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Some Necessary and Sufficient Conditions for Stability of Three-Layer Difference Schemes

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Abstract. This paper investigates two classes of three-layer difference schemes with weights in the form $\alpha y_{\hat{t},n}+\beta\tau^2 y_{\bar{t}\,t,n}+\sigma_1 Ay_{n-1}+(E-\sigma_1-\sigma_2)Ay_n+\sigma_2 Ay_{n+1}=\varphi_n$ and $\alpha y_{\hat{t},n}+\beta\tau^2 y_{\bar{t}\,t,n}+A(\sigma_1 y_{n-1}+(E-\sigma_1-\sigma_2)y_n+\sigma_2 y_{n+1})=\varphi_n$. It obtains some sufficient conditions for the stability in a defined norm and, also, in special cases we achieve conditions for stability which do not depend on the choice of norm.

1. Introduction

Any three-layer operator-differenc scheme can be written in the following canonical form:

$$By_{\hat{t},n}+\tau^2 Ry_{\bar{t}\,t,n}+Ay_n=0,\quad n=1,2,\dots$$
$$y_0, y_1 \text{are given} \tag{1}$$

where $y_n=y(t_n)$, $y_{\hat{t},n}=(y_{n+1}-y_{n-1})/(2\tau)$, $y_{\bar{t}\,t,n}=(y_{n+1}-2y_n+y_{n-1})/\tau^2$, A, B and R are linear operators defined on linear space H of grid function (see [1]). In the space H there is an inner product $(,)$ and the following theorem is obtained in [1].

Theorem 1.1. *If*

$$B\ge 0,\quad A^*=A>0,\quad R^*=R>\tfrac{1}{4}A, \tag{2}$$

then scheme (1) *is stable with respect to the initial data in the norm*

$$\|y_n\|_*=\left(\frac{1}{4}\|y_n+y_{n-1}\|^2+\|y_n+y_{n-1}\|^2_{R-\frac{1}{4}A}\right)^{\frac{1}{2}}. \tag{3}$$

The norm i better examined in [1], where it is obtained that if conditions

$$B\ge 0,\quad A^*=A>0,\quad R^*=R>\tfrac{1}{4}A,$$

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are fulfilled then

$$\sqrt{\frac{\varepsilon}{1+\varepsilon}}\|y_n\|_A \le \|y_n\|_* \le \|y_{n-1}\|_A + \tau\|y_{\bar{t},n}\|_R$$

It is also known that if $A > 0$ and $R > A/4$ that the condition $B \ge 0$ is necessary for stability in norm (3). Using Theorem 1 the fllowing result is not difficult to prove (see [3]):

Theorem 1.2. *Let exist inverse operators for operators K and L and*

$$\begin{aligned} &(LAK)^* = LAK > 0,\\ &(LRK)^* = LRK > \frac{1}{4}LAK,\\ &\quad LBK \ge 0. \end{aligned} \tag{4}$$

Then scheme (1) *is stable with respect to initial data in the norm*

$$\|y_n\|_* = \left(\frac{1}{4}\|y_n + y_{n-1}\|^2_{K^{*-1}LA} + \|y_n + y_{n-1}\|^2_{K^{*-1}L(R-\frac{1}{4}A)}\right)^{\frac{1}{2}}. \tag{5}$$

Using theorem 2 we can free ourselves of, for example, the condition that A s positive. Let $K = E$ and $L = A$ then from (4) we get the condition of stability

$$A^* = A, \quad A^{-1} \text{ exists}, \quad R^*A = AR > \frac{1}{4}A^2, \quad AB \ge 0 \tag{6}$$

in the norm define with operators $K^{*-1}LA = A^2$ and $K^{*-1}L(R - A/4) = A(R - A/4)$.
If $K = A$ and $L = E$ then from (4) we get the condition of stability:

$$A^* = A, \quad A^{-1} \text{ exists}, \quad AR^* = AR > \frac{1}{4}A^2, \quad BA \ge 0 \tag{7}$$

in the norm define with operators $K^{*-1}LA = E$ and $K^{*-1}L(R - A/4) = A^{-1}R - E/4$.

2. Three-layer difference schemes with weights

We consider two classes of three-layer schemes with weights in the form

$$\alpha y_{\hat{t},n} + \beta\tau^2 y_{\bar{t}t,n} + \sigma_1 A y_{n-1} + (E - \sigma_1 - \sigma_2)Ay_n + \sigma_2 A y_{n+1} = \varphi_n \tag{8}$$

and

$$\alpha y_{\hat{t},n} + \beta\tau^2 y_{\bar{t}t,n} + A(\sigma_1 y_{n-1} + (E - \sigma_1 - \sigma_2)y_n + \sigma_2 y_{n+1}) = \varphi_n \tag{9}$$

where α, β are given numbers, σ_1, σ_2 are operators in H.

Theorem 2.1. *If $A^* = A$, operator A^{-1} exists, $\sigma_1^* = \sigma_1$, $\sigma_2^* = \sigma_2$, $\mu = (\sigma_1 + \sigma_2)/2 - E/4$ and operators inequalities*

$$\begin{aligned} &\alpha A + \tau A(\sigma_1 - \sigma_2)A \ge 0,\\ &\beta A\mu A > 0 \end{aligned} \tag{10}$$

are fulfilled then scheme (8) *is stable with respect to the initial data in the norm*

$$\|y_n\|_* = \left(\frac{1}{4}\|y_n + y_{n-1}\|^2_{A^2} + \|y_n + y_{n-1}\|^2_{\beta A^{-1}+\mu)}\right)^{\frac{1}{2}}. \tag{11}$$

Proof. Scheme (8) has a canonic form (1) with $B = \alpha E + \tau(\sigma_1 - \sigma_2)A$, $R = \beta E + 0.5(\sigma_1 + \sigma_2)A$. Then it is not difficult to see that conditions (10) are equivalent with (6) which guarantee stability scheme in the norm (11). □

Theorem 2.2. *If $A^* = A$, operator A^{-1} exists, $\sigma_1^* = \sigma_1$, $\sigma_2^* = \sigma_2$, $\mu = (\sigma_1 + \sigma_2)/2 - E/4$ and operators inequalities* (10) *are fulfilled then scheme* (9) *is stable with respect to the initial data in the norm*

$$\|y_n\|_* = \left(\frac{1}{4}\|y_n + y_{n-1}\|^2 + \|y_n + y_{n-1}\|^2_{\beta A^{-1}+\mu)}\right)^{\frac{1}{2}}. \tag{12}$$

Proof. Scheme (9) has a canonic form (1) with $B = \alpha E + \tau A(\sigma_1 - \sigma_2)$, $R = \beta E + 0.5A(\sigma_1 + \sigma_2)$. For the scheme (9) inequalities (10) are equivalent with (7) which guarantee stability scheme in the norm (12). □

Conditions of stability, formulated in theorems 3 and 4 are connected with the choice of norm. For some schemes (8) and (9) when $\alpha = 0$, $\sigma_1 = \sigma_2 = \sigma$ (symmetric schemes) it is possible to prove that conditions (10) are necessary for stability, independent of the choice of norm.

Theorem 2.3. *Let $A^* = A$, $\sigma_1 = \sigma_2 = \sigma$ - self-adjoint operators, $\alpha = 0$, $\mu = \sigma - E/4$ and operator $(\beta E + \sigma A)^{-1}$ exist. If scheme* (8) *is stable in any space then*

$$\beta A + \tau A\mu A \geq 0. \tag{13}$$

Opposite, if

$$\beta A + \tau A\mu A > 0 \tag{14}$$

then scheme (8) *is stable with respect to the initial data in the norm* (5) *where $K = R^{-1}$ and $L = E$.*

Proof. Scheme (8) we can write in a equivalent form of a two-level scheme

$$Y_{n+1} = SY_n \tag{15}$$

where $S = (S_{\alpha\beta})$ matrix with elements $S_{11} = 0$, $S_{12} = E$, $S_{21} = -B_2^{-1}B_0$, $S_{22} = -B_2^{-1}B_1$ and $B_0 = R - B/(2\tau)$, $B_1 = A - 2R$, $B_2 = R + B/(2\tau)$. If $\sigma_1 = \sigma_2 = \sigma$ and $\alpha = 0$, then scheme has canonic form (1) with $B = 0$, $R = \beta E + \sigma A$. If s is any eigenvalue of operator S and $y = (y_1, y_2)^T$ is a corresponding eigenvector, then the eigenvalue problem $Sy = sy$ is the following

$$(s^2R + s(A - 2R) + R)y = 0. \tag{16}$$

As the condition in the theorem is that $R = \beta E + \sigma A$ has inverse operator therefore the equation (16) can be written in the form

$$(s^2E + s(AR^{-1} - 2E) + E)x = 0, \tag{17}$$

where $x = Ry$.

Now, it is obvious that the number $c = -(1 - s)^2/s$ is eigenvalue of operator AR^{-1}. If the scheme is stable in any norm, then $|s| \leq 1$, that is $0 \leq c \leq 4$. As operator AR^{-1} is self-adjoint inequality $0 \leq c \leq 4$ for all eigenvalues is equivalent to inequality:

$$0 \leq AR^{-1 \leq 4E}. \tag{18}$$

From (18) we got that $AR^{-1}(4E - AR^{-1}) \geq 0$ and

$$4AR^{-1} - (AR^{-1})^*(AR^{-1}) \geq 0$$

or

$$4AR^{-1} - (R^*)^{-1}A^2R^{-1} \geq 0.$$

The last inequality is equivalent to inequality $4R^*A - A^2 \geq 0$, which corresponds to (13). For the proof of the second part of the theorem it is sufficient to prove that (15) follows (4) if $K = R^{-1}$ and $L = E$. Those inequalities give:

$$R^*A > \frac{1}{4}A, \tag{19}$$

and

$$0 < AR^{-1} < 4E. \tag{20}$$

Condition (19) is equivalent to inequality $AR^{-1} > (R^*)^{-1}A^2R^{-1}/4$ or

$$AR^{-1} > (AR^{-1})^2/4. \tag{21}$$

Let λ be any eigenvalue of operator AR^{-1} and x is a corresponding eigenvector. Condition (21) gives $((AR^{-1})^2x, x) < 4(AR^{-1}x, x)$ or $((\lambda^2 - 4\lambda)x, x) < 0$. As $x \neq 0$ follows $\lambda^2 - 4\lambda < 0$ and $0 \leq \lambda_k \leq 4$ for all eigenvalue λ_k. Thath gives us (20). □

Theorem 2.4. *Let $A^* = A$, $\sigma_1 = \sigma_2 = \sigma$ - self-adjoint operators, $\alpha = 0$, $\mu = \sigma - E/4$ and operator $(\beta E + A\sigma)^{-1}$ exist. If scheme* (9) *is stable in any space then* (13) *exists.*

Conversely, if it satisfies (15) *then scheme* (9) *is stable with respect to the initial data in the norm* (5) *where $L = R^{-1}$ and $K = E$.*

Notice, that theorem 5 does not need assumption of the A positive. The following result assumed that A is a positive self-adjoint operator.

It is known that a positive self-adjoint operator can be decomposeable in the form $A = L^*L$, where $L : H \to H_1$, $L^* : H_1 \to H$, and H_1 is placeCityEuclid space different of H, generally. Let $(,)_H$ and $(,)_{H_1}$ be inner products in those spaces. Operator L^* is defined as an operator which satisfies $(Ly, v)_{H_1} = (y, L^*v)_H$ for every $y \in H$, $v \in H_1$.

Lemma 2.5. *Let $L : H \to H_1$, $L^* : H_1 \to H$, $A = L^*L > 0$, $\beta > 0$ and μ is a self-adjoint operator in H. Then inequality* (13) *is equivalent with inequality*

$$\beta E + L\mu L^* \geq 0,$$

and inequality (14) *is equivalent with inequality*

$$\beta E + L\mu L^* > 0.$$

The first part of lemma was proven in the papers [4] and [5], and the second part in principle is not different to the proof in the first part. From theorem 5 and lemma 1 follows

Theorem 2.6. *Let the conditions of theorem 5 be fulfilled and also $A = L^*L > 0$, $\beta > 0$. If scheme* (8) *is stable in any space then*

$$\beta E + \tau L\mu L^* \geq 0. \tag{22}$$

Opposite, if

$$\beta E + \tau L\mu L^* > 0 \tag{23}$$

then the scheme (8) *is stable with respect to the initial data.*

We will examine stability of the schemes with weights for the heat equation.

Consider the difference scheme

$$\begin{aligned} \frac{y_i^{n+1} - 2y_i^n + y_i^{n-1}}{\tau^2} &= \sigma_i y_{\bar{x}x,i}^{n+1} + (1-2\sigma_i) y_{\bar{x}x,i}^n + \sigma_i y_{\bar{x}x,i}^{n-1}, \quad i = 1,2,\ldots,N-1 \\ y_0^n &= y_N^n - 0, \quad n = 0,1,\ldots \\ y_i^0 &= u_0(x_i), \quad y_{\mathring{t},i} = u_1(x_i). \end{aligned} \tag{24}$$

Now $y_i^n = y(x_i, t_n)$, $x_i = ih$, $hN = 1$, $t_n = n\tau$, $\tau > 0$, $y_{\bar{x}x,i} = (y_{i+1} - 2y_i + y_{i+1})/h^2$, σ_i-numbers.

Difference scheme (24) has the form (8) where $y_n = (y_1^N, y_2^N, \ldots, y_{N-1}^N)^T$, $\sigma_1 = \sigma_2 = \sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_{N-1})$, $\alpha = 0$, $\beta = 1/\tau^2$,

$$(Ay)_i = -y_{\bar{x}x,i} \quad i = 1,2,\ldots,N-1 \quad y_0 = y_N = 0.$$

The main space H is space of function y_i with inner product and norm

$$(y,z) = \sum_{i=1}^{N-1} y_i z_i h, \quad \|y\| = \sqrt{(y,y)}.$$

It is known that A is a positive self-adjoint operator in H.

It is easy to see that operators $L : H \to H_1$, $L^* : H_1 \to H$, where H_1 is a set of functions $v_n = v(x_i)$, $x_i = ih$, $i = 1,2,\ldots,N$ and inner product $(v,w) = \sum_{i=1}^{N-1} v_i w_i h$, defined by

$$(Ly)_i = y_{\bar{x},i} = \frac{y_i - y_{i-1}}{h}, \quad i = 1,2,\ldots,N, \quad y_0 = y_N = 0.$$

$$(L^*b)_i = -v_{x,i} = -\frac{v_{i+1} - v_i}{h}, \quad i = 1,2,\ldots,N-1,$$

are both self-adjoint, where $A = L^*L$.

Therefore, scheme (24) satisfies all the conditions of theorem 7 and its stability or instability is defined by the operators

$$Q = E + \tau^2 L\mu L^*, \quad \mu = \sigma - \frac{1}{4}E. \tag{25}$$

In a difference form that operator is defined by the formulas

$$\begin{aligned} (Qv)_1 &= (1+\gamma\mu_1)v_1 - \gamma\mu_1 v_2, \\ (Qv)_2 &= -\gamma\mu_{i-1}v_{i-1} + (1 + \gamma(\mu_i + \mu_{i-1}))v_i \gamma\mu_i v_{i+1} -, \quad i = 2,3,\ldots,N-1, \\ (Qv)_N &= -\gamma\mu_{N-1}v_{N-1} + (1+\gamma\mu)v_N, \end{aligned}$$

where $\gamma = \tau^2/h^2$ and $\mu_i = \sigma_i - 1/4$, $i = 1,2,\ldots,N-1$.

The sign of operator Q is equivalent to the sign (positive or negative) of all eigenvalues of operator Q.

In some cases it is possible to find eigenvalues of matrix (25) in an analytic form. First of all is the case of constant operator σ, where $\sigma_i = \sigma$, $i = 1,2,\ldots,N-1$. Than theorem 7 gives us the following necessary condition for stability

$$\sigma \geq \frac{1}{4} - \frac{h^2}{4\tau^2},$$

and the known sufficient condition

$$\sigma > \frac{1}{4} - \frac{h^2}{4\tau^2}.$$

Now we will present a few subsequents of theorem 7.

Corollary 2.7. *Let $\sigma_1 = 1/2$ when i is odd, and $\sigma_i = 0$ when i is even. Then conditions $\tau \leq \sqrt{2}h$ is necessary for stability of scheme* (24) *in any norm and condition $\tau < \sqrt{2}h$ is sufficient for stability in norm* (5) *where $K = R^{-1}$ and $L = E$.*

Proof. Matrix Q can be written in the form $Q = E+\gamma M$, where $\gamma = \tau^2/h^2$ and M is a symmetric three-diagonal matrix with elements

$$m_11 = \mu_1, \quad m_{ii} = \mu_{i-1} + \mu_i, \quad i = 2, 3, \dots, N-1, \quad m_NN = \mu_{N-1}$$
$$m_{i,i+1} = m_{i+1,i} = -\mu_i, i = 1, 2, \dots, N-1.$$

In this case we have $\mu_i = (-1)^{i-1}/4$, $i = 2, 3, \dots, N-1$ and the eigenvalue problem $Mv = mv$ has the following form

$$(-1)^{i-1}(v_{i-1} - v_{i+1}) = 4mv_i, \quad i = 2, 3, \dots, N-1$$
$$v_1 - v_2 = 4mv_1, \quad (-1)^{N-1}(v_{N-1} - v_N) = 4mv_N.$$

Solutions of this problem are $m_k = 0.5 \sin(\pi k/N)$,

$$v_j^{(k)} = (-1)^{j(j-1)/2} \cos\left[\pi\left(\frac{k}{N} + \frac{1}{2}\right)\left(j - \frac{1}{2}\right)\right], \quad j = 1, 2, \dots, N,$$

where $k = 0, \pm1, \pm2, \dots, \pm(N/2-1), -N/2$, if N is even and $k = 0, \pm1, \pm2, \dots, \pm(N-1)/2$ if N is odd. For eigenvalues $q_k = 1 + 0.5\gamma \sin(\pi k/N)$ matrix (25) minimal is

$$q_{min} = \begin{cases} 1 - 0.5\gamma \cos \pi h, & \text{if N is even} \\ 1 - 0.5\gamma \cos \frac{\pi h}{2}, & \text{if N is odd.} \end{cases}$$

For fixed γ and $h \to 0$, from inequality $q_{min} \geq 0$ ($q_{\min} > 0$) we achieve the necessary condition of stability $\gamma \leq \sqrt{2}$ or $\tau \leq \sqrt{2}h$ and sufficient condition of stability $\tau < \sqrt{2}h$. □

Corollary 2.8. *Let $\sigma_i = 1/4$ when i is even. Then conditions*

$$\sigma_i \geq \frac{1}{4} - \frac{h^2}{2\tau^2} \text{ for odd } i \tag{26}$$

are necessary for the stability of scheme (24)*, and conditions*

$$\sigma_i \geq \frac{1}{4} - \frac{h^2}{2\tau^2} \text{ for odd } i \tag{27}$$

are sufficient for stability in norm (5)*, where $K = R^{-1}$ and $L = E$.*

Proof. As $\mu_i = 0$, when i is even, then equation $Mv = mv$ gives some independent eigen-value problems for the matrix of second order. Eigenvalues m_k for matrix M has the form:

$$m_k = 2\mu_k, \quad k = 1, 2, \dots, N-1$$

and condition $q_{\min} \leq 0$ is equivalent to (26) and $q_{\min} > 0$ to (27). □

Corollary 2.9. *Let $\sigma_2 = \sigma_3 = \dots = \sigma_N = \frac{1}{4}$. Then conditions*

$$\sigma_i \geq \frac{1}{4} - \frac{h^2}{2\tau^2} \text{ and } \sigma_{N-1} \geq \frac{1}{4} - \frac{h^2}{2\tau^2}$$

are necessary for stability, and conditions

$$\sigma_i \geq \frac{1}{4} - \frac{h^2}{2\tau^2} \text{ and } \sigma_{N-1} \geq \frac{1}{4} - \frac{h^2}{2\tau^2}$$

are sufficient for the stability of scheme (24).

3. Stability of three-layer difference schemes with self-adjoint operators

Three layer difference-scheme (1) we can write in an equivalent form of a two-layer difference scheme (see [8]) (15) where $y_n = (y^n, y^{n+1}) \in H^2$ and

$$S = \begin{pmatrix} 0 & E \\ -B_2^{-1}B_0 & -B_2^{-1}B_1 \end{pmatrix}.$$

Let s be any eigenvalue of operator S and $v = (x, y)^T$ is a corresponding eigen-vector. Then from eigenvalue problem $Sy = sy$ we get equations

$$y = sx, \quad -B_2^{-1}B_0x - -B_2^{-1}B_1y = sy,$$

and then

$$(s^2B_2 + sB_1 + B_0)x = 0, \tag{28}$$

where $B_0 = R - B/(2\tau)$, $B_1 = A - 2R$ and $B_2 = R + B/(2\tau)$ and

$$B/(2\tau) = \frac{B_1 - B_0}{2}, \quad R = \frac{B_2 - B_0}{2}, \quad A = B_2 + B_1 + B_0 = R - B/(2\tau).$$

Theorem 3.1. *Let A, B and R are self-adjoint operators and*

$$B_2 = R + B/(2\tau). \tag{29}$$

If difference scheme (1) *is stable in any norm for any eigen-vector x of the problem* (28) *inequalities*

$$(Bx, x) \leq 0, \quad (Ax, x) \leq 0, \quad ((4R - A)x, x) \leq 0, \tag{30}$$

are satisfied.

Proof. it is known (see [8]) that if three layer scheme is stable in any norm then all eigenvalues s of the problem (28) satisfy $|s| < 1$. If we multiply equation (28) with x we get quadratic equation

$$s^2(B_2x, x) + s(B_1x, x) + (B_0x, x) = 0, \tag{31}$$

with a real coefficient (B_ix, x), $i = 0, 1, 2$and positive coefficient (B_2x, x). Both solutions of the (31) have a moduo not greater than 1 if it satisfies

$$(B_1x, x) \leq (B_2x, x) \text{ and } (B_0x, x) \pm (B_1x, x) + (B_2x, x) \geq 0$$

which is equivalent to (30). □

Let us now consider the quadratic problem (28) and linear problem for eigenvalues

$$\begin{aligned} B_0v &= \alpha B_2v \\ B_1w &= \alpha B_2w. \end{aligned} \tag{32}$$

Let operators B_0, B_1, B_2 satisfy

$$\alpha_2B_2 + \alpha_1B_1 + \alpha_0B_0 = 0, \tag{33}$$

where α_0, α_1, α_2 are real numbers.

The following result is obtained (see [7]).

Lemma 3.2. *Let B_0, B_1 and $B_2 > 0$ self-adjoint operators which satisfy* (33) *and $a_0a_2 - a_1^2 \neq 0$. Then if $a_1 \neq 0$ ($a_0 \neq 0$), problems* (28) *and* (32) *have the same set of eigenvalues which generate B_2 the orthogonal basis in H.*

Let us define operator L as $L = l_2B_2 + l_1B_1 + l_0B_0$, where l_0, l_1, and l_2 are real numbers. Let $\{x_k\}_{k=1}^n$ basis of B_2 orthogonal vectors.

Lemma 3.3. *Let all conditions of lemma 2 be fulfilled and* $(Lx_k, x_k) \geq 0$ *for all* $k = 1, 2, \ldots, m$. *Then* $L \leq 0$.

Proof. If $a_0a_2 \neq 0$ then according to lemma 2, the numbers α and β exist such that:

$$B_0x_k = \alpha B_2x_k, \quad B_1x_k = \alpha B_2x_k.$$

Then $Lx_k = \lambda_k B_2x_k$ for $\lambda_k = \alpha l_0 + \beta l_1 + l_2$ and $\{x_k\}_{k=1}^n$ is set of B_2 orthogonal vecotr. From $(Lx_k, x_k) \geq 0$, we get $\lambda_k \geq 0$, $k = 1, 2, \ldots, m$. For $x \in H$, written as $x = \sum_{k=1}^n c_kx_k$ we obtain:

$$(Lx, x) = \sum_{k,i=1}^n c_kc_i\lambda_k(B_2x_k, x_i) = \sum_{k,i=1}^n (c_k)^2\lambda_k(B_2x_k, x_i) \geq 0,$$

which gives $L \geq 0$. If $a_0 = 0$ and $a_1 \neq 0$ according to lemma 2 we have that $B_0x_k = \alpha B_2x_k$, and from (33) subsequently $B_1x_k = -\frac{a_2}{a_1}B_2x_k$. Therefore, $L_kx_k = \lambda_kB_2x_k$, where $\lambda_k = \alpha l_0 + \beta l_1 + l_2$ and $\beta = -a_2/a_1$. Again in the same way we obtain that $\lambda_k \geq 0$ for all k and $L \geq 0$. Analogously, we get the same result for $a_1 = 0$ and $a_0 \neq 0$. □

Equality (33) we can write in the form

$$b_0\frac{B}{2\tau} + b_1R + b_2A = 0, \tag{34}$$

where $b_0 = a_2 - a_0$, $b_2 = a_1$ and $b_1 = a2 - 2a_1 + a_0$. Then condition $a_0a_2 - a_1^2 \neq 0$ is equivalent to

$$b_1^2 + 4b_1b_2 - b_0^2 \neq 0. \tag{35}$$

Theorem 3.4. *Let A, B and R be self adjoint operators and* (34) *is satisfied. Let* $B/(2\tau) + R > 0$ *and* (35) *is satisfied. If scheme* (1) *is stable in any space then*

$$B \leq 0, \quad A \leq 0, \quad R \leq \frac{1}{4}A. \tag{36}$$

Conversely, if

$$B \leq 0, \quad A > 0, \quad R > \frac{1}{4}A, \tag{37}$$

then scheme (1) *is stable with respect to the initial data in the norm* (3).

Proof. Conditions (37) are sufficient wfor stability in norm (3) (theorem 1). To prove the first part of the theorem note that ll conditions of theorem 8 are fulfilled. Therefore for all eigen-vector x_k, $k = 1, 2, \ldots, m$ of the problem (28) inequalities (30) are satisfied. They can be written in the form

$$((B_2 - B_0)x, x) \leq 0 \text{ and } ((B_2 \pm B_1 + B_0)x, x) \leq .$$

Therefore according to lemma 3 we get that inequalities (36) are satisfied. □

Corollary 3.5. *Let A and* $R > 0$ *be self-adjoint operators and* $B = 0$. *If the difference scheme* (1) *is stable in any space then*

$$A \geq 0, \quad R \geq \frac{1}{4}A.$$

Conversely, if

$$A > 0, \quad R > \frac{1}{4}A, \tag{38}$$

then scheme (1) *is stable with respect to the initial data in the norm* (3).

Proof. It is enough to see that if $B = 0$, then the condition (34) is satisfied for $b_0 = 1$ and $b_1 = b_2 = 0$. □

Corollary 3.6. *Let A and $B > 0$ be self-adjoint operators and $R == 0$. If the difference scheme* (1) *i stable in any space then $A = 0$.*

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