Filomat 29:8 (2015), zzz–zzz
DOI (will be added later)

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Buzano Inequality in Inner Product C*-modules via the Operator Geometric Mean

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Abstract. In this paper, by means of the operator geometric mean, we show a Buzano type inequality in an inner product C*-module, which is an extension of the Cauchy-Schwarz inequality in an inner product C*-module.

1. Introduction

The theory of Hilbert C^*-modules over non-commutative C^*-algebras firstly appeared in Paschke [15] and Rieffel [16], and it has contributed greatly to the developments of operator algebras. Recently, many researchers have studied geometric properties of Hilbert C^*-modules from a viewpoint of the operator theory. For examples, Moslehian et al considered in [14] Busano's type inequality in the context of Hilbert C^*-modules. Also, Roukbi considered in [18] norm type inequalities of the Buzano inequality and its generalization in an inner product C^*-module. We showed in [7] the new Cauchy-Schwarz inequality in an inner product C^*-module by means of the operator geometric mean. From the viewpoint, we show a Hilbert C^*-module version of the Buzano inequality which is an extension of the Cauchy-Schwarz inequality in an inner product C^*-module.

We briefly review the Buzano inequality and its generalization in a Hilbert space. Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. Buzano [3] showed an extension of the Cauchy-Schwarz inequality:

$$\frac{|\langle a, x\rangle \langle x, b\rangle|}{\langle x, x\rangle} \leq \frac{1}{2}\left(\langle a, a\rangle^{\frac{1}{2}} \langle b, b\rangle^{\frac{1}{2}} + |\langle a, b\rangle|\right) \tag{1}$$

for all $a, b, x \in H$, also see [8]. In the case of $a = b$, the inequality (1) becomes the Cauchy-Schwarz inequality $|\langle a, x\rangle| \leq \langle a, a\rangle^{\frac{1}{2}} \langle x, x\rangle^{\frac{1}{2}}$ for all $a, x \in H$. For a real inner product space, Richard [17] obtained the following stronger inequality:

$$\left|\frac{\langle a, x\rangle \langle x, b\rangle}{\langle x, x\rangle} - \frac{1}{2}\langle a, b\rangle\right| \leq \frac{1}{2}\langle a, a\rangle^{\frac{1}{2}} \langle x, x\rangle^{\frac{1}{2}} \tag{2}$$

2010 *Mathematics Subject Classification*. Primary 46L08; Secondary 47A63
Keywords. inner product C^*-module, Buzano inequality, operator geometric mean, Caucht-Schwarz inequality
Received: 06 November 2013; Accepted: 12 April 2014
Communicated by Mohammad Sal Moslehian
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for all $a, b, x \in H$. Dragomir [4] showed the following refinement of the Richard inequality (2):

$$\left| \frac{\langle a,x\rangle \langle x,b\rangle}{\langle x,x\rangle} - \frac{\langle a,b\rangle}{\alpha} \right| \le \frac{\langle b,b\rangle^{\frac{1}{2}}}{|\alpha| \langle x,x\rangle^{\frac{1}{2}}} \left(|\alpha - 1|^2 |\langle a,x\rangle|^2 + \langle x,x\rangle \langle a,a\rangle - |\langle a,x\rangle|^2 \right)^{\frac{1}{2}} \tag{3}$$

for all $a, b, x \in H$ with $x \neq 0$ and $\alpha \in \mathbb{C} - \{0\}$. In fact, if we put $\alpha = 2$ in (3), then we have the Richard inequality (2). Moreover, in [5], he showed that if $e_1, \dots, e_n$ is a finite orthonormal system and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ such that $|\alpha_i - 1| = 1$ for $i = 1, \dots, n$, then

$$\left| \langle x,y\rangle - \sum_{i=1}^{n} \alpha_i \langle x,e_i\rangle \langle e_i,y\rangle \right| \le \langle x,x\rangle^{\frac{1}{2}} \langle y,y\rangle^{\frac{1}{2}}. \tag{4}$$

Roukbi [18] considered norm type inequalities of the Dragomir inequality (3) and the Buzano one (1) in an inner product C^*-module. Also, Moslehian et al [14] considered Busano's type inequality in the context of Hilbert C^*-modules.

In this paper, by means of the operator geometric mean, we show inner product C^*-module versions of the Dragomir inequality (3) and the Richard inequality (2). As a result, we have a Buzano type inequality, which are an extension of the Cauchy-Schwarz inequality in an inner product C^*-module.

2. Preliminaries

Let $\mathscr{A}$ be a unital C*-algebra with the unit element e. An element $a \in \mathscr{A}$ is called positive if it is selfadjoint and its spectrum is contained in $[0, \infty)$. For $a \in \mathscr{A}$, we denote the absolute value of a by $|a| = (a^*a)^{\frac{1}{2}}$. For positive elements $a, b \in \mathscr{A}$, the operator geometric mean of a and b is defined by

$$a \,\sharp\, b = a^{\frac{1}{2}} \left(a^{-\frac{1}{2}} b a^{-\frac{1}{2}} \right)^{\frac{1}{2}} a^{\frac{1}{2}}$$

for invertible a, also see [9, 11]. In the case of non-invertible, since $a \,\sharp\, b$ satisfies the upper semicontinuity, we define $a \,\sharp\, b = \lim_{\varepsilon \to +0} (a + \varepsilon e) \,\sharp\, (b + \varepsilon e)$ in the strong operator topology. Hence $a \,\sharp\, b$ belongs to the double commutant $\mathscr{A}''$ of $\mathscr{A}$ in general. If $\mathscr{A}$ is monotone complete in the sense that every bounded increasing net in the self-adjoint part has a supremum with respect to the usual partial order, then we have $a \,\sharp\, b \in \mathscr{A}$, see [10]. The operator geometric mean has the symmetric property: $a \,\sharp\, b = b \,\sharp\, a$. In the case that a and b commute, we have $a \,\sharp\, b = \sqrt{ab}$.

A complex linear space $\mathscr{X}$ is said to be an inner product $\mathscr{A}$-module (or a pre-Hilbert $\mathscr{A}$-module) if $\mathscr{X}$ is a right $\mathscr{A}$-module together with a C*-valued map $(x, y) \mapsto \langle x, y\rangle : \mathscr{X} \times \mathscr{X} \to \mathscr{A}$ such that

(i) $\langle x, \alpha y + \beta z\rangle = \alpha \langle x, y\rangle + \beta \langle x, z\rangle \quad (x, y, x \in \mathscr{X}, \alpha, \beta \in \mathbb{C})$,
(ii) $\langle x, ya\rangle = \langle x, y\rangle a \quad (x, y \in \mathscr{X}, a \in \mathscr{A})$,
(iii) $\langle y, x\rangle = \langle x.y\rangle^* \quad (x, y \in \mathscr{X})$,
(iv) $\langle x, x\rangle \ge 0 \ (x \in \mathscr{X})$ and if $\langle x, x\rangle = 0$, then $x = 0$.

We always assume that the linear structures of $\mathscr{A}$ and $\mathscr{X}$ are compatible. Notice that (ii) and (iii) imply $\langle xa, y\rangle = a^*\langle x, y\rangle$ for all $x, y \in \mathscr{X}, a \in \mathscr{A}$. If $\mathscr{X}$ satisfies all conditions for an inner-product $\mathscr{A}$-module except for the second part of (iv), then we call $\mathscr{X}$ a semi-inner product $\mathscr{A}$-module.

In [7], from a viewpoint of operator theory, we presented the following Cauchy-Schwarz inequality in the framework of a semi-inner product C*-module over a unital C*-algebra: If $x, y \in \mathscr{X}$ such that the inner product $\langle x, y\rangle$ has a polar decomposition $\langle x, y\rangle = u|\langle x, y\rangle|$ with a partial isometry $u \in \mathscr{A}$, then

$$|\langle x, y\rangle| \ \le \ u^*\langle x, x\rangle u \,\sharp\, \langle y, y\rangle. \tag{5}$$

Under the assumption that $\mathscr{X}$ is an inner product $\mathscr{A}$-module and y is nonsingular, the equality in (5) holds if and only if $xu = yb$ for some $b \in \mathscr{A}$, also see [2, 6].

An element x of an inner product C^*-module $\mathscr{X}$ is called nonsingular if the element $\langle x, x\rangle \in \mathscr{A}$ is invertible. The set $\{e_i\} \subset \mathscr{X}$ is called orthonormal if $\left\langle e_i, e_j \right\rangle = \delta_{ij} e$. For more details on Hilbert C^*-modules; see [13].

3. Main Result

First of all, we show an inner product C^*-module version of Dragomir's result (3).

Theorem 3.1. *Let $\mathscr{X}$ be an inner product C^*-module over a unital C^*-algebra $\mathscr{A}$. If $x, y, z \in \mathscr{X}$ such that x is nonsingular and $a \in \mathscr{A}$ and the inner product $\left\langle z, x\langle x,x\rangle^{-1}\langle x,y\rangle - ya\right\rangle$ has a polar decomposition $\left\langle z, x\langle x,x\rangle^{-1}\langle x,y\rangle - ya\right\rangle = u|\left\langle z, x\langle x,x\rangle^{-1}\langle x,y\rangle - ya\right\rangle|$ with a partial isometry $u \in \mathscr{A}$, then*

$$\begin{aligned} &|\langle z,x\rangle\langle x,x\rangle^{-1}\langle x,y\rangle - \langle z,y\rangle a| \\ &\leq u^*\langle z,z\rangle u \,\sharp\, \left((a-e)^*\langle y,x\rangle\langle x,x\rangle^{-1}\langle x,y\rangle(a-e) + a^*\langle y,y\rangle a - a^*\langle y,x\rangle\langle x,x\rangle^{-1}\langle x,y\rangle a\right). \end{aligned} \tag{6}$$

Under the assumption that $x\langle x,x\rangle^{-1}\langle x,y\rangle - ya$ is nonsingular, the equality in (6) *holds if and only if $x\langle x,x\rangle^{-1}\langle x,y\rangle b = zu + yab$ for some $b \in \mathscr{A}$.*

Proof. By the Cauchy-Schwarz inequality (5), it follows that

$$\begin{aligned} &|\langle z,x\rangle\langle x,x\rangle^{-1}\langle x,y\rangle - \langle z,y\rangle a| = |\left\langle z, x\langle x,x\rangle^{-1}\langle x,y\rangle - ya\right\rangle| \\ &\leq u^*\langle z,z\rangle u \,\sharp\, \left\langle x\langle x,x\rangle^{-1}\langle x,y\rangle - ya, x\langle x,x\rangle^{-1}\langle x,y\rangle - ya\right\rangle \\ &= u^*\langle z,z\rangle u \,\sharp\, (a-e)^*\langle y,x\rangle\langle x,x\rangle^{-1}\langle x,y\rangle(a-e) + a^*\langle y,y\rangle a - a^*\langle y,x\rangle\langle x,x\rangle^{-1}\langle x,y\rangle a. \end{aligned}$$

The equality condition in (6) follows from those of the Cauchy-Schwarz inequality (5). □

In particular, if we put $a = \frac{1}{2}e$ in Theorem 3.1, then we have an inner product C^*-module version of the Richard inequality (2):

Theorem 3.2. *If $x, y, z \in \mathscr{X}$ such that x is nonsigular and the inner product $\left\langle z, x\langle x,x\rangle^{-1}\langle x,y\rangle - \frac{1}{2}y\right\rangle$ has a polar decomposition $\left\langle z, x\langle x,x\rangle^{-1}\langle x,y\rangle - \frac{1}{2}y\right\rangle = u|\left\langle z, x\langle x,x\rangle^{-1}\langle x,y\rangle - \frac{1}{2}y\right\rangle|$ with a partial isometry $u \in \mathscr{A}$, then*

$$|\langle z,x\rangle\langle x,x\rangle^{-1}\langle x,y\rangle - \frac{1}{2}\langle z,y\rangle| \leq \frac{1}{2}\left(u^*\langle z,z\rangle u \,\sharp\, \langle y,y\rangle\right). \tag{7}$$

Under the assumption that $x\langle x,x\rangle^{-1}\langle x,y\rangle - \frac{1}{2}y$ is nonsingular, the equality in (7) *holds if and only if $x\langle x,x\rangle^{-1}\langle x,y\rangle b = zub + \frac{1}{2}yb$ for some $b \in \mathscr{A}$.*

Remark 1. Theorem 3.2 is an extension of the Cauchy-Schwarz inequality (5). In fact, if we put $x = y\langle y,y\rangle^{-\frac{1}{2}}$ in Theorem 3.2, then we have the Cauchy-Schwarz inequality (5).

In the case that $a = e$ and $a = 0$ in Theorem 3.1 respectively, we have the following corollary, which is related to the Buzano inequality (1).

Corollary 3.3. *Let $x, y, x \in \mathscr{X}$ be as in Theorem 3.1. Then*

1. $|\langle z,x\rangle\langle x,x\rangle^{-1}\langle x,y\rangle - \langle z,y\rangle| \leq u^*\langle z,z\rangle u \,\sharp\, \left(\langle y,y\rangle - \langle y,x\rangle\langle x,x\rangle^{-1}\langle x,y\rangle\right)$.
2. $|\langle z,x\rangle\langle x,x\rangle^{-1}\langle x,y\rangle| \leq u^*\langle z,z\rangle u \,\sharp\, \langle y,x\rangle\langle x,x\rangle^{-1}\langle x,y\rangle$.

The following theorem is an inner product C^*-module version of Dragomir's result (4):

Theorem 3.4. *Let $\mathscr{X}$ be an inner product C^*-module over a unital C^*-algebra $\mathscr{A}$. If $e_1, \ldots, e_n \in \mathscr{X}$ is an orthonormal system, and $y, z \in \mathscr{X}$ and the inner product $\langle z, \sum_{i=1}^n e_i\langle e_i,y\rangle - y\rangle$ has a polar decomposition $\langle z, \sum_{i=1}^n e_i\langle e_i,y\rangle - y\rangle = u|\langle z, \sum_{i=1}^n e_i\langle e_i,y\rangle - y\rangle|$ with a partial isometry $u \in \mathscr{A}$, then*

$$\left|\sum_{i=1}^n \langle z,e_i\rangle\langle e_i,y\rangle - \langle z,y\rangle\right| \leq u^*\langle z,z\rangle u \,\sharp\, \left(\langle y,y\rangle - \sum_{i=1}^n \langle y,e_i\rangle\langle e_i,y\rangle\right). \tag{8}$$

Under the assumption that $\sum_{i=1}^n e_i\langle e_i,y\rangle - y$ is nonsingular, the equality holds in (8) *if and only if there exists $b \in \mathscr{A}$ such that $\sum_{i=1}^n e_i\langle e_i,y\rangle b = zu + yb$.*

Proof. By the Cauchy-Schwarz inequality (5), it follows that

$$\left|\sum_{i=1}^{n}\langle z,e_i\rangle\langle e_i,y\rangle-\langle z,y\rangle\right|=\left|\left\langle z,\sum_{i=1}^{n}e_i\langle e_i,y\rangle-y\right\rangle\right|\leq u^*\langle z,z\rangle u\ \sharp\ \left\langle\sum_{i=1}^{n}e_i\langle e_i,y\rangle-y,\sum_{i=1}^{n}e_i\langle e_i,y\rangle-y\right\rangle$$
$$=u^*\langle z,z\rangle u\ \sharp\ \left(\langle y,y\rangle-\sum_{i=1}^{n}\langle y,e_i\rangle\langle e_i,y\rangle\right)\quad\text{by the orthonormality of }\{e_i\}.$$

□

It is generally impossible to get the triangle inequality $|a+b|\leq|a|+|b|$ in C^*-algebra. However, Akemann, Anderson and Pedersen [1] showed the following result:

Theorem A. *For each a and b in a unital C^*-algebra $\mathscr{A}$ and $\varepsilon>0$ there are unitaries v and w in $\mathscr{A}$ such that*

$$|a+b|\leq v|a|v^*+w|b|w^*+\varepsilon e.$$

Remark 1. If $\mathscr{A}$ is a von Neumann algebra on a separable Hilbert space, then they moreover showed that for any $x,y\in\mathscr{A}$ there are isometries $v,w\in\mathscr{A}$ such that $|x+y|\leq v|x|v^*+w|y|w^*$.

By Theorem 3.2 and Theorem A, we have the following inner product C^*-module version of the Buzano inequality (1):

Theorem 3.5. *Let $\mathscr{X}$ be an inner product C^*-module over a unital C^*-algebra $\mathscr{A}$. For $\varepsilon>0$ and each $x,y,z\in\mathscr{X}$ such that x is nonsigular and the inner product $\left\langle z,x\langle x,x\rangle^{-1}\langle x,y\rangle-\frac{1}{2}y\right\rangle$ has a polar decomposition $\left\langle z,x\langle x,x\rangle^{-1}\langle x,y\rangle-\frac{1}{2}y\right\rangle=u\left|\left\langle z,x\langle x,x\rangle^{-1}\langle x,y\rangle-\frac{1}{2}y\right\rangle\right|$ with a partial isometry $u\in\mathscr{A}$, then there exist unitaries v_1,w_1,v_2 and w_2 in $\mathscr{A}$ such that*

$$\frac{1}{2}w_2|\langle z,y\rangle|w_2^*-\frac{1}{2}v_2\left(u^*\langle z,z\rangle u\ \sharp\ \langle y,y\rangle\right)v_2^*-\varepsilon e \tag{9}$$
$$\leq|\langle z,x\rangle\langle x,x\rangle^{-1}\langle x,y\rangle|\leq\frac{1}{2}v_1\left(u^*\langle z,z\rangle u\ \sharp\ \langle y,y\rangle\right)v_1^*+\frac{1}{2}w_1|\langle z,y\rangle|w_1^*+\varepsilon e.$$

Proof. By Theorem 3.2 , we have

$$|\langle z,x\rangle\langle x,x\rangle^{-1}\langle x,y\rangle-\frac{1}{2}\langle z,y\rangle|\leq\frac{1}{2}\left(u^*\langle z,z\rangle u\ \sharp\ \langle y,y\rangle\right).$$

By Theorem A, there are unitaries v_1 and w_1 in $\mathscr{A}$ such that

$$|\langle z,x\rangle\langle x,x\rangle^{-1}\langle x,y\rangle|\leq v_1|\langle z,x\rangle\langle x,x\rangle^{-1}\langle x,y\rangle-\frac{1}{2}\langle z,y\rangle|v_1^*+\frac{1}{2}w_1|\langle z,y\rangle|w_1^*+\varepsilon e.$$

Therefore, we have the second part of the desired inequality (9). For the first part, it follows from Theorem A that there are unitaries $\tilde{v}$ and $\tilde{w}$ in $\mathscr{A}$ such that

$$\frac{1}{2}|\langle z,y\rangle|\leq\tilde{v}|\frac{1}{2}\langle z,y\rangle-\langle z,x\rangle\langle x,x\rangle^{-1}\langle x,y\rangle|\tilde{v}^*+\tilde{w}|\langle z,x\rangle\langle x,x\rangle^{-1}\langle x,y\rangle|\tilde{w}^*+\varepsilon e$$
$$\leq\frac{1}{2}\tilde{v}\left(u^*\langle z,z\rangle u\ \sharp\ \langle y,y\rangle\right)\tilde{v}^*+\tilde{w}|\langle z,x\rangle\langle x,x\rangle^{-1}\langle x,y\rangle|\tilde{w}^*+\varepsilon e.$$

If we put $w_2=\tilde{w}^*$ and $v_2=\tilde{w}^*\tilde{v}$, then we have the desired inequality

$$\frac{1}{2}w_2|\langle z,y\rangle|w_2^*-\frac{1}{2}v_2\left(u^*\langle z,z\rangle u\ \sharp\ \langle y,y\rangle\right)v_2^*-\varepsilon e\leq|\langle z,x\rangle\langle x,x\rangle^{-1}\langle x,y\rangle|.$$

□

Remark 2. Theorem 3.5 is an extension of the Cauchy-Schwarz inequality (5) in an inner product C^*-module: As a matter of fact, if we put $x=y\langle y,y\rangle^{-\frac{1}{2}}$ in Theorem 3.2, then we can take $u=w=e$ and $\varepsilon=0$ and hence we have the Cauchy-Schwarz inequality (5).

4. Applications

In this section, as an application, we consider an inequality related to the Selberg inequality in an inner product C^*-module.

Let $\{y_1, \ldots, y_n\}$ be an orthonormal set in $\mathscr{X}$. Then the Bessel inequality says that

$$\sum_{i=1}^{n} |\langle y_i, x\rangle|^2 \le \langle x, x\rangle \tag{10}$$

holds for all $x \in \mathscr{X}$. In [12], we showed the Selberg inequality in an inner product C^*-module, which is a simultaneous extension of the Bessel and the Cauchy-Schwarz inequalities: If $x, y_1, \ldots, y_n$ are nonzero vectors in $\mathscr{X}$ such that $y_1, \ldots, y_n$ are nonsingular, then

$$\sum_{i=1}^{n} \langle x, y_i\rangle \left(\sum_{j=1}^{n} |\left\langle y_j, y_i \right\rangle| \right)^{-1} \langle y_i, x\rangle \le \langle x, x\rangle . \tag{11}$$

By virtue of Theorem 3.5, we show a simultaneous extension of the Selberg inequality (11) and the Buzano inequality (9):

Theorem 4.1. *let $\mathscr{X}$ be an inner product C^*-module over a unital C^*-algebra $\mathscr{A}$ and $x, y, z, w_1, \ldots, w_n \in \mathscr{X}$ be nonzero vectors such that $x, w_1, \ldots, w_n$ are nonsingular. Put $c_i = \sum_{j=1}^{n} |\left\langle w_j, w_i \right\rangle|$ and $h = y - \sum_{i=1}^{n} w_i c_i^{-1} \langle w_i, y\rangle$. Suppose that the inner product $\left\langle z, x\langle x, x\rangle^{-1}\langle x, y\rangle - \frac{1}{2}h \right\rangle$ has a polar decomposition $\left\langle z, x\langle x, x\rangle^{-1}\langle x, y\rangle - \frac{1}{2}h \right\rangle = u\left|\left\langle z, x\langle x, x\rangle^{-1}\langle x, y\rangle - \frac{1}{2}h \right\rangle\right|$ with a partial isometry $u \in \mathscr{A}$. If $\langle x, w_i\rangle = 0$ and $\langle z, w_i\rangle = 0$ for $i = 1, \ldots, n$, then for $\varepsilon > 0$ there are unitaries v and w such that*

$$|\langle z, x\rangle\langle x, x\rangle^{-1}\langle x, y\rangle| \le \frac{1}{2}v^* \left[u^*\langle z, z\rangle u \,\sharp\, \left(\langle y, y\rangle - \sum_{i=1}^{n} \langle y, w_i\rangle c_i^{-1} \langle w_i, y\rangle \right) \right] v + \frac{1}{2}w^* |\langle z, y\rangle| w + \varepsilon e.$$

Proof. Since $\langle z, y\rangle = \langle z, h\rangle$ and $\langle x, y\rangle = \langle x, h\rangle$, we have

$$\begin{aligned}
&|\langle z, x\rangle\langle x, x\rangle^{-1}\langle x, y\rangle - \frac{1}{2}\langle z, y\rangle| = |\langle z, x\rangle\langle x, x\rangle^{-1}\langle x, h\rangle - \frac{1}{2}\langle z, h\rangle| \\
&\le \frac{1}{2}(u^*\langle z, z\rangle u \,\sharp\, \langle h, h\rangle) \qquad \text{by Theorem 3.2} \\
&\le \frac{1}{2}\left(u^*\langle z, z\rangle u \,\sharp\, (\langle y, y\rangle - \sum_{i=1}^{n} \langle y, w_i\rangle c_i^{-1} \langle w_i, y\rangle) \right)
\end{aligned}$$

and the last inequality follows from [12, Theorem 3.1]. By Theorem A, there are uniatries v and w in $\mathscr{A}$ such that

$$\begin{aligned}
&|\langle z, x\rangle\langle x, x\rangle^{-1}\langle x, y\rangle| \le v|\langle z, x\rangle\langle x, x\rangle^{-1}\langle x, y\rangle - \frac{1}{2}\langle z, y\rangle| v^* + \frac{1}{2}w|\langle z, y\rangle| w^* + \varepsilon e \\
&\le \frac{1}{2}v\left[u^*\langle z, z\rangle u \,\sharp\, \left(\langle y, y\rangle - \sum_{i=1}^{n} \langle y, w_i\rangle c_i^{-1} \langle w_i, y\rangle \right) \right] v^* + \frac{1}{2}w|\langle z, y\rangle| w^* + \varepsilon e
\end{aligned}$$

as desired. □

References

[1] C.H. Akemann, J. Anderson and G.K. Pedersen, *Triangle inequalities in operator algebras*, Linear Multilinear Algebra, **11** (1982), 167–178.

[2] L. Arambašić, D. Bakić and M.S. Moslehian, *A treatment of the Cauchy-Schwarz inequality in C^*-modules*, J. Math. Anal. Appl., **381** (2011), 546–556.

[3] M.L. Buzano, *Generalizzazione della diseguaglianza di Cauchy-Schwarz*, Rend. Sem. Mat. Univ. e Politech. Torino **31** (1971–73). (1974), 405–409.

[4] S.S. Dragomir, *Refinement of Buzano's and Kurepa's inequalities in inner product spaces*, Facta Universitatis (NIS). Ser. Math. Inform. **20** (2005), 63–73.

[5] S.S. Dragomir, *A potpourri of Schwarz related inequalities in inner product spaces (II)*, J. Inequal. Pure Appl. Math., **7** (2006), Article 14.

[6] J.I. Fujii, *Operator-valued inner product and operator inequalities*, Banach J. Math. Anal., **2** (2008), 59–67.

[7] J.I. Fujii, M. Fujii, M.S. Moslehian and Y. Seo, *Cauchy-Schwarz inequality in semi-inner product C*-modules via polar decomposition*, J. Math. Anal. Appl., **394** (2012), 835-840.

[8] M. Fujii and F. Kubo, *Buzano's inequality and bounds for roots of algebraic equations*, Proc. Amer. Math. Soc., **117** (1993), 359–361.

[9] M. Fujii, J. Mićić Hot, J. Pečarić and Y. Seo, *Recent developments of Mond-Pečarić method in operator inequalities*, Monographs in Inequalities 4, Element, Zagreb, 2012.

[10] M. Hamana, *Partial $*$-automorphisms, normalizers, and submodules in monotome complete C^*-algebras*, Canad. J. Math., **58** (2006), 1144–1202.

[11] F. Kubo and T. Ando, *Means of positive linear operators*, Math. Ann., **246** (1980), 205–224.

[12] K. Kubo, F. Kubo and Y. Seo, *Selberg type inequalities in a Hilbert C^*-module and its applications*, to appear in Sci. Math. Jpn.

[13] E.C. Lance, *Hilbert C^*-Modules*, London Math. Soc. Lecture Note Series 210, Cambridge Univ. Press, 1995.

[14] M.S. Moslehian, M. Khosravi and R. Drnovsek, *A commutator approach to Buzuno's inequality*, Filomat, **26** (2012), 827–832.

[15] W.L. Paschke, *Inner product modules over B^*-algebras*, Trans. Amer. Math. Soc. **182** (1973), 443-468.

[16] M.A. Rieffel, *Morita equivalence for C^*-algebras and W^*-algebras*, J. Pure Applied Algebra, **5** (1974), 51-96.

[17] U. Richard, *Sur des inégalités du type Wirtinger et leurs application aux équationes diffrentielles ordinarires*, Colloquium of Anaysis held in Rio de Janeiro, (1972), 233–244.

[18] A. Roukbi, *Dragomir's, Buzano's and Kerupa's inequalities in Hilbert C^*-modules*, Facta Universitis (NIS) Ser. Math. Inform., **27**, No.1 (2012), 117–129.