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Simultaneous Approximation of Conformal Mappings on Smooth Domains

Cem Koşar[a]

[a]*Department of Mathematics and Science Education, Gaziantep University, Gaziantep, TURKEY*

Abstract. Some estimates for the simultaneous polynomial approximation of the conformal mapping of the finite simple connected domain onto the disc in the complex plane $\mathbb{C}$ and its derivatives are obtained. The approximation rate in dependence of the differential parameters of the considered smooth domain is estimated.

1. Introduction and Background

Let G be a finite region in the complex plane, bounded by rectifiable Jordan curve $\Gamma := \partial G$ and $\Omega := ext\overline{G}$ and $\Delta := ext\mathbb{T}$. Let also $\mathbb{T} := \{w \in \mathbb{C} : |w| = 1\}$, $\mathbb{D} := int\mathbb{T}$. By the Riemann mapping theorem, there exists a unique conformal mapping $w = \varphi_0(z)$ of G onto the disk $D_{r_o} := \{w \in \mathbb{C} : |w| < r_0\}$, normalized by $\varphi_0(z_0) = 0$, $\varphi_0'(z_0) = 1$, where $r_0 := r_0(z_0; G)$ is called the conformal radius of G with respect to z_0 and having the inverse mapping ψ_0.

Similarly $w = \varphi(z)$ is conformal mapping of Ω onto Δ with normalizations $\varphi(\infty) = \infty$ and $lim_{z\to\infty} \frac{\varphi(z)}{z} > 0$. We denote by ψ the inverse mapping of φ.

For an arbitrary analytic function f given on G we set

$$\left\|f\right\|_{L^2(G)} := \left(\int\int_G \left|f(z)\right|^2 d\sigma_z\right)^{\frac{1}{2}}$$

where $d\sigma_z$ stands for area measure on G.

If the function f has a continuous extension to $\overline{G}$, we use also the uniform norm

$$\left\|f\right\|_{C(\overline{G})} := sup\left\{\left|f(z)\right| : z \in \overline{G}\right\}.$$

It is well known that the function $w = \varphi_0(z)$ minimizes the integral $\left\|f'\right\|_{L^2(G)}$ in the class of all analytic functions in G, normalized by $f(z_0) = 0, f'(z_0) = 1$. Definition of the Bieberbach polynomials clearly, let $\wp_n$ be the class of polynomials $p_n(z)$ of degree at most n and satisfying the conditions $p_n(z_0) = 0, p_n'(z_0) = 1$. A polynomial $\pi_n \in \wp_n$ is called $n-$ th Bieberbach polynomial for pair (G, z_0) if it minimizes the norm $\left\|p'\right\|_{L^2(G)}$ in the class $\wp_n$. It is easy to check that π_n also minimizes the norm $\left\|\varphi_0' - p_n'\right\|_{L^2(G)}$ in the class $\wp_n$.

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As follows from the results due to Farrel [11] and Markushevich [22], if G is a Caratheodory region, then $\left\|\varphi_0' - p_n'\right\|_{L^2(G)}$ tends to zero as n approaches infinity and so of the sequence $\{\pi_n\}$ converges uniformly to φ_0 on compact subsets of G.

The first study was done by Keldych [21] on the uniform convergence $\{\pi_n(z)\}$ polynomials to the function $\varphi_0(z)$ in the closure of G. He showed that if the boundary L of G is a smooth Jordan curve with bounded curvature, then for every small $\varepsilon > 0$ and $\gamma = 1 - \varepsilon$ there exists a constant $c(\varepsilon)$ independent of n such that

$$\left\|\varphi_0 - \pi_n\right\|_{C(\overline{G})} := max_{z \in \overline{G}} \left|\varphi_0(z) - \pi_n(z)\right| \leqslant \frac{c(\varepsilon)}{n^\gamma}, \tag{1}$$

where γ depends on the geometric properties of the boundary $\Gamma := \partial G$, holds for every natural number n.

In [21] the author also constructed an example of a starlike region, bounded by piecewise analytic curve with one singular point, where Bieberbach polynomials diverge. Therefore, the uniform convergence of the sequence $\{\pi_n(z)\}$ in $\overline{G}$ depends on the properties of $\Gamma := \partial G$. Later, Keldych's counterexample was given by Andrievkii and Pritsker [9], for more generalized region. Bisedes, its geometry is made more clearly than Keldych's counterexample.

Furthermore, Mergelyan [23] showed that $\gamma = \frac{1}{2} - \varepsilon$ for arbitrary small $\varepsilon > 0$, whenever $\Gamma := \partial G$ is a smooth Jordan curve. Additionally, Mergelyan stated it as a conjecture that the exponent $\gamma = \frac{1}{2} - \varepsilon$ in (1) could be replaced by $\gamma = 1 - \varepsilon$. Mergelyan's conjecture was proved for a smooth domain of bounded boundary rotation by Israfilov in [17].

A considerable progress in this area has been achieved by Mergelyan [23], Suetin [26], Simonenko [25], Wu [29], Andrievskii [7, 8], Gaier [12, 13], Abdullayev [1, 2, 4, 5], Israfilov [16, 17] and the others.

In the paper [23] S.N. Mergelyan also noted without an estimate the convergence of any derivatives of the Bieberbach polynomials π_n to the phenoma is also true for derivatives of conformal mapping function φ_0, so called the simultaneous approximation.

In our opinion, first results related on the simultaneous approximation of Bieberbach polynomials were obtained by Suetin [26] and Israfilov [20].

A smooth Jordan curve Γ is called Dini-smooth if $\vartheta(s)$ be its tangent direction angle expressed via arclenght of Γ, satisfying the condition

$$\int_0^c \frac{\omega(\vartheta, u)}{u} du < \infty \tag{2}$$

where $\omega(\vartheta, u)$ is the modulus of continuity of $\vartheta(s)$, for some $c > 0$.

Definition 1.1. *[18] Let $r = 0, 1, 2, ..., \alpha \in (0, 1]$ and $\beta \in [0, \infty)$. If the tangent direction angle ϑ of Γ fulfills*

$$\omega\left(\vartheta^{(r)}, \delta\right) \leqslant c\delta^\alpha \ln^\beta\left(\frac{4}{\delta}\right), \quad \delta \in (0, \pi]$$

with a positive constant c independent of δ, then we say that the Jordan curve Γ belongs to the class $C^{r,\alpha,\beta}$.

We also say that $f \in C^{\alpha,\beta}$ if $\omega(f, \delta) \leqslant c\delta^\alpha \ln^\beta\left(\frac{4}{\delta}\right)$, $\alpha \in (0, 1]$, $\beta \in [0, \infty)$ for some constant c.

The class $C^{r,\alpha,\beta}$ is generalization of the class $B(\alpha, \beta)$, defined in [19]. In particular, the class $C^{0,\alpha,\beta}$ coincides with $B(\alpha, \beta)$ and the class $C^{0,\alpha,0}$, $\alpha \in (0, 1)$, coincides with the class of Lyapunov curves.

The aim of this article, we study the estimation (1) for simultaneous approximation in domains with a subclass of smooth Jordan curves.

2. Main Results

We consider domains $C^{r,\alpha,\beta}$ with $r = 0, 1, 2, ...,$ $0 < \alpha \leq 1$ and $\beta \geq 0$ in this section. The following main results contain estimates for the rates of uniform convergence of the derivatives of Bieberbach polynomials.

Theorem 2.1. *Let* $\Gamma \in C^{r,\,\alpha,\,\beta}$ *with* $r = 2, 3, ...,\ \ 0 < \alpha < 1$ *and* $0 \leqslant \beta < r + \alpha - \frac{3}{2}$. *If* $1 \leqslant k \leqslant r - 1$, *then*

$$\left\|\varphi_0^{(k)} - \pi_n^{(k)}\right\|_{C(\overline{G})} \leqslant \frac{c \ln^{\beta}(n)}{n^{r+\alpha-k-\frac{1}{2}}}, \quad n \geqslant k \tag{3}$$

with some positive constant $c = c(r)$.

Corollary 2.2. *Let* $\Gamma \in C^{r,\,\alpha,\,0}$ *with* $r = 2, 3, ...$ *and* $0 < \alpha < 1$. *If* $1 \leqslant k \leqslant r - 1$ *then*

$$\left\|\varphi_0^{(k)} - \pi_n^{(k)}\right\|_{C(\overline{G})} \leqslant \frac{c}{n^{r+\alpha-k-\frac{1}{2}}}, \quad n \geqslant k \tag{4}$$

with some positive constant $c = c(r)$.

Theorem 2.3. *Let* $\Gamma \in C^{r,\,1,\,\beta}$ *with* $r = 3, 4, ...,$ *and* $0 \leqslant \beta < r - \frac{5}{2}$. *If* $1 \leqslant k \leqslant r - 1$ *then*

$$\left\|\varphi_0^{(k)} - \pi_n^{(k)}\right\|_{C(\overline{G})} \leqslant \frac{c \ln^{\beta+1}(n)}{n^{r-k-\frac{1}{2}}}, \quad n \geqslant k \tag{5}$$

with some positive constant $c = c(r)$.

Theorem 2.4. *Let* $\Gamma \in C^{r,\,\alpha,\,\beta}$ *with* $r = 1, 2, 3...,\ \frac{1}{2} < \alpha < 1$ *and* $0 \leqslant \beta < \alpha - \frac{1}{2}$. *Then*

$$\left\|\varphi_0^{(r)} - \pi_n^{(r)}\right\|_{C(\overline{G})} \leqslant \frac{c \ln^{\beta}(n)}{n^{\alpha-\frac{1}{2}}}, \quad n \geqslant r \tag{6}$$

with some positive constant $c = c(r)$.

3. Some Auxiliary Facts

The following some auxiliary results are given spaces $L^p(G)$ and $E^p(G)$ with $p > 1$, but in this work, our interest is focused on the Hilbert spaces $L^2(G)$ and $E^2(G)$. Throughout this paper $c, c_1, c_2, ...$ are positive constants which in general depend on G. By $L^p(G)$ and $E^p(G)$ we denote the set of all measurable complex valued functions such that $|f|^p$ is lebesque integrable with respect to arclenght, and Simirnov class of analytic functions in G, respectively.

Each function $f \in E^p(G)$ has a non-tangential limit almost everywhere on Γ and if we use the same notation for the non-tangential limit of f, then $f \in L^p(\Gamma)$.

For $p \geqslant 1$, $L^p(G)$ and $E^p(G)$ are Banach spaces with respect to the norm

$$\|f\|_{E^p(G)} = \|f\|_{L^p(\Gamma)} := \left(\int_\Gamma |f(z)|^p\,|dz|\right)^{\frac{1}{p}}.$$

For the further fundamental properties see [15, p.438-453]. For the mapping φ_0 and a weight function ω defined on Γ we set

$$\varepsilon_n\left(\varphi_0'\right)_p := \inf_{p_n}\left\|\varphi_0' - p_n\right\|_{L^p(G)} \quad and \quad E_n^0\left(\varphi_0',\ \omega\right)_p := \inf_{p_n}\left\|\varphi_0' - p_n\right\|_{L^p(\Gamma,\ \omega)},$$

where infimum is taken over all algebraic polynomials p_n of degree at most n and

$$L^p(\Gamma,\ \omega) := \left\{f \in L^1(\Gamma) : |f|^p\,\omega \in L^1(\Gamma)\right\}, \qquad E^p(G,\ \omega) := \left\{f \in E^1(G) : f \in L^p(\Gamma,\ \omega)\right\}.$$

According to Dynkn's result [10], in case of $\omega := \left|\varphi'\right|^{-1}$ between the best approximation numbers $\varepsilon_n\left(\varphi_0'\right)_p$ and $E_n^0\left(\varphi_0', \left|\varphi'\right|^{-1}\right)_p$ the following relation holds

$$\varepsilon_n\left(\varphi_0'\right)_p \leqslant cn^{-\frac{1}{p}} E_n^0\left(\varphi_0', \frac{1}{\left|\varphi'\right|}\right)_p. \tag{7}$$

Let $f \in E^p(G)$ and let

$$\omega_p(f,\delta) = \sup_{|h|\leqslant\delta}\left\|(fo\psi)\left(e^{i(\theta+h)}\right)-(fo\psi)\left(e^{i\theta}\right)\right\|_{L^p[0,2\pi]} = \sup_{|h|\leqslant\delta}\left\{\int_0^{2\pi}\left|(fo\psi)\left(e^{i(\theta+h)}\right)-(fo\psi)\left(e^{i\theta}\right)\right|^p d\theta\right\}^{\frac{1}{p}}$$

be the generalized modulus of continuity of f. We use the following approximation theorem by polynomials in the Simirnov class $E^p(G)$, $1 < p < \infty$.

Theorem 3.1. *[6] Let $k \in \mathbb{N}$ and $f^{(k)} \in E^p(G)$ with $1 < p < \infty$. If Γ satisfies the condition (2), then for an arbitrary algebraic polynomial $p_n(z,f)$ we have*

$$\left\|f - p_n(z,f)\right\|_{L^p(\Gamma)} \leqslant \frac{c}{n^k}\omega_p\left(f^{(k)},\delta\right),$$

where $\omega_p\left(f^{(k)},\delta\right) := \sup_{|h|\leqslant\delta}\left\|\left(f^{(k)}o\psi\right)\left(e^{i(\theta+h)}\right)-\left(f^{(k)}o\psi\right)\left(e^{i\theta}\right)\right\|_{L^p[0,2\pi]}.$

Definition 3.2. *[24] A bounded Jordan region G is called a $k-$quasidisk, $0 \leqslant k < 1$, if any conformal mapping ψ_0 can be extended to a $K-$ quasiconformal, $K = \frac{1+k}{1-k}$, homeomorphizm of the plane $\overline{\mathbb{C}}$ on the $\overline{\mathbb{C}}$. In that case the curve $\Gamma := \partial G$ is called a $K-$quasicircle. The region G (curve Γ) is called a quasidisk (quasicircle), if it is $k-$quasidisk ($k-$ quasicircle) with some $0 \leqslant k < 1$.*

Theorem 3.3. *[3] Let G be a $k-$quasidisk, $0 \leq k < 1$. Then for arbitrary $q_n \in \wp_n$ and any $m = 0,1,2,\ldots$we have*

$$\left\|q_n^{(m)}\right\|_{C(\overline{G})} \leqslant cn^{\left(m+\frac{2}{p}\right)(1+k)}\left\|q_n\right\|_{L^p(G)},\ p > 1.$$

Corollary 3.4. *Let $\Gamma \in C^{r,\alpha,\beta}$ with $0 < \alpha < 1$, $\beta \geq 0$ and $r = 1,2,\ldots$. Then for arbitrary $p_n \in \wp_n$ we have*

$$\left\|p_n^{(r)}\right\|_{C(\overline{G})} \leqslant cn^r\left\|p_n'\right\|_{L^2(G)}.$$

Proof. Since φ_0 is a conformal mapping we can get $k = 0$ by taking $K = 1$ into account in Definition 3.2. Moreover substituting $q_n = p_n'$, $p = 2$ and $m = r-1$ into the Theorem 3.3, we easily obtain Corollary 3.4. □

Lemma 3.5. *[18] If $\Gamma \in C^{r,\alpha,\beta}$, with $r = 0,1,2,\ldots$, $\alpha \in (0,1]$, $\beta \in [0,\infty)$, then for $\Phi^{(r)}(w) := \varphi_o^{(r+1)}(\psi(w))$, we have*

$$\omega\left(\Phi^{(r)},\delta\right) \leqslant \begin{cases} c\delta^\alpha \ln^\beta\left(\frac{4}{\delta}\right) & ; if\ \ 0 < \alpha < 1 \\ c\delta\ln^{\beta+1}\left(\frac{4}{\delta}\right) & ; if\ \ \alpha = 1. \end{cases}$$

Lemma 3.6. *[27] Suppose that $\sum_{k=1}^\infty a_k$ converges and s is the value of the series. If $r_n := \frac{a_{n+1}}{a_n}$ is a decreasing sequence and $r_{n+1} < 1$, then*

$$0 \leqslant R_n = s - \sum_{k=1}^n a_k \leqslant \frac{a_{n+1}}{1-r_{n+1}}.$$

4. Proof of the Main Results

For the proofs of the main results we use a traditional method based on the extremal property of Bieberbach polynomials and also the inequality (7)

Proof of Theorem 2.1. Let $1 \leqslant k \leqslant r-1$. Since $\pi_n \to \varphi_0$, as $n \to \infty$, uniformly in G, for any $z \in G$, $n \in \mathbb{N}$ with $n \geqslant k$ and $2^j \leqslant n \leqslant 2^{j+1}$ we have

$$\varphi_0(z) - \pi_n(z) = \left[\pi_{2^{j+1}}(z) - \pi_n(z)\right] + \sum_{m=j+1}^{\infty} \left[\pi_{2^{m+1}}(z) - \pi_{2^m}(z)\right]$$

and

$$\varphi_0^{(k)}(z) - \pi_n^{(k)}(z) = \left[\pi_{2^{j+1}}^{(k)}(z) - \pi_n^{(k)}(z)\right] + \sum_{m=j+1}^{\infty} \left[\pi_{2^{m+1}}^{(k)}(z) - \pi_{2^m}^{(k)}(z)\right].$$

Therefore, the inequality

$$\left\|\varphi_0^{(k)} - \pi_n^{(k)}\right\|_{C(\overline{G})} \leqslant \left\|\pi_{2^{j+1}}^{(k)} - \pi_n^{(k)}\right\|_{C(\overline{G})} + \sum_{m=j+1}^{\infty} \left\|\pi_{2^{m+1}}^{(k)} - \pi_{2^m}^{(k)}\right\|_{C(\overline{G})}$$

holds. If we use Corollary 3.4, also we have

$$\left\|\varphi_0^{(k)} - \pi_n^{(k)}\right\|_{C(\overline{G})} \leqslant c_1 2^{(j+1)k} \left\|\pi'_{2^{j+1}} - \pi'_n\right\|_{L^2(G)} + c_2 \sum_{m=j+1}^{\infty} 2^{(m+1)k} \left\|\pi'_{2^{m+1}} - \pi'_{2^m}\right\|_{L^2(G)}. \tag{8}$$

Setting

$$Q_n(z) := \int_{z_0}^{z} q_n(t)\,dt \text{ and } t_n(z) := Q_n(z) + \left[1 - q_n(z_0)\right](z - z_0)$$

for the polynomial q_n, best approximating φ'_0 in the norm $\|.\|_{L^2(G)}$. We have $t_n(z_0) = 0$ and $t'_n(z_0) = 1$. Then,

$$\begin{aligned} \left\|\varphi'_0 - t'_n\right\|_{L^2(G)} &= \left\|\varphi'_0 - q_n - 1 + q_n(z_0)\right\|_{L^2(G)} \leqslant \left\|\varphi'_0 - q_n\right\|_{L^2(G)} + \left\|1 - q_n(z_0)\right\|_{L^2(G)} \\ &= \varepsilon_n\left(\varphi'_0\right)_2 + \left\|1 - q_n(z_0)\right\|_{L^2(G)}. \end{aligned} \tag{9}$$

Considering the inequality (see [14, p.4])

$$\left|f(z_0)\right| \leqslant \frac{\|f\|_{L^2(G)}}{\sqrt{\pi}\, dist\,(z_0, \Gamma)}$$

which holds for every analytic function f with $\|f\|_{L^2(G)} < \infty$, we can write $\varphi'_0 - q_n$ instead of f, and we get

$$\left\|1 - q_n(z_0)\right\|_{L^2(G)} \leqslant \frac{\left\|\varphi'_0 - q_n\right\|_{L^2(G)}}{\sqrt{\pi}\, dist\,(z_0, \Gamma)} = c_3 \varepsilon_n\left(\varphi'_0\right)_2.$$

Using last inequality, the minimization property of the Bieberbach polynomials and substituting (7) into (9), we have

$$\left\|\varphi'_0 - \pi'_n\right\|_{L^2(G)} \leqslant \left\|\varphi'_0 - t'_n\right\|_{L^2(G)} \leqslant c_4 \varepsilon_n\left(\varphi'_0\right)_2 \leqslant c_5 n^{-\frac{1}{2}} E_n^0\left(\varphi'_0, \left|\varphi'\right|^{-1}\right)_2.$$

Then for a natural number $n \in \mathbb{N}$ with $n \geqslant k$ and $2^j \leqslant n \leqslant 2^{j+1}$, by applying Theorem 3.1 for φ_0' we have

$$\begin{aligned}\left\|\pi_{2^{j+1}}' - \pi_n'\right\|_{L^2(G)} &\leqslant \left\|\pi_{2^{j+1}}' - \varphi_0'\right\|_{L^2(G)} + \left\|\varphi_0' - \pi_n'\right\|_{L^2(G)} \\ &= c_6\varepsilon_{2^{j+1}}\left(\varphi_0'\right)_2 + c_5\varepsilon_n\left(\varphi_0'\right)_2 \\ &\leqslant c_7\varepsilon_n\left(\varphi_0'\right)_2 \\ &\leqslant c_8 n^{-\frac{1}{2}} E_n^0\left(\varphi_0', \left|\varphi'\right|^{-1}\right)_2 \\ &\leqslant c_9 n^{-\frac{1}{2}}\left\|p_n\left(z,\varphi_0'\right) - \varphi_0'\right\|_{L^2\left(\Gamma,\, \left|\varphi'\right|^{-1}\right)}.\end{aligned}$$

We know that from [28], for $\Gamma \in C^{r,\,\alpha,\,\beta}$

$$0 < c_{10} \leqslant \left|\varphi'(z)\right| \leqslant c_{11}\ ,\ z \in \Gamma.$$

Furthermore, substituting $f = \varphi_0'$, $k = r-1$ into Theorem 3.1 and from Lemma 3.5 for $0 < \alpha < 1$, $\beta \geqslant 0$ we get

$$\begin{aligned}\left\|\pi_{2^{j+1}}' - \pi_n'\right\|_{L^2(G)} &\leqslant c_{12} n^{-\frac{1}{2}}\left\|p_n\left(z,\varphi_0'\right) - \varphi_0'\right\|_{L^2(\Gamma)} \\ &\leqslant c_{13} n^{-\frac{1}{2}}\frac{1}{n^{r-1}}\omega_2\left(\varphi_0^{(r)} o\psi, \frac{1}{n}\right) \\ &\leqslant \frac{c_{14}\ln^\beta(n)}{n^{r+\alpha-\frac{1}{2}}}.\end{aligned}$$

By the similar way we can show that

$$\left\|\pi_{2^{j+1}}' - \pi_{2^j}'\right\|_{L^2(G)} \leqslant \frac{c_{15}\ln^\beta(2^j)}{2^{j\left(r+\alpha-\frac{1}{2}\right)}}.$$

Using these estimations in (8) and lemma 3.6 we obtain the required estimation

$$\begin{aligned}\left\|\varphi_0^{(k)} - \pi_n^{(k)}\right\|_{C(\overline{G})} &\leqslant \frac{c_{16}2^{(j+1)k}\ln^\beta(n)}{n^{r+\alpha-\frac{1}{2}}} + c_{17}\sum_{m=j+1}^{\infty}\frac{2^{(m+1)k}\ln^\beta(2^m)}{2^{m\left(r+\alpha-\frac{1}{2}\right)}} \\ &\leqslant \frac{c_{18}\ln^\beta(n)}{n^{r+\alpha-k-\frac{1}{2}}} + c_{19}\sum_{m=j+1}^{\infty}\frac{\ln^\beta(2^m)}{2^{m\left(r+\alpha-k-\frac{1}{2}\right)}} \\ &\leqslant \frac{c_{20}\ln^\beta(n)}{n^{r+\alpha-k-\frac{1}{2}}}.\end{aligned}$$

Thus the proof of Theorem 2.1 is completed. □

Proof of Theorem 2.3. As in the case of of Theorem 2.1, we obtain the following estimations

$$\left\|\pi_{2^{j+1}}' - \pi_n'\right\|_{L^2(G)} \leqslant \frac{c_{21}\ln^{\beta+1}(n)}{n^{r-\frac{1}{2}}}\ ,\ \left\|\pi_{2^{j+1}}' - \pi_{2^j}'\right\|_{L^2(G)} \leqslant \frac{c_{22}\ln^{\beta+1}(2^j)}{2^{j\left(r-\frac{1}{2}\right)}}. \tag{10}$$

Combining (8), (10) and lemma 3.6 we have

$$\left\|\varphi_0^{(k)} - \pi_n^{(k)}\right\|_{C(\overline{G})} \leqslant \frac{c_{23}\ln^{\beta+1}(n)}{n^{r-k-\frac{1}{2}}}.$$

This gives the desired inequality. □

Proof of Theorem 2.4. The proof of Theorem 2.4 is similar to that of Theorem 2.1. □

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