bounded subsets of E and the family of all nonempty and relatively compact sets, respectively.

Definition 1.1. *[6] A mapping $\mu : M_E \longrightarrow [0, \infty)$ is called a measure of noncompactness if it satisfies the following conditions:*

(1) The family $Ker\mu = \{X \in M_E : \mu(X) = 0\}$ is nonempty and $Ker\mu \subseteq N_E$.

(2) $X \subseteq Y \Longrightarrow \mu(X) \leq \mu(Y)$.

(3) $\mu(\overline{X}) = \mu(X)$.

(4) $\mu(Conv(X)) = \mu(X)$.

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(5) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ *for* $\lambda \in [0,1]$.

(6) If $\{X_n\}$ *is sequence of closed sets from* M_E *such that* $X_{n+1} \subseteq X_n$ *for* $n = 1,2,\ldots$ *and* $\lim_{n\to\infty}\mu(X_n) = 0$, *then* $\bigcap_{n=1}^{\infty} X_n$ *is nonempty.*

Now we state Darbo and Schauder fixed point theorems of [3, 11] which are used in the main results.

Theorem 1.2. *(Schauder) Let* U *be a nonempty, bounded, closed and convex subset of a Banach space* E*. Then every continuous and compact map* $F : U \longrightarrow U$ *has at least one fixed point in* U*.*

Theorem 1.3. *(Darbo) Let* Q *be a nonempty, closed, bounded and convex subset of the Banach space* E *and* $F : Q \longrightarrow Q$ *be a continuous function. Assume that there exists a constant* $k \in [0,1)$ *such that* $\mu(FX) \leq k\mu(X)$ *for any nonempty subset* X *of* Q*. Then* F *has a fixed point in* Q*.*

Aghajani et al. [1] generalized Darbo fixed point theorem as follows.

Theorem 1.4. *Let* Ω *be a nonempty, bounded, closed and convex subset of a Banach space* E*. Assume* $T : \Omega \longrightarrow \Omega$ *be a continuous operator such that*

$$\psi(\mu(TX)) \leq \psi(\mu(X)) - \varphi(\mu(X))$$

for any nonempty subset X *of* Ω*, where* μ *is an arbitrary measure of noncompactness and* $\varphi, \psi : [0,\infty) \longrightarrow [0,\infty)$ *are given functions such that* φ *is lower semicontinuous and* ψ *is continuous on* $[0,\infty)$*. Moreover,* $\varphi(0) = 0$ *and* $\varphi(t) > 0$ *for* $t > 0$*. Then* T *has at least one fixed point in* Ω*.*

Theorem 1.5. *Let* Ω *be a nonempty, bounded, closed and convex subset of a Banach space* E*. Suppose* $T : \Omega \longrightarrow \Omega$ *be a continuous operator satisfying*

$$\mu(TX) \leq \varphi(\mu(X))$$

for any nonempty subset X *of* Ω*, where* μ *is an arbitrary measure of noncompactness and* $\varphi : [0,\infty) \longrightarrow [0,\infty)$ *is nondecreasing function such that* $\lim_{n\to\infty}\varphi^n(t) = 0$ *for each* $t \geq 0$*. Then* T *has at least one fixed point in the set* Ω*.*

The following definition, theorem and example are used to prove the main result.

Definition 1.6. *([7]) An element* $(x,y) \in X \times X$ *is called coupled fixed point of a mapping* $F : X \times X \longrightarrow X$ *if* $F(x,y) = x$ *and* $F(y,x) = y$*.*

Theorem 1.7. *([6]) Assume* $\mu_1, \mu_2, \ldots, \mu_n$ *be the measure of noncompactness in* $E_1, E_2, \ldots, E_n$ *respectively. Suppose* $F : [0,\infty)^n \longrightarrow [0,\infty)$ *is convex and*
$F(x_1, x_2, \ldots, x_n) = 0$ *if and only if* $x_i = 0$ *for* $i = 1,2,\ldots,n$*. Then*

$$\mu(X) = F(\mu_1(X_1), \mu_2(X_2), \ldots, \mu_n(X_n))$$

defines a measure of noncompactness in $E_1 \times E_2 \times \cdots \times E_n$ *where* X_i *denote the natural projection of* X *into* E_i *for* $i = 1,2,\ldots,n$*.*

Example 1.8. *([2]) Let* μ *be a measure of noncompactness and* $F(x,y) = x + y$ *for* $(x,y) \in [0,\infty)^2$*. Then* F *has all the existent properties in Theorem 1.7. Hence* $\overline{\mu}(X) = \mu(X_1) + \mu(X_2)$ *is a measure of noncompactness in the space* $E \times E$ *where* $X_i, i = 1,2$ *denote the natural projections of* X *into* E*.*

2. Some Results Concerning the Darbo Fixed Point Theorem

Now motivated and inspired by Theorems 1.4, 1.5 and the existing contractive condition in [9], we state the main result of this paper which extends and generalizes Darbo fixed point theorem.

Theorem 2.1. *Let U be a nonempty, bounded, closed and convex subset of the Banach space E. Assume $F : U \longrightarrow U$ be a continuous operator satisfying*

$$\psi(\mu(F(X))) \leq \phi(\psi(\mu(X)))\psi(\mu(X)) \tag{1}$$

for all nonempty subset X of U, where μ is an arbitrary measure of noncompactness defined in E, $\psi : [0,\infty) \longrightarrow [0,\infty)$ is nondecreasing such that $\psi(t) = 0$ if and only if $t = 0$ and $\phi : [0,\infty) \longrightarrow [0,1)$ with $\limsup_{t\to r^+}\phi(t) < 1$ for all $r \geq 0$. Then F has a fixed point in U.

Proof. Let us define a sequence $\{U_n\}$ such that $U_0 = U$ and $U_n = Conv(FU_{n-1})$ for $n = 1, 2, \ldots$. If there exists an integer number n_0 such that $\mu(U_{n_0}) = 0$, then U_{n_0} is compact. So by Theorem 1.2 F has a fixed point in U. Assume that $\mu(U_n) > 0$ for $n \geq 0$. Due to (1), we deduce that

$$\psi(\mu(U_{n+1})) = \psi(\mu(Conv(FU_n))) = \psi(\mu(FU_n)) \leq \phi(\psi(\mu(U_n)))\psi(\mu(U_n)). \tag{2}$$

By (2), the sequence $\{\psi(\mu(U_n))\}$ is non-increasing. So, there exists $r \geq 0$ such that $\psi(\mu(U_n)) \longrightarrow r$. We claim that $r = 0$. Let $\beta_n = \psi(\mu(U_n))$ and $\beta = \sup_{n\in N}\phi(\beta_n)$. Thus, from [[14], Theorem 2.1] we have $\beta \in [0,1)$. Due to (2), we have

$$\beta_{n+1} \leq \phi(\beta_n)\beta_n \leq \beta\beta_n. \tag{3}$$

This implies that

$$\beta_{n+1} \leq \phi(\beta_n)\beta_n \leq \beta\beta_n \leq \beta^2\beta_{n-1} \leq \cdots \leq \beta^n\beta_1. \tag{4}$$

Letting $n \longrightarrow \infty$ in (4), we infer that $\beta_n = \psi(\mu(U_n)) \longrightarrow 0$. Now we indicate that $\mu(U_n) \longrightarrow 0$. Since $\{\psi(\mu(U_n)\}$ is a decreasing sequence and ψ is nondecreasing, we obtain that $\{\mu(U_n)\}$ is a decreasing sequence of positive numbers. This yields that there exists $t \geq 0$ such that $\lim_{n\to\infty}\mu(U_n) = t^+$. Since ψ is nondecreasing, we arrive that

$$\psi(t) \leq \psi(\mu(U_n)). \tag{5}$$

Letting $n \longrightarrow \infty$ in (5), we have $\psi(t) \leq 0$. So, $\psi(t) = 0$ which implies that $t = 0$ and we have $\mu(U_n) \longrightarrow 0$. Now keeping in mind that $U_{n+1} \subseteq U_n$ and $\lim_{n\to\infty}\mu(U_n) = 0$, from condition (6) of Definition 1.1 we conclude that the set $U_\infty = \cap_{n=1}^{\infty}U_n$ is nonempty, closed, convex and $U_\infty \subseteq U$. Moreover, taking in to account our assumptions, we infer that U_∞ is invariant under the operator F and $U_\infty \in Ker\mu$. Consequently, from Theorem 1.2 we deduce that F has a fixed point. □

The Darbo fixed point theorem is followed if $\psi = I$ and $\phi = k$ in the Theorem 2.1.

Corollary 2.2. *Let U be a nonempty, bounded, closed and convex subset of the Banach space E. Assume $F : U \longrightarrow U$ be a continuous operator such that*

$$\mu(F(X)) \leq k\mu(X) \tag{6}$$

for all nonempty subset X of U, where μ is an arbitrary measure of noncompactness defined in E and k is a constant in $[0,1)$. Then F has a fixed point in U.

Corollary 2.3. *Let U be a nonempty, bounded, closed and convex subset of the Banach space E. Assume $F : U \longrightarrow U$ be a continuous operator satisfying*

$$\mu(F(X)) \leq \alpha(\mu(X))\mu(X) \tag{7}$$

for all nonempty subset X of U, where μ is an arbitrary measure of noncompactness defined in E and $\alpha : [0,\infty) \longrightarrow [0,1)$ is a non-decreasing function. Then F has a fixed point in U.

Proof. Since α is non-decreasing, we have $\limsup_{t\to r^+} \alpha(t) < 1$ for all $r \geq 0$. Thus, by taking $\alpha = \phi$ and $\psi = I$ in (7) we have

$$\psi(\mu(F(X))) \leq \phi(\psi((\mu(X))))\psi(\mu(X)).$$

Now, Theorem 2.1 completes the proof. □

Corollary 2.4. *Let U be a nonempty, bounded, closed and convex subset of the Banach space E and $F : U \longrightarrow U$ be a continuous operator such that*

$$\mu(F(X)) \leq \phi(2\mu(X))\mu(X) \tag{8}$$

for all nonempty subset X of U, where μ is an arbitrary measure of noncompactness defined in E and $\phi : [0,\infty) \longrightarrow [0,1)$ such that $\limsup_{t\to r^+} \phi(t) < 1$ for all $r \geq 0$. Then F has a fixed point in U.

Proof. The result is followed by applying Theorem 2.1 with $\psi(t) = 2t$. □

3. Application to Coupled Fixed Point

Very recently, as an applications of measure of noncompactness, Aghajani and Sabzali [2] derived the following coupled fixed point theorem.

Theorem 3.1. *Assume Ω be a nonempty, bounded, closed and convex subset of the Banach space E and $G : \Omega\times\Omega \longrightarrow \Omega$ be a continuous function such that*

$$\mu(G(X_1 \times X_2)) \leq k \max\{\mu(X_1), \mu(X_2)\}$$

for any $X_1, X_2 \subseteq \Omega$, where μ is an arbitrary measure of noncompactness and k is a constant with $0 \leq k < 1$. Then G has at least a coupled fixed point.

Proof. [2] □

Let Ψ denote all functions $\psi : [0,\infty) \longrightarrow [0,\infty)$ such that

(1) ψ is nondecreasing and $\psi(t) = 0$ if and only if $t = 0$,

(2) $\psi(t + s) \leq \psi(t) + \psi(s)$ for all $t, s \geq 0$.

Now motivated and inspired by Theorem 3.1, we apply Theorem 2.1 and the concept of measure of noncompactness to prove the existence of a coupled fixed point for the function $F : U \times U \longrightarrow U$.

Theorem 3.2. *Let U be a nonempty, bounded, closed and convex subset of the Banach space E. Assume $F : U\times U \longrightarrow U$ be a continuous function such that*

$$\psi(\mu(F(X_1 \times X_1))) \leq \frac{1}{2}\phi(\psi(\mu(X_1) + \mu(X_2)))\psi(\mu(X_1) + \mu(X_2)) \tag{9}$$

for any $X_1, X_2 \subseteq U$, where μ is an arbitrary measure of noncompactness defined in E, $\psi \in \Psi$ and $\phi : [0,\infty) \longrightarrow [0,1)$ such that $\limsup_{t\to r^+} \phi(t) < 1$ for all $r \geq 0$. Then F has a coupled fixed point.

Proof. By taking $\widetilde{\mu}(X) = \mu(X_1) + \mu(X_2)$, we know that $\widetilde{\mu}$ is a measure of noncompactness in the space $E \times E$, where $X_i, i = 1, 2$ indicate the natural projections of X. Let us define the map $\widetilde{F} : U \times U \longrightarrow U \times U$ by $\widetilde{F}(x, y) = (F(x, y), F(y, x))$. Since F is continuous, $\widetilde{F}$ is also continuous. Let $X \subseteq U \times U$ be a nonempty subset. Hence, due to (9) and the condition (2) of Definition 1.1 we conclude that

$$\begin{aligned}
\psi(\widetilde{\mu}(\widetilde{F}(X))) &\leq \psi(\widetilde{\mu}(F(X_1 \times X_2) \times F(X_2 \times X_1))) \\
&= \psi(\mu(F(X_1 \times X_2)) + \psi(\mu(F(X_2 \times X_1))) \\
&\leq \phi(\psi(\mu(X_1) + \mu(X_2)))\psi(\mu(X_1) + \mu(X_2)) \\
&= \phi(\psi(\widetilde{\mu}(X)))\psi(\widetilde{\mu}(X)).
\end{aligned}$$

So

$$\psi(\widetilde{\mu}(\widetilde{F}(X))) \leq \phi(\psi(\widetilde{\mu}(X)))\psi(\widetilde{\mu}(X)). \tag{10}$$

Consequentially, by Theorem 2.1 $\widetilde{F}$ has a fixed point, which means that F has a coupled fixed point. □

Corollary 3.3. *Let U be a nonempty, bounded, closed and convex subset of the Banach space E. Assume $F : U \times U \longrightarrow U$ be a continuous function such that*

$$\mu(F(X_1 \times X_1)) \leq \frac{1}{2}\phi(2\mu(X_1) + 2\mu(X_2))(\mu(X_1) + \mu(X_2)) \tag{11}$$

for any $X_1, X_2 \subseteq U$, where μ is an arbitrary measure of noncompactness defined in E and $\phi : [0,\infty) \longrightarrow [0,1)$ is a function satisfying $\limsup_{t\to r^+}\phi(t) < 1$ for all $r \geq 0$. Then F has a coupled fixed point.

Proof. Applying Theorem 3.2 with $\psi(t) = 2t$, we have the result. □

4. Existence of Solution for an Integral Equation

In this section as an application of Theorem 2.1, we investigate the existence of solutions for the integral equation of the form

$$x(t) = f(t, x(t)) + \int_0^t g(t, s, x(s))ds, \tag{12}$$

where $t \geq 0$. Let $x \in BC(R^+)$ where $BC(R^+)$ is the space of all real functions defined on R^+ equipped with the norm

$$\|x\| = \sup\{|\, x \,|: t \geq 0\}.$$

Banas [5] constructed a measure of noncompactness in the space $BC(R^+)$. To present that measure, let us fix a nonempty bounded subset X of $BC(R^+)$ and a positive number $K > 0$. For $x \in X$ and $\varepsilon \geq 0$ Put

$$\omega^K(x,\varepsilon) = \sup\{|\, x(t) - y(t) \,|: t, s \in [0,K], |\, t - s \,| \leq \varepsilon\},$$

$$\omega^K(X,\varepsilon) = \sup\{\omega^K(x,\varepsilon) : x \in X\},$$

$$\omega_0^K(X) = \lim_{\varepsilon\to 0}\omega^K(X,\varepsilon),$$

$$\omega_0(X) = \lim_{K\to\infty}\omega_0^K(X).$$

Moreover, for a fixed number $t \in R^+$, let us put

$$X(t) = \{x(t) : x \in X\}$$

and

$$diamX(t) = \sup\{|\, x(t) - y(t) \,|: x, y \in X\}.$$

Now let the function μ is defined on $M_{BC(R^+)}$ as the follows:

$$\mu(X) = \omega_0(X) + \limsup_{t\to\infty} diamX(t).$$

Banas [5] showed that the function $\mu(X)$ defines a measure of noncompactness in the space $BC(R^+)$.
Let Φ^* denote all nondecreasing and upper semicontinuous functions $\phi : [0,\infty) \longrightarrow [0,1)$. Moreover, denote by Ψ^* all functions $\psi : R^+ \longrightarrow R^+$ such that

(1) ψ is nondecreasing and upper semicontinuous,

(1) $\psi(t) < t$ for all $t > 0$ and $\psi(t) = 0$ if and only if $t = 0$.

Assume that the functions applied in (12) satisfy the following conditions:

(1) $f : R^+ \times R \longrightarrow R$ is continuous and the function $t \longrightarrow f(t, 0)$ is a number of the space $BC(R^+)$.

(2) There exist functions $\phi \in \Phi^*$ and $\psi \in \Psi^*$ such that for any $t \in R^+$ and for all $x, y \in R$ we have

$$|f(t,x) - f(t,y)| \leq \frac{1}{2}\phi(\psi(|x-y|))\psi(|x-y|).$$

(3) The function $g : R^+ \times R^+ \times R \longrightarrow R$ is a continuous function and there exist continuous functions $a, b : R^+ \longrightarrow R^+$ such that

$$\lim_{t\to\infty} a(t)\int_0^t b(s)ds = 0 \quad \textit{and} \quad |g(t,s,x)| \leq a(t)b(s)$$

for all $t, s \in R^+$ with $s \leq t$ and for each $x \in R$.

(4) There exists a positive solution r_0 of the inequality

$$\frac{1}{2}\phi(\psi(r))\psi(r) + q \leq r,$$

where $q = \sup\{\mid f(t,0) \mid + a(t) \int_0^t b(s)ds\}$.

Now we can state the following theorem.

Theorem 4.1. *Let assumptions* (1) − (4) *are satisfied. Then the equation (12) has at least one solution* $x = x(t)$ *belonging to the space* $BC(R^+)$.

Proof. Let the operator T has been defined on the space $BC(R^+)$ as the follows:

$$(Tx)(t) = f(t, x(t)) + \int_0^t g(t, s, x(s))ds. \tag{13}$$

Keeping in mind our assumptions, we infer that Tx is continuous on R^+. Let x be an arbitrary element of $BC(R^+)$. Taking into account our assumptions, we deduce that

$$\begin{aligned} |(Tx)(t)| &\leq |f(t,x(t)) - f(t,0)| + |f(t,0)| + \textstyle\int_0^t \mid g(t,s,x(s) \mid ds \\ &\leq \tfrac{1}{2}\phi(\psi(\mid x(t) \mid))\psi(\mid x(t) \mid) + |f(t,0)| + a(t) \textstyle\int_0^t b(s)ds. \end{aligned} \tag{14}$$

Since ψ and ϕ are nondecreasing, in view of (14) we conclude that

$$\| Tx \| \leq \frac{1}{2}\phi(\psi(\| x \|))\psi(\| x \|) + \mid f(t,0) \mid + q, \tag{15}$$

where $q = \sup\{|f(t,0)| + a(t) \int_0^t b(s)ds\}$. Thus T transforms continuously $BC(R^+)$ into itself. From (15) and assumption (4) we conclude that T maps B_{r_0} into itself where r_0 is the existing constant in the assumption (4). Now we indicate that T is continuous on B_{r_0}. Let $\varepsilon > 0$ be an arbitrary fixed number and $x, y \in B_{r_0}$ such that $\| x - y \| \leq \varepsilon$. Taking into account our assumptions, we have

$$\begin{aligned} |(Tx)(t) - (Ty)(t)| &\leq \tfrac{1}{2}\phi(\psi(\mid x(t) - y(t) \mid))\psi(\mid x(t) - y(t) \mid) + \textstyle\int_0^t \mid g(t,s,x(s) - g(t,s,y(s) \mid ds \\ &\leq \tfrac{1}{2}\phi(\psi(\varepsilon))\psi(\varepsilon) + \textstyle\int_0^t \mid g(t,s,x(s) \mid ds \\ &\quad + \textstyle\int_0^t \mid g(t,s,y(s) \mid ds \leq \tfrac{1}{2}\phi(\psi(\varepsilon))\psi(\varepsilon) + 2c(t), \end{aligned} \tag{16}$$

where $c(t) = a(t)\int_0^t b(s)ds$. Moreover, from(3) we conclude that there exists $K > 0$ such that for all $t \geq K$ we have $2a(t)\int_0^t b(s)ds \leq \varepsilon$. So, from (16) we obtain that

$$|(Tx)(t) - (Ty)(t)| \leq \varepsilon + \frac{1}{2}\phi(\psi(\varepsilon))\psi(\varepsilon). \tag{17}$$

Now let the quantity $\omega^K(g,\varepsilon)$ is defined as follows:

$$\omega^K(g,\varepsilon) = \sup\{|g(t,s,x) - g(t,s,y)| : t,s \in [0,K], x,y \in [-r_0,r_0], |x-y| \leq \varepsilon\}.$$

The function $g(t,s,x)$ is uniformly continuous on the set $[0,K]\times[0,K]\times[-r_0,r_0]$. So $\omega^K(g,\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. On the other hand from (16) for an arbitrary fixed $t \in [0,K]$ we obtain

$$\begin{aligned} |(Tx)(t) - (Ty)(t)| &\leq \phi(\psi(\varepsilon))\psi(\varepsilon) + \int_0^K |\omega^K(g,\varepsilon)|ds \\ &= \phi(\psi(\varepsilon))\psi(\varepsilon) + K\omega^K(g,\varepsilon). \end{aligned} \tag{18}$$

Due to (17) and (18) we infer that T is continuous on B_{r_0}. Now let X be a nonempty subset of B_{r_0}, $\varepsilon > 0$ and $K > 0$ be arbitrary fixed numbers. Assume $t,s \in [0,K]$ such that $s \leq t$ and $|t-s| \leq \varepsilon$. In view of our assumptions, for $x \in X$ we get

$$\begin{aligned} |(Tx)(t) - (Tx)(s)| &\leq |f(t,x(t)) - f(s,x(s))| + \left|\int_0^t g(t,\tau,x(\tau))d\tau - \int_0^s g(s,\tau,x(\tau))d\tau\right| \\ &\leq |f(t,x(t)) - f(t,x(s))| + |f(t,x(s)) - f(s,x(s))| \\ &\quad + \left|\int_0^t g(t,\tau,x(\tau))d\tau - \int_0^t g(s,\tau,x(\tau))d\tau\right| + \left|\int_0^t g(s,\tau,x(\tau))d\tau - \int_0^s g(s,\tau,x(\tau))d\tau\right| \\ &\leq \omega_1^K(f,\varepsilon) + \tfrac{1}{2}\phi(\psi(|x(t)-x(s)|))\psi(|x(t)-x(s)|) \\ &\quad + \int_0^t |g(t,\tau,x(\tau)) - g(s,\tau,x(\tau))|d\tau + \int_s^t |g(s,\tau,x(\tau))|d\tau \\ &\leq \omega_1^K(f,\varepsilon) + \tfrac{1}{2}\phi(\psi(\omega^K(x,\varepsilon)))\psi(\omega^K(x,\varepsilon)) + \int_0^t \omega_1^K(g,\varepsilon)d\tau + a(s)\int_s^t b(\tau)d\tau \\ &\leq \omega_1^K(f,\varepsilon) + \tfrac{1}{2}\phi(\psi(\omega^K(x,\varepsilon)))\psi(\omega^K(x,\varepsilon)) + K\omega_1^K(g,\varepsilon) + \varepsilon\sup\{a(s)b(t), t,s \in [0,K]\}, \end{aligned} \tag{19}$$

where

$$\omega_1^K(f,\varepsilon) = \sup\{|f(t,x) - f(s,x)| : t,s \in [0,K], x \in [-r_0,r_0], |t-s| \leq \varepsilon\}$$

and

$$\omega_1^K(g,\varepsilon) = \sup\{|g(t,\tau,x) - g(s,\tau,x)| : t,s \in [0,K], x \in [-r_0,r_0], |t-s| \leq \varepsilon\}.$$

Since f is uniformly continuous on $[0,K]\times[-r_0,r_0]$ and the function g on the set $[0,K]\times[0,K]\times[-r_0,r_0]$, we obtain that $\omega_1^K(f,\varepsilon) \longrightarrow 0$ and $\omega_1^K(g,\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Hence, from (19) we obtain that

$$\omega_0^K(TX) \leq \frac{1}{2}\lim_{\varepsilon\to 0}\phi(\psi(\omega^K(x,\varepsilon)))\psi(\omega^K(x,\varepsilon)). \tag{20}$$

Since ϕ and ψ are upper semicontinuous, so from (20) we derive that

$$\omega_0^K(TX) \leq \frac{1}{2}\phi(\psi(\omega_0^K(X)))\psi(\omega_0^K(X)).$$

So

$$\omega_0(TX) \leq \frac{1}{2}\phi(\psi(\omega_0(X)))\psi(\omega_0(X)). \tag{21}$$

Let x, y be two arbitrary functions of X. Thus, for $t \in R^+$ we have

$$\begin{aligned} |(Tx)(t) - (Ty)(t)| &\leq |f(t,x(t)) - f(t,y(t))| + \int_0^t |g(t,s,x(s))|ds \\ &\quad + \int_0^t |g(t,s,y(s))|ds \leq \tfrac{1}{2}\phi(\psi(|x(t)-y(t)|))\psi(|x(t)-y(t)|) \\ &\quad + 2c(t), \end{aligned} \tag{22}$$

where $c(t) = a(t) \int_0^t b(s)ds$. Since ψ and ϕ are nondecreasing functions, so from (22) we have

$$diam(TX)(t) \leq \frac{1}{2}\phi(\psi(diam(X)(t)))\psi(diam(X)(t)) + 2c(t). \tag{23}$$

Now since ϕ and ψ are upper semicontinuous, from (23) we get

$$\limsup_{t\to\infty} diam(X)(t) \leq \frac{1}{2}\phi(\psi(\limsup_{t\to\infty} diam(X)(t)))\psi(\limsup_{t\to\infty} diam(X)(t)). \tag{24}$$

Using (21) and (24) and our assumptions we have

$$\begin{aligned} \psi(\omega_0(X) + \limsup_{t\to\infty} diam(X)(t)) &\leq \omega_0(X) + \limsup_{t\to\infty} diam(X)(t) \\ &\leq \phi(\psi(\omega_0(X) + \limsup_{t\to\infty} diam(X)(t)))\psi(\omega_0(X) \\ &\quad + \limsup_{t\to\infty} diam(X)(t)). \end{aligned}$$

So

$$\psi(\mu(TX)) \leq \phi(\psi(\mu(X)))\psi(\mu(X)).$$

From the above inequality and Theorem 2.1, we infer that there exists a solution of the the equation (12). □

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