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Parabolic-Hyperbolic Transmission Problem in Disjoint Domains

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Abstract. In applications, especially in engineering, often are encountered composite or layered structures, where the properties of individual layers can vary considerably from the properties of the surrounding material. Layers can be structural, thermal, electromagnetic or optical, etc. Mathematical models of energy and mass transfer in domains with layers lead to so called transmission problems.
In this paper we investigate a mixed parabolic-hyperbolic initial-boundary value problem in two non-adjacent rectangles with nonlocal integral conjugation conditions. It was considered more examples of physical and engineering tasks which are reduced to transmission problems of similar type. For the model problem the existence and uniqueness of its weak solution in appropriate Sobolev-like space is proved. A finite difference scheme approximating this problem is proposed and analyzed.

1. Formulation of the problem

In this paper we consider the transmission problem whose solution is defined in two separated and disconnected areas. Such a situation occurs when the solution in the intermediate region is known, or can be determined from a simpler equation. The effect of the intermediate region can be modelled [2, 3, 5–7, 16, 24] by means of nonlocal jump conditions across the intermediate region. Transmission problems in separated and disconnected domains are represented by systems of partial differential equations [26] with additional conditions (initial and boundary conditions, nonlocal jump conditions). The problems of such types were investigated by many authors [5, 6, 8, 16, 24]. In particular, an one and two-dimensional elliptic problem in two disjoint domains was studied in [14, 17, 22]. Analogous parabolic problem with Dirichlet and Robin boundary condition, in one and two-dimensional case, and its numerical approximation, was investigated in [10, 11, 15, 20, 21, 23] and hyperbolic problem with Dirichlet boundary condition, in two-dimensional case, was investigated in [12]. Real problems that can be presented as transmission problems are heat transfer by radiation in the black body system, heat transfer in a wall in which there is a cavity, heat transfer in a wall that is a layered or composite structure. Also, problem arising from the interaction of electromagnetic waves traveling in air with a metallic body and many others. As a model example,we consider the following parabolic-hyperbolic system equations:

$$\frac{\partial u_1}{\partial t} - \frac{\partial}{\partial x}\left(p_1(x,y)\frac{\partial u_1}{\partial x}\right) - \frac{\partial}{\partial y}\left(q_1(x,y)\frac{\partial u_1}{\partial y}\right) = f_1(x,y,t), \tag{1}$$

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$$(x,y) \in \Omega_1 = (a_1,b_1)\times(c_1,d_1), \quad t>0,$$
$$\frac{\partial^2 u_2}{\partial t^2} - \frac{\partial}{\partial x}\left(p_2(x,y)\frac{\partial u_2}{\partial x}\right) - \frac{\partial}{\partial y}\left(q_2(x,y)\frac{\partial u_2}{\partial y}\right) = f_2(x,y,t), \tag{2}$$
$$(x,y) \in \Omega_2 = (a_2,b_2)\times(c_2,d_2), \quad t>0,$$

where $-\infty < a_1 < b_1 < a_2 < b_2 < +\infty$ and $c_2 < c_1 < d_1 < d_2$,
the initial conditions

$$\begin{aligned} &u_1(x,y,0) = u_{10}(x,y), \quad (x,y)\in\Omega_1,\\ &u_2(x,y,0) = u_{20}(x,y), \quad \frac{\partial u_2}{\partial t}(x,y,0) = u_{21}(x,y), \quad (x,y)\in\Omega_2 \end{aligned} \tag{3}$$

the simplest external Dirichlet boundary conditions

$$\begin{aligned} &u_1(a_1,y,t) = 0, \quad y\in(c_1,d_1), \quad u_2(x,c_2,t)=0, \quad x\in(a_2,b_2),\\ &u_2(b_2,y,t) = 0, \quad y\in(c_2,d_2), \quad u_2(x,d_2,t)=0, \quad x\in(a_2,b_2), \end{aligned} \tag{4}$$

and the internal conjugation conditions of non-local Robin-Dirichlet type

$$p_1(b_1,y)\frac{\partial u_1}{\partial x}(b_1,y,t) + \alpha_1(y)u_1(b_1,y,t) = \int_{c_2}^{d_2}\beta_1(y,y')u_2(a_2,y',t)\,dy', \tag{5}$$

$$\begin{aligned} &-p_2(a_2,y)\frac{\partial u_2}{\partial x}(a_2,y,t) + \alpha_2(y)u_2(a_2,y,t) =\\ &\begin{cases} \int_{c_1}^{d_1}\beta_2(y,y')u_1(b_1,y',t)\,dy' + \int_{a_1}^{b_1}\check\beta_2(y,x')u_1(x',c_1,t)\,dx', & y\in(c_2,c_1),\\ \int_{c_1}^{d_1}\beta_2(y,y')u_1(b_1,y',t)\,dy', & y\in(c_1,d_1),\\ \int_{c_1}^{d_1}\beta_2(y,y')u_1(b_1,y',t)\,dy' + \int_{a_1}^{b_1}\hat\beta_2(y,x')u_1(x',d_1,t)\,dx', & y\in(d_1,d_2), \end{cases} \end{aligned} \tag{6}$$

$$-q_1(x,c_1)\frac{\partial u_1}{\partial y}(x,c_1,t) + \check\alpha_1(x)u_1(x,c_1,t) = \int_{c_2}^{c_1}\check\beta_1(x,y')u_2(a_2,y',t)\,dy', \tag{7}$$

$$q_1(x,d_1)\frac{\partial u_1}{\partial y}(x,d_1,t) + \hat\alpha_1(x)u_1(x,d_1,t) = \int_{d_1}^{d_2}\hat\beta_1(x,y')u_2(a_2,y',t)\,dy'. \tag{8}$$

Throughout the paper we assume that the input data satisfy the usual regularity and ellipticity conditions

$$p_i(x,y),\ q_i(x,y)\in L_\infty(\Omega_i), \quad i=1,2, \tag{9}$$

$$0 < p_{i0} \le p_i(x,y),\ \ 0 < q_{i0} \le q_i(x,y), \ \ \text{a.e in } \Omega_i, \quad i=1,2 \tag{10}$$

and

$$\begin{aligned} &\alpha_i \in L_\infty(c_i,d_i), \quad i=1,2, \qquad \check\alpha_1,\ \hat\alpha_1 \in L_\infty(a_1,b_1),\\ &\beta_i \in L_\infty((c_i,d_i)\times(c_{3-i},d_{3-i})), \quad i=1,2,\\ &\check\beta_1 \in L_\infty((a_1,b_1)\times(c_2,c_1)), \ \ \hat\beta_1 \in L_\infty((a_1,b_1)\times(d_1,d_2)),\\ &\check\beta_2 \in L_\infty((c_2,c_1)\times(a_1,b_1)), \ \ \hat\beta_2 \in L_\infty((d_1,d_2)\times(a_1,b_1)). \end{aligned} \tag{11}$$

In real physical problems [1] we also often have

$$\alpha_i>0, \ \ \beta_i>0, \ \ \hat\alpha_1>0, \ \ \check\alpha_1>0, \ \ \hat\beta_i>0, \ \ \check\beta_i>0, \quad i=1,2. \tag{12}$$

2. Existence and uniqueness of weak solutions

We introduce the product space

$$L_2 = L_2(\Omega_1) \times L_2(\Omega_2) = \{v = (v_1, v_2) | v_i \in L_2(\Omega_i)\},$$

endowed with the inner product and associated norm

$$(u, v)_{L_2} = (u_1, v_1)_{L_2(\Omega_1)} + (u_2, v_2)_{L_2(\Omega_2)}, \quad \|v\|_{L_2} = (v, v)_{L_2}^{1/2},$$

where

$$(u_i, v_i)_{L_2(\Omega_i)} = \iint_{\Omega_i} u_i v_i \, dxdy, \quad i = 1, 2.$$

We also define the spaces

$$H^k = \{v = (v_1, v_2) | \; v_i \in H^k(\Omega_i)\}, \quad k = 1, 2, ...$$

endowed with the inner product and associated norm

$$(u, v)_{H^k} = (u_1, v_1)_{H^k(\Omega_1)} + (u_2, v_2)_{H^k(\Omega_2)}, \quad \|v\|_{H^k} = (v, v)_{H^k}^{1/2},$$

where $H^k(\Omega_i)$ are the standard Sobolev spaces. In particular, we set

$$H_0^1 = \left\{ v \in H^1 \,|\, v_1(a_1, y) = 0, \;\; v_2(b_2, y) = v_2(x, c_2) = v_2(x, d_2) = 0 \right\}.$$

Finally, with $u = (u_1, u_2)$ and $v = (v_1, v_2)$ we define the following bilinear form:

$$\begin{aligned}
A(u, v) &= \iint_{\Omega_1} \Big(p_1 \frac{\partial u_1}{\partial x} \frac{\partial v_1}{\partial x} + q_1 \frac{\partial u_1}{\partial y} \frac{\partial v_1}{\partial y} \Big) \, dxdy \\
&+ \iint_{\Omega_2} \Big(p_2 \frac{\partial u_2}{\partial x} \frac{\partial v_2}{\partial x} + q_2 \frac{\partial u_2}{\partial y} \frac{\partial v_2}{\partial y} \Big) \, dxdy \\
&+ \int_{c_1}^{d_1} \alpha_1(y) \, u_1(b_1, y) v_1(b_1, y) \, dy + \int_{c_2}^{d_2} \alpha_2(y) \, u_2(a_2, y) v_2(a_2, y) \, dy \\
&+ \int_{a_1}^{b_1} \check{\alpha}_1(x) \, u_1(x, c_1) v_1(x, c_1) \, dx + \int_{a_1}^{b_1} \hat{\alpha}_1(x) \, u_1(x, d_1) v_1(x, d_1) \, dx \\
&- \int_{c_1}^{d_1}\int_{c_2}^{d_2} \beta_1(y, y') u_2(a_2, y') v_1(b_1, y) \, dy' dy - \int_{c_2}^{d_2}\int_{c_1}^{d_1} \beta_2(y, y') u_1(b_1, y') v_2(a_2, y) \, dy' dy \\
&- \int_{a_1}^{b_1}\int_{c_2}^{c_1} \check{\beta}_1(x, y) u_2(a_2, y) v_1(x, c_1) \, dydx - \int_{a_1}^{b_1}\int_{d_1}^{d_2} \hat{\beta}_1(x, y) u_2(a_2, y) v_1(x, d_1) \, dydx \\
&- \int_{c_2}^{c_1}\int_{a_1}^{b_1} \check{\beta}_2(y, x) u_1(x, c_1) v_2(a_2, y) \, dxdy - \int_{d_1}^{d_2}\int_{a_1}^{b_1} \hat{\beta}_2(y, x) u_1(x, d_1) v_2(a_2, y) \, dxdy.
\end{aligned} \tag{13}$$

Lemma 2.1. *Under the conditions* (9) *and* (11) *the bilinear form A, defined by* (13), *is bounded on $H^1 \times H^1$. If besides it the conditions* (10) *are fulfilled, this form satisfies the Gårding's inequality on H_0^1, i.e. there exist positive constants m and κ such that*

$$A(u, u) + \kappa \|u\|_{L_2}^2 \geq m \|u\|_{H^1}^2, \quad \forall \, u \in H_0^1.$$

Proof. The proof is analogous to the proof of Lemma 1 in [20]. Boundedness of A follows from (9) and (11) and the trace theorem for the Sobolev spaces

$$\|u_i\|_{L_2(\partial\Omega_i)} \leq C\|u_i\|_{H^1(\Omega_i)}.$$

Gårding's inequality follows from (10), (13), multiplicative trace inequality (see, e.g., Prop. 1.6.3 in [4])

$$\|u_i\|^2_{L_2(\partial\Omega_i)} \leq C\|u_i\|_{L_2(\Omega_i)}\|u_i\|_{H^1(\Omega_i)}, \ i = 1, 2.$$

Poincaré, Cauchy-Schwartz and ϵ-inequalities, for sufficiently small $\epsilon > 0$. □
It is easy to prove the following assertions.

Lemma 2.2. *If the condition* $\beta_1(y, y') = \beta_2(y', y)$, $\breve{\beta}_1(x, y) = \breve{\beta}_2(y, x)$ *and* $\hat{\beta}_1(x, y) = \hat{\beta}_2(y, x)$ *are satisfied the bilinear form A, defined by* (13)*, is symmetric on* $H^1 \times H^1$.

Lemma 2.3. *If* α_k, $\breve{\alpha}_1(x)$ *and* $\hat{\alpha}_1(x)$ *are positive and* β_k, $\breve{\beta}_k$ *and* $\hat{\beta}_k$ *sufficiently small, then the bilinear form A is coercive, i.e.* $\kappa = 0$. *Sufficient conditions are*

$$\begin{aligned} &|\beta_1(y, y') + \beta_2(y', y)| \leq \frac{2\sqrt{\alpha_1(y)\,\alpha_2(y')}}{\sqrt{3(d_2 - c_2)(d_1 - c_1)}}, \quad |\breve{\beta}_1(x, y') + \breve{\beta}_2(y', x)| \leq \frac{2\sqrt{\breve{\alpha}_1(x)\,\alpha_2(y')}}{\sqrt{3(b_1 - a_1)(c_2 - c_1)}} \\ &\textit{and} \quad |\hat{\beta}_1(x, y') + \hat{\beta}_2(y', x)| \leq \frac{2\sqrt{\hat{\alpha}_1(x)\,\alpha_2(y')}}{\sqrt{3(b_1 - a_1)(d_2 - d_1)}}. \end{aligned} \tag{14}$$

Let Ω be a domain in $\mathbb{R}^n$ and $u(t)$ a function mapping Ω into Hilbert space H. In the usual manner [18] we define the spaces $L_p(\Omega, H)$, $p \geq 1$, endowed with the norm

$$\|u\|_{L_p(\Omega,H)} = \Big(\int_\Omega \|u(t)\|^p_H \,\mathrm{d}t\Big)^{1/p}, \quad 1 \leq p < \infty,$$

with the standard modification for $p = \infty$. We also define Sobolev spaces of vector-valued functions $H^k(\Omega, H)$, endowed with the inner product

$$(u, v)_{H^k(\Omega,H)} = \int_\Omega \sum_{|\alpha|\leq k} (D^\alpha u(t), D^\alpha v(t))_H \,\mathrm{d}t, \quad k = 0, 1, 2, ...$$

where denoted $D^\alpha = \frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_n}}{\partial t_1^{\alpha_1}\partial t_2^{\alpha_2}...\partial t_n^{\alpha_n}}$. We set $L_2(\Omega, H) = H^0(\Omega, H)$. Multiplying equation (1) by $v_1(x, y, t)$ and equation (2) by $v_2(x, y, t)$, and integrating by parts using condition (4)-(8), we get the weak form of (1)-(8):

$$\Big(\frac{\partial u_1}{\partial t}(\cdot, t), v_1(\cdot, t)\Big)_{L_2(\Omega_1)} + \Big(\frac{\partial^2 u_2}{\partial t^2}(\cdot, t), v_2(\cdot, t)\Big)_{L_2(\Omega_2)} + A(u(\cdot, t), v(\cdot, t)) = (f(\cdot, t), v(\cdot, t))_{L_2}. \tag{15}$$

Using a similar technique as in [13] and Lemma 2.1 one immediately obtains the following result.

Theorem 2.4. *Let the assumptions* (9)-(12) *and* (14) *holds and suppose that* $u_0 = (u_{10}, u_{20}) \in H^1_0$, $u_{21} \in L_2(\Omega_2)$ *and* $f_1 \in L_2((0, T), L_2(\Omega_1))$, $f_2 \in L_1((0, T), L_2(\Omega_2))$. *Then for* $0 < T < +\infty$ *the initial-boundary-value problem* (1)-(8) *has a unique weak solution* $u \in L_\infty((0, T), H^1)$ *such that* $\frac{\partial u_1}{\partial t} \in L_2((0, T), L_2(\Omega_1))$ *and* $\frac{\partial u_2}{\partial t} \in L_\infty((0, T), L_2(\Omega_2))$.
If $\frac{\partial p_1}{\partial x} \in L_\infty(\Omega_1)$ *and* $\frac{\partial q_1}{\partial y} \in L_\infty(\Omega_1)$, *then also* $\frac{\partial^2 u_1}{\partial x^2} \in L_2((0, T), L_2(\Omega_1))$ *and* $\frac{\partial^2 u_1}{\partial y^2} \in L_2((0, T), L_2(\Omega_1))$.

3. Finite difference approximation

Let $\overline{\omega}_{i,h_i}$ be a uniform mesh in $[a_i, b_i]$, with step size $h_i = (b_i - a_i)/(n_i - 1/2)$, $i = 1, 2$. We denote $\omega_{i,h_i} := \overline{\omega}_{i,h_i} \cap (a_i, b_i)$, $\omega^-_{i,h_i} := \omega_{i,h_i} \cup \{a_i\}$, $\omega^+_{i,h_i} := \omega_{i,h_i} \cup \{b_i\}$, $\hat{\omega}_{1,h_1} := \{x_{1,j_1-1/2} = a_1 + (j_1 - 1/2)h_1 | j_1 = 1, 2, \ldots, n_1\}$, $\hat{\omega}_{2,h_2} := \{x_{2,j_2-1/2} = b_2 - (j_2 - 1/2)h_2 | j_2 = 1, 2, \ldots, n_2\}$. Analogously we define a uniform mesh $\overline{\omega}_{i,k_i}$ in $[c_i, d_i]$, with the step size $k_i = (d_i - c_i)/(m_i - 1/2)$, $i = 1, 2$ and its submeshes $\omega_{i,k_i} := \overline{\omega}_{i,k_i} \cap (c_i, d_i)$, $\omega^-_{i,k_i} := \omega_{i,k_i} \cup \{c_i\}$, $\omega^+_{i,k_i} := \omega_{i,k_i} \cup \{d_i\}$, $\hat{\omega}_{2,k_2} := \{x_{2,j_3-1/2} = c_2 + (j_3 - 1/2)k_2 | j_3 = 1, 2, \ldots, m_2\}$, $\hat{\omega}_{2,k_2} := \{x_{2,j_3-1/2} = d_2 - (j_3 - 1/2)k_2 | j_3 = 1, 2, \ldots, m_2\}$. We assume that $h_1 \asymp h_2 \asymp k_1 \asymp k_2$. Finally, we introduce a uniform mesh $\overline{\omega}_\tau$ in $[0, T]$with step size $\tau = T/n$ and set $\omega_\tau := \overline{\omega}_\tau \cap (0, T)$, $\omega^-_\tau := \omega_\tau \cup \{0\}$, $\omega^+_\tau := \omega_\tau \cup \{T\}$, $\hat{\omega}_\tau := \{t_{j-1/2} = (j-1/2)\tau | j = 1, 2, \ldots, n\}$. We will consider vector-functions of the form $v = (v_1, v_2)$ where v_i is a mesh function defined on $\overline{\omega}_{i,h_i} \times \overline{\omega}_{i,k_i} \times \overline{\omega}_\tau$, $i = 1, 2$. We define difference quotients in the following way:

$$v_i = v_i(x, y, t)$$

$$\delta_x v_i = \frac{v_i(x + \frac{h_i}{2}, y, t) - v_i(x - \frac{h_i}{2}, y, t)}{h_i}, \quad \delta_x^2 v_i = \delta_x(\delta_x v_i)$$

$$\delta_y v_i = \frac{v_i(x, y + \frac{k_i}{2}, t) - v_i(x, y - \frac{k_i}{2}, t)}{k_i}, \quad \delta_y^2 v_i = \delta_y(\delta_y v_i)$$

$$\hat{v}_i = \frac{v_i(x, y, t + \frac{\tau}{2}) + v_i(x, y, t - \frac{\tau}{2})}{2}$$

$$\hat{\hat{v}}_i = \frac{\hat{v}_i(x, y, t + \frac{\tau}{2}) + \hat{v}_i(x, y, t - \frac{\tau}{2})}{2} = \frac{v_i(x, y, t + \tau) + 2v_i(x, y, t) + v_i(x, y, t - \tau)}{4}$$

$$\delta_t v_i = \frac{v_i(x, y, t + \frac{\tau}{2}) - v_i(x, y, t - \frac{\tau}{2})}{\tau}, \quad \delta_t^2 v_i = \delta_t(\delta_t v_i), \quad \hat{\delta}_t v_i = \delta_t \hat{v}_i = \frac{v_i(x, y, t + \tau) - v_i(x, y, t - \tau)}{2\tau}.$$

We define the following discrete inner products and norms:

$$(v_i, w_i)_{L_2(\omega_{i,h_i} \times \omega_{i,k_i})} = h_i k_i \sum_{x \in \omega_{i,h_i}} \sum_{y \in \omega_{i,k_i}} v_i(x, y) w_i(x, y), \quad \|v_i\|^2_{L_2(\omega_{i,h_i} \times \omega_{i,k_i})} = (v_i, v_i)_{L_2(\omega_{i,h_i} \times \omega_{i,k_i})}$$

$$(v_i, w_i)_{L_2(\hat{\omega}_{i,h_i} \times \omega_{i,k_i})} = h_i k_i \sum_{x \in \hat{\omega}_{i,h_i}} \sum_{y \in \omega_{i,k_i}} v_i(x, y) w_i(x, y), \quad \|v_i\|^2_{L_2(\hat{\omega}_{i,h_i} \times \omega_{i,k_i})} = (v_i, v_i)_{L_2(\hat{\omega}_{i,h_i} \times \omega_{i,k_i})}$$

$$(v_i, w_i)_{L_2(\omega_{i,h_i} \times \hat{\omega}_{i,k_i})} = h_i k_i \sum_{x \in \omega_{i,h_i}} \sum_{y \in \hat{\omega}_{i,k_i}} v_i(x, y) w_i(x, y), \quad \|v_i\|^2_{L_2(\omega_{i,h_i} \times \hat{\omega}_{i,k_i})} = (v_i, v_i)_{L_2(\omega_{i,h_i} \times \hat{\omega}_{i,k_i})}$$

$$(v, w)_{L_h} = (v_1, w_1)_{L_2(\omega_{1,h_1} \times \omega_{1,k_1})} + (v_2, w_2)_{L_2(\omega_{2,h_2} \times \omega_{2,k_2})}, \quad \|v\|^2_{L_h} = (v, v)_{L_h}$$

$$(v, w)_{L_{\hat{h}}} = (v_1, w_1)_{L_2(\hat{\omega}_{1,h_1} \times \omega_{1,k_1})} + (v_2, w_2)_{L_2(\hat{\omega}_{2,h_2} \times \omega_{2,k_2})}, \quad \|v\|^2_{L_{\hat{h}}} = (v, v)_{L_{\hat{h}}}$$

$$(v, w)_{L_{\hat{\hat{h}}}} = (v_1, w_1)_{L_2(\omega_{1,h_1} \times \hat{\omega}_{1,k_1})} + (v_2, w_2)_{L_2(\omega_{2,h_2} \times \hat{\omega}_{2,k_2})}, \quad \|v\|^2_{L_{\hat{\hat{h}}}} = (v, v)_{L_{\hat{\hat{h}}}}$$

$$(v, w)_{H^1_h} = (v, w)_{L_h} + (\delta_x v, \delta_x w)_{L_{\hat{h}}} + (\delta_y v, \delta_y w)_{L_{\hat{\hat{h}}}} \quad \|v\|^2_{H^1_h} = \|v\|^2_{L_h} + \|\delta_x v\|^2_{L_{\hat{h}}} + \|\delta_y v\|^2_{L_{\hat{\hat{h}}}}.$$

Also, we introduce the discrete time-dependent norms:

$$\|w\|^2_{L_2(\omega)} = \tau \sum_{t \in \omega} w^2(t), \quad \|w\|_{L_\infty(\omega)} = \max_{t \in \omega} |w(t)|$$

$$\|v_i\|^2_{L_2(\omega, L_2(\omega_i))} = \tau \sum_{t \in \omega} \|v_i(\cdot, t)\|^2_{L_2(\omega_i)}, \quad \|v\|^2_{L_2(\omega, H)} = \tau \sum_{t \in \omega} \|v(\cdot, t)\|^2_H, \quad \|v\|_{L_\infty(\omega, H)} = \max_{t \in \omega} \|v(t)\|_H,$$

where

$$\omega \in \{\omega_\tau, \hat{\omega}_\tau\}, \quad \omega_i \in \{\omega_{i,h_i} \times \omega_{i,k_i},\ \hat{\omega}_{i,h_i} \times \omega_{i,k_i},\ \omega_{i,h_i} \times \hat{\omega}_{i,k_i}\}, \quad H \in \{L_h, L_{\hat{h}}, L_{\hat{\hat{h}}}, H^1_h\}.$$

Finite difference approximation of problem (1)-(8) is:

$$\delta_t v_1 - \delta_x(p_1\delta_x\hat{v}_1) - \delta_y(q_1\delta_y\hat{v}_1) = \hat{f}_1,\ x\in\omega_{1,h_1},\ y\in\omega_{1,k_1},\ t\in\hat{\omega}_\tau \tag{16}$$

$$\delta_t^2 v_2 - \delta_x(p_2\delta_x\hat{\hat{v}}_2) - \delta_y(q_2\delta_y\hat{\hat{v}}_2) = \hat{\hat{f}}_2,\ x\in\omega_{2,h_2},\ y\in\omega_{2,k_2},\ t\in\omega_\tau \tag{17}$$

$$\begin{aligned}&\delta_t v_1(b_1,y,t) + \frac{2}{h_1}\Big[p_1(b_1-h_1/2,y)\delta_x\hat{v}_1(b_1-h_1/2,y,t) + \alpha_1(y)\hat{v}_1(b_1,y,t)\\&-k_2\sum_{y'\in\omega_{2,k_2}}\beta_1(y,y')\hat{v}_2(a_2,y',t)\Big] - \delta_y(q_1\delta_y\hat{v}_1)(b_1,y,t) = \hat{f}_1(b_1,y,t),\ y\in\omega_{1,k_1},\ t\in\hat{\omega}_\tau\end{aligned} \tag{18}$$

$$\begin{aligned}&\delta_t v_1(x,c_1,t) - \frac{2}{k_1}\Big[q_1(x,c_1+k_1/2)\delta_y\hat{v}_1(x,c_1+k_1/2,t) - \check{\alpha}_1(x)\hat{v}_1(x,c_1,t)\\&+k_2\sum_{y'\in\omega_{2,k_2}}\check{\beta}_1(x,y')\hat{v}_2(a_2,y',t)\Big] - \delta_x(p_1\delta_x\hat{v}_1)(x,c_1,t) = \hat{f}_1(x,c_1,t),\ x\in\omega_{1,h_1},\ t\in\hat{\omega}_\tau\end{aligned} \tag{19}$$

$$\begin{aligned}&\delta_t v_1(x,d_1,t) + \frac{2}{k_1}\Big[q_1(x,d_1-k_1/2)\delta_y\hat{v}_1(x,d_1-k_1/2,t) + \hat{\alpha}_1(x)\hat{v}_1(x,d_1,t)\\&-k_2\sum_{y'\in\omega_{2,k_2}}\hat{\beta}_1(x,y')\hat{v}_2(a_2,y',t)\Big] - \delta_x(p_1\delta_x\hat{v}_1)(x,d_1,t) = \hat{f}_1(x,d_1,t),\ x\in\omega_{1,h_1},\ t\in\hat{\omega}_\tau\end{aligned} \tag{20}$$

$$\begin{aligned}&\delta_t^2 v_2(a_2,y,t) - \frac{2}{h_2}\Big[p_2(a_2+h_2/2,y)\delta_x\hat{\hat{v}}_2(a_2+h_2/2,y,t) - \alpha_2(y)\hat{\hat{v}}_2(a_2,y,t) + k_1\sum_{y'\in\omega_{1,k_1}}\beta_2(y,y')\hat{\hat{v}}_1(b_1,y',t)\\&+h_1\sum_{x'\in\omega_{1,h_1}}\check{\beta}_2(y,x')\hat{\hat{v}}_1(x',c_1,t)\Big] - \delta_y(q_2\delta_y\hat{\hat{v}}_2)(a_2,y,t) = \hat{\hat{f}}_2(a_2,y,t),\ y\in(c_2,c_1),\ t\in\omega_\tau\end{aligned} \tag{21}$$

$$\begin{aligned}&\delta_t^2 v_2(a_2,y,t) - \frac{2}{h_2}\Big[p_2(a_2+h_2/2,y)\delta_x\hat{\hat{v}}_2(a_2+h_2/2,y,t) - \alpha_2(y)\hat{\hat{v}}_2(a_2,y,t)\\&+k_1\sum_{y'\in\omega_{1,k_1}}\beta_2(y,y')\hat{\hat{v}}_1(b_1,y',t)\Big] - \delta_y(q_2\delta_y\hat{\hat{v}}_2)(a_2,y,t) = \hat{\hat{f}}_2(a_2,y,t),\ y\in(c_1,d_1),\ t\in\omega_\tau\end{aligned} \tag{22}$$

$$\begin{aligned}&\delta_t^2 v_2(a_2,y,t) - \frac{2}{h_2}\Big[p_2(a_2+h_2/2,y)\delta_x\hat{\hat{v}}_2(a_2+h_2/2,y,t) - \alpha_2(y)\hat{\hat{v}}_2(a_2,y,t) + k_1\sum_{y'\in\omega_{1,k_1}}\beta_2(y,y')\hat{\hat{v}}_1(b_1,y',t)\\&+h_1\sum_{x'\in\omega_{1,h_1}}\hat{\beta}_2(y,x')\hat{\hat{v}}_1(x',d_1,t)\Big] - \delta_y(q_2\delta_y\hat{\hat{v}}_2)(a_2,y,t) = \hat{\hat{f}}_2(a_2,y,t),\ y\in(d_1,d_2),\ t\in\omega_\tau\end{aligned} \tag{23}$$

$$\begin{aligned}&v_1(a_1,y,t)=0,\ y\in\omega_{1,k_1},\ t\in\overline{\omega}_\tau,\\&v_2(x,c_2,t)=0,\ x\in\omega_{2,h_2},\ t\in\overline{\omega}_\tau\\&v_2(b_2,y,t)=0,\ y\in\omega_{2,k_2},\ t\in\overline{\omega}_\tau,\\&v_2(x,d_2,t)=0,\ x\in\omega_{2,h_2},\ t\in\overline{\omega}_\tau\end{aligned} \tag{24}$$

$$v_i(x,y,0) = u_{i0}(x,y),\ x\in\omega_{i,h_i},\ y\in\omega_{i,k_i},\ i=1,2 \tag{25}$$

$$\delta_t v_2(x,y,\frac{\tau}{2}) = u_{21}(x,y) + \frac{\tau}{2}\Big[\frac{\partial}{\partial x}\Big(p_2\frac{\partial u_{20}}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(q_2\frac{\partial u_{20}}{\partial y}\Big) + f_2(x,y,0)\Big],\ x\in\omega_{2,h_2},\ y\in\omega_{2,k_2}. \tag{26}$$

From the general theory of difference schemes [25] immediately follows that the finite difference scheme (16)-(26) is numerically stable. Namely, parabolic part of the scheme is a Crank-Nicolson scheme while its hyperbolic part is a symmetric three level scheme with the weight parameter 1/4. Both schemes are unconditionally stable.

4. Convergence of the finite difference scheme

Let $u = (u_1, u_2)$ be the solution of the problem (1)-(8) and $v = (v_1, v_2)$ the solution of finite difference scheme (16)-(26).
Then the error $z = u - v$ satisfies the following finite difference scheme:

$$\delta_t z_1 - \delta_x(p_1\delta_x\hat{z}_1) - \delta_y(q_1\delta_y\hat{z}_1) = \chi_1,\ x \in \omega_{1,h_1},\ y \in \omega_{1,k_1},\ t \in \hat{\omega}_\tau \tag{27}$$

$$\delta_t^2 z_2 - \delta_x(p_2\delta_x\hat{\hat{z}}_2) - \delta_y(q_2\delta_y\hat{\hat{z}}_2) = \xi_2,\ x \in \omega_{2,h_2},\ y \in \omega_{2,k_2},\ t \in \omega_\tau \tag{28}$$

$$\begin{aligned}&\delta_t z_1(b_1, y, t) + \frac{2}{h_1}\Big[p_1(b_1 - h_1/2, y)\delta_x\hat{z}_1(b_1 - h_1/2, y, t) + \alpha_1(y)\hat{z}_1(b_1, y, t)\\ &-k_2\sum_{y'\in\omega_{2,k_2}}\beta_1(y, y')\hat{z}_2(a_2, y', t)\Big] - \delta_y(q_1\delta_y\hat{z}_1)(b_1, y, t) = \eta_1,\ y \in \omega_{1,k_1},\ t \in \hat{\omega}_\tau\end{aligned} \tag{29}$$

$$\begin{aligned}&\delta_t z_1(x, c_1, t) - \frac{2}{k_1}\Big[q_1(x, c_1 + k_1/2)\delta_y\hat{z}_1(x, c_1 + k_1/2, t) - \check{\alpha}_1(x)\hat{z}_1(x, c_1, t)\\ &+k_2\sum_{y'\in\omega_{2,k_2}}\check{\beta}_1(x, y')\hat{z}_2(a_2, y', t)\Big] - \delta_x(p_1\delta_x\hat{z}_1)(x, c_1, t) = \mu_1,\ x \in \omega_{1,h_1},\ t \in \hat{\omega}_\tau\end{aligned} \tag{30}$$

$$\begin{aligned}&\delta_t z_1(x, d_1, t) + \frac{2}{k_1}\Big[q_1(x, d_1 - k_1/2)\delta_y\hat{z}_1(x, d_1 - k_1/2, t) + \hat{\alpha}_1(x)\hat{z}_1(x, d_1, t)\\ &-k_2\sum_{y'\in\omega_{2,k_2}}\hat{\beta}_1(x, y')\hat{z}_2(a_2, y', t)\Big] - \delta_x(p_1\delta_x\hat{z}_1)(x, d_1, t) = \psi_1,\ x \in \omega_{1,h_1},\ t \in \hat{\omega}_\tau\end{aligned} \tag{31}$$

$$\begin{aligned}&\delta_t^2 z_2(a_2, y, t) - \frac{2}{h_2}\Big[p_2(a_2 + h_2/2, y)\delta_x\hat{\hat{z}}_2(a_2 + h_2/2, y, t) - \alpha_2(y)\hat{\hat{z}}_2(a_2, y, t) + k_1\sum_{y'\in\omega_{1,k_1}}\beta_2(y, y')\hat{\hat{z}}_1(b_1, y', t)\\ &+h_1\sum_{x'\in\omega_{1,h_1}}\check{\beta}_2(y, x')\hat{\hat{z}}_1(x', c_1, t)\Big] - \delta_y(q_2\delta_y\hat{\hat{z}}_2)(a_2, y, t) = \mu_2,\ y \in (c_2, c_1),\ t \in \omega_\tau\end{aligned} \tag{32}$$

$$\begin{aligned}&\delta_t^2 z_2(a_2, y, t) - \frac{2}{h_2}\Big[p_2(a_2 + h_2/2, y)\delta_x\hat{\hat{z}}_2(a_2 + h_2/2, y, t) - \alpha_2(y)\hat{\hat{z}}_2(a_2, y, t)\\ &+k_1\sum_{y'\in\omega_{1,k_1}}\beta_2(y, y')\hat{\hat{z}}_1(b_1, y', t)\Big] - \delta_y(q_2\delta_y\hat{\hat{z}}_2)(a_2, y, t) = \mu_2^{'},\ y \in (c_1, d_1),\ t \in \omega_\tau\end{aligned} \tag{33}$$

$$\begin{aligned}&\delta_t^2 z_2(a_2, y, t) - \frac{2}{h_2}\Big[p_2(a_2 + h_2/2, y)\delta_x\hat{\hat{z}}_2(a_2 + h_2/2, y, t) - \alpha_2(y)\hat{\hat{z}}_2(a_2, y, t) + k_1\sum_{y'\in\omega_{1,k_1}}\beta_2(y, y')\hat{\hat{z}}_1(b_1, y', t)\\ &+h_1\sum_{x'\in\omega_{1,h_1}}\hat{\beta}_2(y, x')\hat{\hat{z}}_1(x', d_1, t)\Big] - \delta_y(q_2\delta_y\hat{\hat{z}}_2)(a_2, y, t) = \mu_2^{''},\ y \in (d_1, d_2),\ t \in \omega_\tau\end{aligned} \tag{34}$$

$$\begin{aligned}&z_1(a_1, y, t) = 0,\ y \in \omega_{1,k_1},\ t \in \overline{\omega}_\tau,\\ &z_2(x, c_2, t) = 0,\ x \in \omega_{2,h_2},\ t \in \overline{\omega}_\tau\\ &z_2(b_2, y, t) = 0,\ y \in \omega_{2,k_2},\ t \in \overline{\omega}_\tau,\\ &z_2(x, d_2, t) = 0,\ x \in \omega_{2,h_2},\ t \in \overline{\omega}_\tau,\end{aligned} \tag{35}$$

$$z_i(x, y, 0) = 0,\ x \in \omega_{i,h_i},\ y \in \omega_{i,k_i},\ i = 1, 2 \tag{36}$$

$$\delta_t z_2(x, y, \frac{\tau}{2}) = \zeta_2 \tag{37}$$

where

$$\chi_1 = \Big[\delta_t u_1 - \widehat{\Big(\frac{\partial u_1}{\partial t}\Big)}\Big] - \Big[\delta_x(p_1\delta_x\hat{u}_1) - \widehat{\frac{\partial}{\partial x}\Big(p_1(x, y)\frac{\partial u_1}{\partial x}\Big)}\Big] - \Big[\delta_y(q_1\delta_y\hat{u}_1) - \widehat{\frac{\partial}{\partial y}\Big(q_1(x, y)\frac{\partial u_1}{\partial y}\Big)}\Big]$$

$$\xi_2 = \Big[\delta_t^2 u_2 - \widehat{\widehat{\frac{\partial^2 u_2}{\partial t^2}}}\Big] - \Big[\delta_x(p_2\delta_x\hat{u}_2) - \widehat{\widehat{\frac{\partial}{\partial x}}}\Big(p_2(x,y)\frac{\partial u_2}{\partial x}\Big)\Big] - \Big[\delta_y(q_2\delta_y\hat{u}_2) - \widehat{\widehat{\frac{\partial}{\partial y}}}\Big(q_2(x,y)\frac{\partial u_2}{\partial y}\Big)\Big]$$

$$\eta_1 = \frac{2}{h_1}\Big[p_1(b_1-h_1/2,y)\delta_x\hat{u}_1(b_1-h_1/2,y,t) - p_1(b_1,y)\widehat{\frac{\partial u_1}{\partial x}} - k_2\sum_{y'\in\omega_{2,k_2}}\beta_1(y,y')\hat{u}_2(a_2,y',t)$$
$$+\int_{c_2}^{d_2}\beta_1(y,y')\hat{u}_2(a_2,y',t)\,dy'\Big] - \delta_y(q_1\delta_y\hat{u}_1)(b_1,y,t) + \widehat{\frac{\partial}{\partial y}}\Big(q_1(b_1,y)\frac{\partial u_1}{\partial y}\Big)$$

$$\mu_1 = \frac{2}{k_1}\Big[-q_1(x,c_1+k_1/2)\delta_y\hat{u}_1(x,c_1+k_1/2,t) + q_1(x,c_1)\widehat{\frac{\partial u_1}{\partial y}} - k_2\sum_{y'\in\omega_{2,k_2}}\check{\beta}_1(x,y')\hat{u}_2(a_2,y',t)$$
$$+\int_{c_2}^{c_1}\check{\beta}_1(x,y')\hat{u}_2(a_2,y',t)\,dy'\Big] - \delta_x(p_1\delta_x\hat{u}_1)(x,c_1,t) + \widehat{\frac{\partial}{\partial x}}\Big(p_1(x,c_1)\frac{\partial u_1}{\partial x}\Big)$$

$$\psi_1 = \frac{2}{k_1}\Big[q_1(x,d_1-k_1/2)\delta_y\hat{u}_1(x,d_1-k_1/2,t) - q_1(x,d_1)\widehat{\frac{\partial u_1}{\partial y}} - k_2\sum_{y'\in\omega_{2,k_2}}\hat{\beta}_1(x,y')\hat{u}_2(a_2,y',t)$$
$$+\int_{d_1}^{d_2}\hat{\beta}_1(x,y')\hat{u}_2(a_2,y',t)\,dy'\Big] - \delta_x(p_1\delta_x\hat{u}_1)(x,d_1,t) + \widehat{\frac{\partial}{\partial x}}\Big(p_1(x,d_1)\frac{\partial u_1}{\partial x}\Big)$$

$$\mu_2 = \frac{2}{h_2}\Big[-p_2(a_2+h_2/2,y)\delta_x\hat{\hat{u}}_2(a_2+h_2/2,y,t) + p_2(a_2,y)\widehat{\widehat{\frac{\partial u_2}{\partial x}}}$$
$$-k_1\sum_{y'\in\omega_{1,k_1}}\beta_2(y,y')\hat{\hat{u}}_1(b_1,y',t) + \int_{c_1}^{d_1}\beta_2(y,y')\hat{\hat{u}}_1(b_1,y',t)\,dy' + \int_{a_1}^{b_1}\check{\beta}_2(y,x')\hat{\hat{u}}_1(x',c_1,t)dx'$$
$$-h_1\sum_{x'\in\omega_{1,h_1}}\check{\beta}_2(y,x')\hat{\hat{u}}_1(x',c_1,t)\Big] - \delta_y(q_2\delta_y\hat{\hat{u}}_2)(a_2,y,t) + \widehat{\widehat{\frac{\partial}{\partial y}}}\Big(q_2(a_2,y)\frac{\partial u_2}{\partial y}\Big)$$

$$\mu_2' = \frac{2}{h_2}\Big[-p_2(a_2+h_2/2,y)\delta_x\hat{\hat{u}}_2(a_2+h_2/2,y,t) + p_2(a_2,y)\widehat{\widehat{\frac{\partial u_2}{\partial x}}}$$
$$-k_1\sum_{y'\in\omega_{1,k_1}}\beta_2(y,y')\hat{\hat{u}}_1(b_1,y',t) + \int_{c_1}^{d_1}\beta_2(y,y')\hat{\hat{u}}_1(b_1,y',t)\,dy'\Big] - \delta_y(q_2\delta_y\hat{\hat{u}}_2)(a_2,y,t) + \widehat{\widehat{\frac{\partial}{\partial y}}}\Big(q_2(a_2,y)\frac{\partial u_2}{\partial y}\Big)$$

$$\mu_2'' = \frac{2}{h_2}\Big[-p_2(a_2+h_2/2,y)\delta_x\hat{\hat{u}}_2(a_2+h_2/2,y,t) + p_2(a_2,y)\widehat{\widehat{\frac{\partial u_2}{\partial x}}}$$
$$-k_1\sum_{y'\in\omega_{1,k_1}}\beta_2(y,y')\hat{\hat{u}}_1(b_1,y',t) - h_1\sum_{x'\in\omega_{1,h_1}}\hat{\beta}_2(y,x')\hat{\hat{u}}_1(x',d_1,t) + \int_{c_1}^{d_1}\beta_2(y,y')\hat{\hat{u}}_1(b_1,y',t)\,dy'$$
$$+\int_{a_1}^{b_1}\hat{\beta}_2(y,x')\hat{\hat{u}}_1(x',d_1,t)\,dx'\Big] - \delta_y(q_2\delta_y\hat{\hat{u}}_2)(a_2,y,t) + \widehat{\widehat{\frac{\partial}{\partial y}}}\Big(q_2(a_2,y)\frac{\partial u_2}{\partial y}\Big)$$

$$\zeta_2 = \frac{1}{\tau}\int_0^{\tau}\int_0^{t'}\int_0^{t''}\frac{\partial^3 u_2}{\partial t^3}dt'''dt''dt.$$

Taking halfsum of (27) from two adjacent time levels we obtain

$$\hat{\delta}_t z_1 - \delta_x(p_1\delta_x\hat{\hat{z}}_1) - \delta_y(q_1\delta_y\hat{z}_1) = \xi_1,\ x\in\omega_{1,h_1},\ y\in\omega_{1,k_1},\ t\in\hat{\omega}_\tau \tag{38}$$

where $\xi_1 = \hat{\chi}_1$.
In the sequel we shall assume that the generalized solution of the problem (1)-(8) belongs to the Sobolev space H^s, $s > 4.5$, while the data satisfy the following smoothness conditions:

$$\begin{gathered} p_i,\ q_i \in H^{s-1}(\Omega_i), \quad \alpha_i \in H^{s-1}(c_i, d_i), \quad \beta_i \in H^{s-1}((c_i, d_i) \times (c_{3-i}, d_{3-i}), \\ \check{\beta}_1 \in H^{s-1}((a_1, b_1) \times (c_2, c_1)), \quad \hat{\beta}_1 \in H^{s-1}((a_1, b_1) \times (d_1, d_2)), \quad \check{\beta}_2 \in H^{s-1}((c_2, c_1) \times (a_1, b_1)), \\ \check{\beta}_2 \in H^{s-1}((d_1, d_2) \times (a_1, b_1)), \quad f_i \in H^{s-2}(\Omega_i), \quad i = 1, 2. \end{gathered} \tag{39}$$

Using discrete Poincaré, Cauchy-Schwartz and Gårding's-inequalities, Gronwall lemma and summation by part over the mesh ω_τ follows the next result.

Theorem 4.1. *Let the assumptions* (9),(10) *and* (11) *hold and let the coefficient of finite difference scheme are well defined in the mesh nodes . Then the solution z of finite difference scheme satisfies a priori estimate*

$$\begin{aligned} &\|\hat{z}\|_{L_\infty(\hat{\omega}_\tau, H_h^1)} + \|\delta_t z_2\|_{L_\infty(\hat{\omega}_\tau, L_2(\omega_{2,h_2} \times \omega_{2,k_2}))} + \|\hat{\delta}_t z_1\|_{L_2(\hat{\omega}_\tau, L_2(\omega_{1,h_1} \times \omega_{1,k_1}))} \le C(T)\Big(\|\xi\|_{L_2(\omega_\tau, L_h)} + \|\eta\|_{L_\infty(\overline{\omega}_\tau, L_h)} \\ &+\|\zeta_2\|_{L_2(\omega_{2,h_2} \times \omega_{2,k_2})} + \sqrt{\tau}\|\chi_1\|_{L_2(\omega_{1,h_1} \times \omega_{1,k_1})} + \|\delta_t \eta\|_{L_2(\hat{\omega}_\tau, L_h)} + \tau\|\delta_x \zeta_2\|_{L_2(\omega_{2,h_2} \times \omega_{2,k_2})}\Big). \end{aligned} \tag{40}$$

Here is

$$\begin{aligned} &\|\eta\|_{L_\infty(\overline{\omega}_\tau, L_h)} = \|\eta_1\|_{L_\infty(\overline{\omega}_\tau, L_2(\omega_{1,h_1} \times \omega_{1,k_1}))} + \|\mu_1\|_{L_\infty(\overline{\omega}_\tau, L_2(\omega_{1,h_1} \times \omega_{1,k_1}))} + \|\psi_1\|_{L_\infty(\overline{\omega}_\tau, L_2(\omega_{1,h_1} \times \omega_{1,k_1}))} \\ &+\|\mu_2\|_{L_\infty(\overline{\omega}_\tau, L_2(\omega_{2,h_2} \times \omega_{2,k_2}))} + \|\mu_2'\|_{L_\infty(\overline{\omega}_\tau, L_2(\omega_{2,h_2} \times \omega_{2,k_2}))} + \|\mu_2''\|_{L_\infty(\overline{\omega}_\tau, L_2(\omega_{2,h_2} \times \omega_{2,k_2}))}, \\ &\|\xi\|_{L_2(\omega_\tau, L_h)} = \|\xi_1\|_{L_2(\omega_\tau, L_2(\omega_{1,h_1} \times \omega_{1,k_1}))} + \|\xi_2\|_{L_2(\omega_\tau, L_2(\omega_{2,h_2} \times \omega_{2,k_2}))} \end{aligned}$$

and

$$\begin{aligned} &\|\delta_t \eta\|_{L_2(\hat{\omega}_\tau, L_h)} = \|\delta_t \eta_1\|_{L_2(\hat{\omega}_\tau, L_2(\omega_{1,h_1} \times \omega_{1,k_1}))} + \|\delta_t \mu_1\|_{L_2(\hat{\omega}_\tau, L_2(\omega_{1,h_1} \times \omega_{1,k_1}))} + \|\delta_t \psi_1\|_{L_2(\hat{\omega}_\tau, L_2(\omega_{1,h_1} \times \omega_{1,k_1}))} \\ &+\|\delta_t \mu_2\|_{L_2(\hat{\omega}_\tau, L_2(\omega_{2,h_2} \times \omega_{2,k_2}))} + \|\delta_t \mu_2'\|_{L_2(\hat{\omega}_\tau, L_2(\omega_{2,h_2} \times \omega_{2,k_2}))} + \|\delta_t \mu_2''\|_{L_2(\hat{\omega}_\tau, L_2(\omega_{2,h_2} \times \omega_{2,k_2}))}. \end{aligned}$$

In order to determine the convergence rate finite difference scheme it is necessary to estimate the right side of the inequality (40). Using Bramble-Hilbert lemma [3] and procedure proposed in [9, 12, 19, 25] we obtain the following result.

Theorem 4.2. *Let $\tau \asymp h_1 \asymp h_2 \asymp k_1 \asymp k_2 \asymp h$ and the assumptions* (39) *hold. The finite difference scheme converge in the norm $L_\infty(\hat{\omega}_\tau, H_h^1)$ and the convergence rate estimate*

$$\|\hat{z}\|_{L_\infty(\hat{\omega}_\tau, H_h^1)} + \|\delta_t z_2\|_{L_\infty(\hat{\omega}_\tau, L_2(\omega_{2,h_2} \times \omega_{2,k_2}))} + \|\hat{\delta}_t z_1\|_{L_2(\hat{\omega}_\tau, L_2(\omega_{1,h_1} \times \omega_{1,k_1}))} \le Ch^2\|u\|_{H^s}, \quad s > 4.5$$

holds.

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