Buzano Inequality in Inner Product C*-modules via the Operator Geometric Mean

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Abstract. In this paper, by means of the operator geometric mean, we show a Buzano type inequality in an inner product C*-module, which is an extension of the Cauchy-Schwarz inequality in an inner product C*-module.

1. Introduction

The theory of Hilbert C*-modules over non-commutative C*-algebras firstly appeared in Paschke [15] and Rieffel [16], and it has contributed greatly to the developments of operator algebras. Recently, many researchers have studied geometric properties of Hilbert C*-modules from a viewpoint of the operator theory. For examples, Moslehian et al considered in [14] Busano’s type inequality in the context of Hilbert C*-modules. Also, Roukbi considered in [18] norm type inequalities of the Buzano inequality and its generalization in an inner product C*-module. We showed in [7] the new Cauchy-Schwarz inequality in an inner product C*-module by means of the operator geometric mean. From the viewpoint, we show a Hilbert C*-module version of the Buzano inequality which is an extension of the Cauchy-Schwarz inequality in an inner product C*-module.

We briefly review the Buzano inequality and its generalization in a Hilbert space. Let $H$ be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. Buzano [3] showed an extension of the Cauchy-Schwarz inequality:

$$\frac{|\langle a, x \rangle \langle x, b \rangle|}{\langle x, x \rangle} \leq \frac{1}{2} \left( \langle a, a \rangle^{\frac{1}{2}} \langle b, b \rangle^{\frac{1}{2}} + |\langle a, b \rangle| \right)$$

for all $a, b, x \in H$, also see [8]. In the case of $a = b$, the inequality (1) becomes the Cauchy-Schwarz inequality $|\langle a, x \rangle| \leq \langle a, a \rangle^{\frac{1}{2}} \langle x, x \rangle^{\frac{1}{2}}$ for all $a, x \in H$. For a real inner product space, Richard [17] obtained the following stronger inequality:

$$\left| \frac{\langle a, x \rangle \langle x, b \rangle}{\langle x, x \rangle} - \frac{1}{2} \langle a, b \rangle \right| \leq \frac{1}{2} \langle a, a \rangle^{\frac{1}{2}} \langle x, x \rangle^{\frac{1}{2}}$$

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for all \(a, b, x \in H\). Dragomir [4] showed the following refinement of the Richard inequality (2):

\[
\left| \frac{\langle a, x \rangle \langle x, b \rangle}{\langle x, x \rangle} - \frac{\langle a, b \rangle}{a} \right| \leq \frac{\langle b, b \rangle}{\langle a, a \rangle} \left( |a - 1|^2 |\langle a, x \rangle|^2 + |\langle a, x \rangle|^2 \right)^{\frac{1}{2}}
\]

(3)

for all \(a, b, x \in H\) with \(x \neq 0\) and \(a \in \mathbb{C} - \{0\}\). In fact, if we put \(a = 2\) in (3), then we have the Richard inequality (2). Moreover, in [5], he showed that if \(e_1, \ldots, e_n\) is a finite orthonormal system and \(a_1, \ldots, a_n \in \mathbb{C}\) such that \(\alpha_i - 1 = 1\) for \(i = 1, \ldots, n\), then

\[
\left| \langle x, y \rangle - \sum_{i=1}^{n} a_i \langle x, e_i \rangle \langle e_i, y \rangle \right| \leq |\langle x, x \rangle|^2 |\langle y, y \rangle|^2 \sum_{i=1}^{n} a_i^2 \langle e_i, e_i \rangle
\]

(4)

Roukbi [18] considered norm type inequalities of the Dragomir inequality (3) and the Buzano one (1) in an inner product \(C^\ast\)-module. Also, Moslehian et al [14] considered Busano’s type inequality in the context of Hilbert \(C^\ast\)-modules.

In this paper, by means of the operator geometric mean, we show inner product \(C^\ast\)-module versions of the Dragomir inequality (3) and the Richard inequality (2). As a result, we have a Buzano type inequality, which are an extension of the Cauchy-Schwarz inequality in an inner product \(C^\ast\)-module.

2. Preliminaries

Let \(\mathcal{A}\) be a unital \(C^\ast\)-algebra with the unit element \(e\). An element \(a \in \mathcal{A}\) is called positive if it is selfadjoint and its spectrum is contained in \([0, \infty)\). For \(a \in \mathcal{A}\), we denote the absolute value of \(a\) by \(|a| = (a^*a)^{\frac{1}{2}}\). For positive elements \(a, b \in \mathcal{A}\), the operator geometric mean of \(a\) and \(b\) is defined by

\[
a \# b = a^\frac{1}{2} (a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^\frac{1}{2} a^\frac{1}{2}
\]

for invertible \(a\), also see [9, 11]. In the case of non-invertible, since \(a \# b\) satisfies the upper semicontinuity, we define \(a \# b = \lim_{n \to 0} (a + \epsilon e)^\frac{1}{2} (b + \epsilon e)^\frac{1}{2}\) in the strong operator topology. Hence \(a \# b\) belongs to the double commutant \(\mathcal{A}''\) of \(\mathcal{A}\) in general. If \(\mathcal{A}\) is monotone complete in the sense that every bounded increasing net in the self-adjoint part has a supremum with respect to the usual partial order, then we have \(a \# b \in \mathcal{A}\), see [10]. The operator geometric mean has the symmetric property: \(a \# b = b \# a\). In the case that \(a\) and \(b\) commute, we have \(a \# b = \sqrt{ab}\).

A complex linear space \(\mathcal{X}\) is said to be an inner product \(\mathcal{A}\)-module (or a pre-Hilbert \(\mathcal{A}\)-module) if \(\mathcal{X}\) is a right \(\mathcal{A}\)-module together with a \(C^\ast\)-valued map \((x, y) \mapsto \langle x, y \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}\) such that

\[
\begin{align*}
(i) & \quad \langle ax + \beta x, y \rangle = a \langle x, y \rangle + \beta \langle x, z \rangle, \\
(ii) & \quad \langle x, ya \rangle = \langle x, y \rangle a, \\
(iii) & \quad \langle y, x \rangle = \langle x, y \rangle, \\
(iv) & \quad \langle x, x \rangle \geq 0 (x \in \mathcal{X}) \text{ and if } \langle x, x \rangle = 0, \text{ then } x = 0.
\end{align*}
\]

We always assume that the linear structures of \(\mathcal{A}\) and \(\mathcal{X}\) are compatible. Notice that (ii) and (iii) imply \(\langle xa, y \rangle = a^* \langle x, y \rangle\) for all \(x, y \in \mathcal{X}, a \in \mathcal{A}\). If \(\mathcal{X}\) satisfies all conditions for an inner-product \(\mathcal{A}\)-module except for the second part of (iv), then we call \(\mathcal{X}\) a semi-inner product \(\mathcal{A}\)-module.

In [7], from a viewpoint of operator theory, we presented the following Cauchy-Schwarz inequality in the framework of a semi-inner product \(C^\ast\)-module over a unital \(C^\ast\)-algebra: If \(x, y \in \mathcal{X}\) such that the inner product \((x, y)\) has a polar decomposition \((x, y) = uu^\dagger\) with a partial isometry \(u \in \mathcal{A}\), then

\[
|\langle x, y \rangle| \leq u^\dagger \langle x, x \rangle u \# \langle y, y \rangle.
\]

(5)

Under the assumption that \(\mathcal{X}\) is an inner product \(\mathcal{A}\)-module and \(y\) is nonsingular, the equality in (5) holds if and only if \(xu = yb\) for some \(b \in \mathcal{A}\), also see [2, 6].

An element \(x\) of an inner product \(C^\ast\)-module \(\mathcal{X}\) is called nonsingular if the element \(\langle x, x \rangle \in \mathcal{A}\) is invertible. The set \(\{e_i\} \subset \mathcal{X}\) is called orthonormal if \(\langle e_i, e_j \rangle = \delta_{ij}\). For more details on Hilbert \(C^\ast\)-modules; see [13].
3. Main Result

First of all, we show an inner product $C^*$-module version of Dragomir’s result (3).

**Theorem 3.1.** Let $\mathcal{X}$ be an inner product $C^*$-module over a unital $C^*$-algebra $\mathcal{A}$. If $x, y, z \in \mathcal{X}$ such that $x$ is nonsingular and $a \in \mathcal{A}$ and the inner product $\langle z, x(x,x)^{-1} \langle x, y \rangle - ya \rangle$ has a polar decomposition $\langle z, x(x,x)^{-1} \langle x, y \rangle - ya \rangle = u|\langle z, x(x,x)^{-1} \langle x, y \rangle - ya \rangle|$ with a partial isometry $u \in \mathcal{A}$, then

$$
|\langle z, x(x,x)^{-1} \langle x, y \rangle - ya \rangle| \leq u^* \langle z, z \rangle u \# \left( (a - e)^* \langle y, x \rangle(x,x)^{-1} \langle x, y \rangle (a - e) + a^* \langle y, y \rangle a - a^* \langle y, x \rangle(x,x)^{-1} \langle x, y \rangle a \right).
$$

Under the assumption that $\langle x(x,x)^{-1} \langle x, y \rangle - ya \rangle$ is nonsingular, the equality in (6) holds if and only if $\langle x(x,x)^{-1} \langle x, y \rangle b = zu + ya \rangle$ for some $b \in \mathcal{A}$.

**Proof.** By the Cauchy-Schwarz inequality (5), it follows that

$$
|\langle z, x(x,x)^{-1} \langle x, y \rangle - ya \rangle| = |\langle z, x(x,x)^{-1} \langle x, y \rangle - ya \rangle| \leq u^* \langle z, z \rangle u \# \left( (a - e)^* \langle y, x \rangle(x,x)^{-1} \langle x, y \rangle (a - e) + a^* \langle y, y \rangle a - a^* \langle y, x \rangle(x,x)^{-1} \langle x, y \rangle a \right).
$$

The equality condition in (6) follows from those of the Cauchy-Schwarz inequality (5). □

In particular, if we put $a = \frac{1}{2} e$ in Theorem 3.1, then we have an inner product $C^*$-module version of the Richard inequality (2):

**Theorem 3.2.** If $x, y, z \in \mathcal{X}$ such that $x$ is nonsingular and the inner product $\langle z, x(x,x)^{-1} \langle x, y \rangle - \frac{1}{2} y \rangle$ has a polar decomposition $\langle z, x(x,x)^{-1} \langle x, y \rangle - \frac{1}{2} y \rangle = u|\langle z, x(x,x)^{-1} \langle x, y \rangle - \frac{1}{2} y \rangle|$ with a partial isometry $u \in \mathcal{A}$, then

$$
|\langle z, x(x,x)^{-1} \langle x, y \rangle - \frac{1}{2} \rangle (z, y) | \leq \frac{1}{2} \left( u^* \langle z, z \rangle u \# \langle y, y \rangle \right).
$$

Under the assumption that $\langle x(x,x)^{-1} \langle x, y \rangle - \frac{1}{2} y \rangle$ is nonsingular, the equality in (7) holds if and only if $\langle x(x,x)^{-1} \langle x, y \rangle b = zu + \frac{1}{2} y \rangle$ for some $b \in \mathcal{A}$.

**Remark 1.** Theorem 3.2 is an extension of the Cauchy-Schwarz inequality (5). In fact, if we put $x = y \langle y, y \rangle^{-\frac{1}{2}}$ in Theorem 3.2, then we have the Cauchy-Schwarz inequality (5).

In the case that $a = e$ and $a = 0$ in Theorem 3.1 respectively, we have the following corollary, which is related to the Buzano inequality (1).

**Corollary 3.3.** Let $x, y, z \in \mathcal{X}$ be as in Theorem 3.1. Then

1. $|\langle z, x(x,x)^{-1} \langle x, y \rangle - \frac{1}{2} y \rangle | \leq u^* \langle z, z \rangle u \# \left( \langle y, y \rangle - \langle y, x \rangle(x,x)^{-1} \langle x, y \rangle \right).
$}

2. $|\langle z, x(x,x)^{-1} \langle x, y \rangle | \leq u^* \langle z, z \rangle u \# \langle y, x \rangle(x,x)^{-1} \langle x, y \rangle \rangle$.

The following theorem is an inner product $C^*$-module version of Dragomir’s result (4):

**Theorem 3.4.** Let $\mathcal{X}$ be an inner product $C^*$-module over a unital $C^*$-algebra $\mathcal{A}$. If $e_1, \ldots, e_n \in \mathcal{X}$ is an orthonormal system, and $y, z \in \mathcal{X}$ and the inner product $\langle z, \sum_{i=1}^{n} e_i \langle e_i, y \rangle - y \rangle$ has a polar decomposition $\langle z, \sum_{i=1}^{n} e_i \langle e_i, y \rangle - y \rangle = u|\langle z, \sum_{i=1}^{n} e_i \langle e_i, y \rangle - y \rangle|$ with a partial isometry $u \in \mathcal{A}$, then

$$
\sum_{i=1}^{n} \langle z, e_i \rangle \langle e_i, y \rangle - \langle z, y \rangle \leq u^* \langle z, z \rangle u \# \left( \langle y, y \rangle - \sum_{i=1}^{n} \langle y, e_i \rangle \langle e_i, y \rangle \right).
$$

Under the assumption that $\sum_{i=1}^{n} e_i \langle e_i, y \rangle - y$ is nonsingular, the equality holds in (8) if and only if there exists $b \in \mathcal{A}$ such that $\sum_{i=1}^{n} e_i \langle e_i, y \rangle b = zu + yb$. 


Proof. By the Cauchy-Schwarz inequality (5), it follows that
\[
\left| \sum_{i=1}^{n} \langle z, e_i \rangle \langle e_i, y \rangle - \langle z, y \rangle \right| = \left| \sum_{i=1}^{n} e_i \langle e_i, y \rangle - y \right| \leq u^* \langle z, z \rangle u \# \left( \sum_{i=1}^{n} e_i \langle e_i, y \rangle - y \right)
\]
\[
= u^* \langle z, z \rangle u \# \left( \langle y, y \rangle - \sum_{i=1}^{n} \langle y, e_i \rangle \langle e_i, y \rangle \right) \text{ by the orthonormality of } \{ e_i \}.
\]
\]

It is generally impossible to get the triangle inequality \(|a + b| \leq |a| + |b|\) in C*-algebra. However, Akemann, Anderson and Pedersen [1] showed the following result:

**Theorem A.** For each \(a \) and \(b \) in a unital C*-algebra \(\mathcal{A}\) and \(\epsilon > 0\) there are unitaries \(v \) and \(w \) in \(\mathcal{A}\) such that
\[
|a + b| \leq v|a^* + w|b^* + \epsilon.
\]

**Remark.** If \(\mathcal{A}\) is a von Neumann algebra on a separable Hilbert space, then they moreover showed that for any \(x, y \in \mathcal{A}\) there are isometries \(v, w \in \mathcal{A}\) such that \(|x + y| \leq v|x|^* + w|y|^*\).

By Theorem 3.2 and Theorem A, we have the following inner product C*-module version of the Buzano inequality (1):

**Theorem 3.5.** Let \(\mathcal{X}\) be an inner product C*-module over a unital C*-algebra \(\mathcal{A}\). For \(\epsilon > 0\) and each \(x, y, z \in \mathcal{X}\) such that \(x\) is nonsigular and the inner product
\[
\langle z, x \langle x, x \rangle^{-1} \langle x, y \rangle - \frac{1}{2} y \rangle \text{ has a polar decomposition } \langle z, x \langle x, x \rangle^{-1} \langle x, y \rangle - \frac{1}{2} y \rangle = u \langle z, x \langle x, x \rangle^{-1} \langle x, y \rangle - \frac{1}{2} y \rangle
\]
with a partial isometry \(u \in \mathcal{A}\), then there exist unitaries \(v_1, w_1, v_2 \) and \(w_2 \) in \(\mathcal{A}\) such that
\[
\frac{1}{2} \langle w_2 \rangle \langle z, y \rangle \langle w_2 \rangle^* - \frac{1}{2} v_2 \langle u^* \langle z, z \rangle u \# \langle y, y \rangle \rangle v_2^* - \epsilon
\]
\[
\leq |\langle z, x \rangle \langle x, x \rangle^{-1} \langle x, y \rangle| \leq \frac{1}{2} v_1 \langle u^* \langle z, z \rangle u \# \langle y, y \rangle \rangle v_1^* + \frac{1}{2} w_1 \langle z, y \rangle \langle w_1 \rangle + \epsilon.
\]

**Proof.** By Theorem 3.2, we have
\[
|\langle z, x \rangle \langle x, x \rangle^{-1} \langle x, y \rangle - \frac{1}{2} \langle z, y \rangle| \leq \frac{1}{2} \langle u^* \langle z, z \rangle u \# \langle y, y \rangle \rangle.
\]

By Theorem A, there are unitaries \(v_1 \) and \(w_1 \) in \(\mathcal{A}\) such that
\[
|\langle z, x \rangle \langle x, x \rangle^{-1} \langle x, y \rangle| \leq v_1 |\langle z, x \rangle \langle x, x \rangle^{-1} \langle x, y \rangle - \frac{1}{2} \langle z, y \rangle| v_1^* + \frac{1}{2} w_1 |\langle z, y \rangle | w_1^* + \epsilon.
\]

Therefore, we have the second part of the desired inequality (9). For the first part, it follows from Theorem A that there are unitaries \(\tilde{v} \) and \(\tilde{w} \) in \(\mathcal{A}\) such that
\[
\frac{1}{2} |\langle z, y \rangle| \leq |\langle z, y \rangle - \langle z, x \rangle \langle x, x \rangle^{-1} \langle x, y \rangle| |\tilde{v}^* + \tilde{w} |\langle z, x \rangle \langle x, x \rangle^{-1} \langle x, y \rangle |\tilde{w}^* + \epsilon.
\]
\[
\leq \frac{1}{2} \tilde{v} \langle u^* \langle z, z \rangle u \# \langle y, y \rangle \rangle \tilde{v}^* + \tilde{w} |\langle z, x \rangle \langle x, x \rangle^{-1} \langle x, y \rangle |\tilde{w}^* + \epsilon.
\]

If we put \(w_2 = \tilde{w}^*\) and \(v_2 = \tilde{v}^*\), then we have the desired inequality
\[
\frac{1}{2} \langle w_2 \rangle \langle z, y \rangle \langle w_2 \rangle^* - \frac{1}{2} v_2 \langle u^* \langle z, z \rangle u \# \langle y, y \rangle \rangle v_2^* - \epsilon \leq |\langle z, x \rangle \langle x, x \rangle^{-1} \langle x, y \rangle|.
\]

**Remark.** Theorem 3.5 is an extension of the Cauchy-Schwarz inequality (5) in an inner product C*-module:
As a matter of fact, if we put \(x = y \langle y, y \rangle^{-1/2}\) in Theorem 3.2, then we can take \(u = w = \epsilon\) and \(\epsilon = 0\) and hence we have the Cauchy-Schwarz inequality (5).
4. Applications

In this section, as an application, we consider an inequality related to the Selberg inequality in an inner product C*-module.

Let \(|y_1, \ldots, y_n|\) be an orthonormal set in \(\mathcal{X}\). Then the Bessel inequality says that

\[
\sum_{j=1}^{n} |\langle y_j, x \rangle|^2 \leq \langle x, x \rangle
\]

holds for all \(x \in \mathcal{X}\). In [12], we showed the Selberg inequality in an inner product C*-module, which is a simultaneous extension of the Bessel and the Cauchy-Schwarz inequalities: If \(x, y_1, \ldots, y_n\) are nonzero vectors in \(\mathcal{X}\) such that \(y_1, \ldots, y_n\) are nonsingular, then

\[
\sum_{i=1}^{n} \langle x, y_i \rangle \left( \sum_{j=1}^{n} |\langle y_j, y_i \rangle|^2 \right)^{-1} |\langle y_i, x \rangle| \leq \langle x, x \rangle.
\]

By virtue of Theorem 3.5, we show a simultaneous extension of the Selberg inequality (11) and the Buzano inequality (9):

**Theorem 4.1.** Let \(\mathcal{X}\) be an inner product C*-module over a unital C*-algebra \(\mathcal{A}\) and \(x, y, z, w_1, \ldots, w_n \in \mathcal{X}\) be nonzero vectors such that \(x, w_1, \ldots, w_n\) are nonsingular. Put \(c_i = \sum_{j=1}^{n} |\langle w_i, w_j \rangle|\) and \(h = y - \sum_{j=1}^{n} w_i c_i^{-1} \langle w_i, y \rangle\). Suppose that the inner product \(\langle z, x (x, x)^{-1} (x, y) - \frac{1}{2} h \rangle\) has a polar decomposition \(\langle z, x (x, x)^{-1} (x, y) - \frac{1}{2} h \rangle = u \left[ \langle z, x (x, x)^{-1} (x, y) - \frac{1}{2} h \rangle \right] \) with a partial isometry \(u \in \mathcal{A}\). If \(\langle x, w_i \rangle = 0\) and \(\langle z, w_i \rangle = 0\) for \(i = 1, \ldots, n\), then for \(\varepsilon > 0\) there are unitaries \(v\) and \(w\) such that

\[
\langle z, x (x, x)^{-1} (x, y) \rangle \leq \frac{1}{2} v \left[ u^* \langle z, z \rangle u \right] \langle y, y \rangle \left( \langle y, y \rangle - \sum_{i=1}^{n} \langle y, w_i \rangle c_i^{-1} \langle w_i, y \rangle \right) v + \frac{1}{2} \langle z, z \rangle |\langle w, w \rangle| + \varepsilon^2.
\]

**Proof.** Since \(\langle z, y \rangle = \langle z, h \rangle\) and \(\langle x, y \rangle = \langle x, h \rangle\), we have

\[
\langle z, x (x, x)^{-1} (x, y) \rangle \leq \frac{1}{2} \langle z, z \rangle \langle h, h \rangle\quad \text{by Theorem 3.2}
\]

and the last inequality follows from [12, Theorem 3.1]. By Theorem A, there are unitaries \(v\) and \(w\) in \(\mathcal{A}\) such that

\[
\langle z, x (x, x)^{-1} (x, y) \rangle \leq v \langle z, x (x, x)^{-1} (x, y) \rangle \left( \langle y, y \rangle - \sum_{i=1}^{n} \langle y, w_i \rangle c_i^{-1} \langle w_i, y \rangle \right) v + \frac{1}{2} \langle z, z \rangle |\langle w, w \rangle| + \varepsilon^2
\]

as desired. \(\square\)

References


