n-Dimensional Ostrowski Type Inequalities
with Applications on Time Scales

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Abstract. An $n-$dimensional Montgomery identity in the form of generalized polynomial has been derived and as a consequence we find Ostrowski type inequalities. Our established results generalized some results in [12]. Some new applications are also given.

1. Introduction

The classical Ostrowski inequality [11] gives estimate of the absolute deviation of a differentiable function $f : [a, b] \to \mathbf{R}$ from its integral mean as:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a)M, \tag{1}$$

provided that

$$M := \sup_{a \le x \le b} |f'(x)|.$$

Inequality (1) has triggered a huge amount of interest over the years, due to vide applications in numerical analysis, theory of special means. We discuss its recent studies over the last decade. It has been extended, generalized and refined in a number of ways [2–4, 7, 8, 12–14]. The development of the theory of time scales was initiated by Stefan Hilger [5] in his Ph. D thesis, as a theory capable of containing both difference and differential calculus, in a consistent way. Since then, many authors have studied certain integral inequalities or dynamic equations on time scales [2, 9, 10]. For more information about time scales see [1]. Motivated by the idea generated by Bin Zheng et al. [12], we establish a generalized $n-$dimensional Montgomery identity and subsequently find an Ostrowski type inequalities, unifying the continuous, discrete and quantum cases. In the whole discussion $\mathbb{T}_i$, $i \in \mathbb{N}$, is considered to be a time scale. For $n = 1$, the sum $\sum_{k=1}^{n-1}$ is vacuously considered to be zero and take $I_{\mathbb{T}} = I \bigcap \mathbb{T}$ for $I \subseteq \mathbf{R}$. This paper is organized as follows. After this Introduction in Section 2 we present our main results regarding Ostrowski type inequalities and in Section 3 we give some applications of the results in Section 2.

2. Main Results

Before stating the main results we come across to some notations for the presentation to be in compact form.

$$\mathfrak{A}_1(x_1, ..., x_n) := \sum_{r=1}^{n-1} \sum_{1 \le m_1 < ... < m_r \le n} \frac{(-1)^{r+1} \mathfrak{L}^{(n)}}{\prod\limits_{j=1}^{r} H_k(a_{m_j}, x_{m_j}, b_{m_j})}$$

$$\times \int_{a_{m_1}}^{b_{m_1}} ... \int_{a_{m_r}}^{b_{m_r}} f(x_1, ..., x_{m_1-1}, \sigma(s_{m_1}), x_{m_1+1}, ..., \sigma(s_{m_r}), x_{m_r+1}..., x_n)$$

$$\times \prod_{j=1}^{r} P_{m_j, k-1}(x_{m_j}, s_{m_j}) \Delta s_{m_j}$$

$$\mathfrak{L}^{(n)} := \prod_{i=1}^{n} H_k(a_i, x_i, b_i); \quad H_k(a_i, x_i, b_i) = h_k(x_i, a_i) - h_k(x_i, b_i)$$

$$P_{u,k-1}^c(x_u, s_u) := \begin{cases} \dfrac{(s_u - a_u)^{k-1}}{(k-1)!}, & s_u \in [a_u, x_u]_\mathbf{R} \\ \dfrac{(s_u - b_u)^{k-1}}{(k-1)!}, & s_u \in (x_u, b_u]_\mathbf{R}; \end{cases} \qquad \mathfrak{L}_c^{(n)} := \prod_{i=1}^{n} \frac{(x_i - a_i)^k - (x_i - b_i)^k}{k!}$$

$$P_{u,k-1}^d(x_u, s_u) := \begin{cases} \dfrac{(s_u - a_u)^{(k-1)}}{(k-1)!}, & s_u \in [a_u, x_u]_\mathbb{Z} \\ \dfrac{(s_u - b_u)^{(k-1)}}{(k-1)!}, & s_u \in [x_u + 1, b_u]_\mathbb{Z}; \end{cases} \qquad \mathfrak{M}_1 := \sup_{\substack{a_i \le s_i \le b_i \\ 1 \le i \le n}} |\frac{\partial^n f(s_1, ..., s_n)}{\partial s_1 ... \partial s_n}|.$$

$$\mathfrak{L}_d^{(n)} := \prod_{i=1}^{n} \frac{(x_i - a_i)^{(k)} - (x_i - b_i)^{(k)}}{k!}; \quad \mathfrak{L}_q^{(n)} := \prod_{i=1}^{n} \frac{(x_i - a_i)_{q_i}^k - (x_i - b_i)_{q_i}^k}{[k]!}$$

$$P_{u,k}^q(x_u, s_u) := \begin{cases} \dfrac{(s_u - a_u)_{q_i}^k}{[k]!}, & s_u \in [a_u, x_u/q_i]_{\mathbb{T}_i} \\ \dfrac{(s_u - b_u)^k}{[k]!}, & s_u \in (x_u, b_u]_{\mathbb{T}_i}; \end{cases}$$

Lemma 2.1. *Let $a_i, b_i \in \mathbb{T}_i$, $x_i \in [a_i, b_i]_{\mathbb{T}_i}$ for some $i \in \mathbb{N}$; let $f \in C_{rd}\left(\prod_{i=1}^{n}[a_i, b_i]_{\mathbb{T}_i}, \mathbf{R}\right)$ be such that f^{Δ^n} exists. If $h_k(.,.)$ is a generalized polynomial, for some $k \in \mathbb{N}_0$, such that:*

$$P_{i,k}(x_i, s_i) = \begin{cases} h_k(s_i, a_i), & s_i \in [a_i, x_i] \\ h_k(s_i, b_i), & s_i \in (x_i, b_i], \end{cases}$$

then

$$f(x_1, ..., x_n)\mathfrak{L}^{(n)} - \mathfrak{A}_1(x_1, ..., x_n) = \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} \frac{\partial^n f(s_1, ..., s_n)}{\Delta s_1 ... \Delta s_n} \prod_{i=1}^{n} P_{i,k}(x_i, s_i) \Delta s_i \qquad (2)$$

$$+ (-1)^{n+1} \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} f(\sigma(s_1), ..., \sigma(s_n)) \prod_{i=1}^{n} P_{i,k-1}(x_i, s_i) \Delta s_i$$

Proof. The proof is followed by induction on n.
For $n = 1$, relation (2) reduces to [6, lemma A]. For $n = 2$, we have

$$
\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\partial^2 f(s_1, s_2)}{\Delta s_1 \Delta s_2} \prod_{i=1}^{2} P_{i,k}(x_i, s_i) \Delta s_i \;=\; \int_{a_1}^{b_1} \int_{a_2}^{x_2} h_k(s_2, a_2) \frac{\partial^2 f(s_1, s_2)}{\Delta s_1 \Delta s_2} P_{1,k}(x_1, s_1) \Delta s_2 \Delta s_1
$$

$$
+ \int_{a_1}^{b_1} \int_{x_2}^{b_2} h_k(s_2, b_2) \frac{\partial^2 f(s_1, s_2)}{\Delta s_1 \Delta s_2} P_{1,k}(x_1, s_1) \Delta s_2 \Delta s_1
$$

$$
= \prod_{i=1}^{2} H_k(a_i, x_i, b_i) f(x_1, x_2) - H_k(a_2, x_2, b_2) \int_{a_1}^{b_1} P_{1,k-1}(x_1, s_1) f(\sigma(s_1), x_2) \Delta s_1
$$

$$
- H_k(a_1, x_1, b_1) \int_{a_2}^{b_2} P_{2,k-1}(x_2, s_2) f(x_1, \sigma(s_2)) \Delta s_2
$$

$$
+ \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(\sigma(s_1), \sigma(s_2)) \prod_{i=1}^{2} P_{i,k-1}(x_i, s_i) \Delta s_i.
$$

Equivalently:

$$
f(x_1, x_2) \mathfrak{L}^{(2)} - \mathfrak{A}_1(x_1, x_2) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\partial^2 f(s_1, s_2)}{\Delta s_1 \Delta s_2} \prod_{i=1}^{2} P_{i,k}(x_i, s_i) \Delta s_i + (-1)^3 \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(\sigma(s_1), \sigma(s_2)) \prod_{i=1}^{2} P_{i,k-1}(x_i, s_i) \Delta s_i
$$

Suppose the statement is true for $n = v - 1$, that is

$$
\int_{a_1}^{b_1} \cdots \int_{a_{v-1}}^{b_{v-1}} \frac{\partial^{v-1} f(s_1, \ldots, s_{v-1})}{\Delta s_1 \ldots \Delta s_{v-1}} \prod_{i=1}^{v-1} P_{i,k}(x_i, s_i) \Delta s_i = \mathfrak{L}^{(v-1)} f(x_1, \ldots, x_{v-1}) \tag{3}
$$

$$
+ \sum_{r=1}^{v-2} \sum_{1 \le m_1 < \ldots < m_r \le v-1} \frac{(-1)^r \mathfrak{L}^{(v-1)}}{\prod_{q=1}^{r} H_k(a_{m_q}, x_{m_q}, b_{m_q})} \int_{a_{m_1}}^{b_{m_1}} \cdots \int_{a_{m_r}}^{b_{m_r}} f(x_1, \ldots, x_{m_1-1}, \sigma(s_{m_1}), x_{m_1+1}, \ldots, \sigma(s_{m_r}), x_{m_r+1}, \ldots, x_{v-1})
$$

$$
\times \prod_{q=1}^{r} P_{m_q, k-1}(x_{m_q}, s_{m_q}) \Delta s_{m_q} + (-1)^{v-1} \int_{a_1}^{b_1} \cdots \int_{a_{v-1}}^{b_{v-1}} f(\sigma(s_1), \ldots, \sigma(s_{v-1})) \prod_{j=1}^{v-1} P_{j,k-1}(x_j, s_j) \Delta s_j.
$$

To prove the relation (2) is true for $n = v$, we proceed as follows

$$
\int_{a_1}^{b_1} \cdots \int_{a_v}^{b_v} \frac{\partial^v f(s_1, \ldots, s_v)}{\Delta s_1 \ldots \Delta s_v} \prod_{i=1}^{v} P_{i,k}(x_i, s_i) \Delta s_i
$$

$$
= \int_{a_1}^{b_1} \cdots \int_{a_{v-1}}^{b_{v-1}} \left[\int_{a_v}^{b_v} \frac{\partial^v f(s_1, \ldots, s_v)}{\Delta s_1 \ldots \Delta s_v} P_{v,k}(x_v, s_v) \Delta s_v \right] \prod_{i=1}^{v-1} P_{i,k}(x_i, s_i) \Delta s_i
$$

$$
= \int_{a_1}^{b_1} \cdots \int_{a_{v-1}}^{b_{v-1}} \left[\int_{a_v}^{x_v} \frac{\partial^v f(s_1, \ldots, s_v)}{\Delta s_1 \ldots \Delta s_v} h_k(s_v, a_v) \Delta s_v + \int_{x_v}^{b_v} \frac{\partial^v f(s_1, \ldots, s_v)}{\Delta s_1 \ldots \Delta s_v} h_k(s_v, b_v) \Delta s_v \right] \prod_{i=1}^{v-1} P_{i,k}(x_i, s_i) \Delta s_i
$$

$$= H_k(a_v, x_v, b_v) \int_{a_1}^{b_1} ... \int_{a_{v-1}}^{b_{v-1}} \frac{\partial^{v-1} f(s_1, ..., s_{v-1}, x_v)}{\Delta s_1 ... \Delta s_{v-1}} \prod_{i=1}^{v-1} P_{i,k}(x_i, s_i) \Delta s_i$$

$$- \int_{a_v}^{b_v} P_{v,k-1}(x_v, s_v) \{ \int_{a_1}^{b_1} ... \int_{a_{v-1}}^{b_{v-1}} \frac{\partial^{v-1} f(s_1, ..., s_{v-1}, \sigma(s_v))}{\Delta s_1 ... \Delta s_{v-1}} \prod_{i=1}^{v-1} P_{i,k}(x_i, s_i) \Delta s_i \} \Delta s_v$$

$$= H_k(a_v, x_v, b_v) \{ \mathfrak{L}^{(v-1)} f(x_1, ..., x_{v-1}, x_v)$$

$$+ \sum_{r=1}^{v-2} \sum_{1 \le m_1 < ... < m_r \le v-1} \frac{(-1)^r \mathfrak{L}^{(v-1)}}{\prod_{q=1}^r H_k(a_{m_q}, x_{m_q}, b_{m_q})} \int_{a_{m_1}}^{b_{m_1}} ... \int_{a_{m_r}}^{b_{m_r}} f(x_1, ..., x_{m_1-1}, \sigma(s_{m_1}), x_{m_1+1}, ..., \sigma(s_{m_r}), x_{m_r+1}, ..., x_{v-1}, x_v)$$

$$\times \prod_{q=1}^r P_{m_q, k-1}(x_{m_q}, s_{m_q}) \Delta s_{m_q} + (-1)^{v-1} \int_{a_1}^{b_1} ... \int_{a_{v-1}}^{b_{v-1}} f(\sigma(s_1), ..., \sigma(s_{v-1}), x_v) \prod_{j=1}^{v-1} P_{j,k-1}(x_j, s_j) \Delta s_j \}$$

$$- \int_{a_v}^{b_v} P_{v,k-1}(x_v, s_v) \{ \mathfrak{L}^{(v-1)} f(x_1, ..., x_{v-1}, \sigma(s_v))$$

$$+ \sum_{r=1}^{v-2} \sum_{1 \le m_1 < ... < m_r \le v-1} \frac{(-1)^r \mathfrak{L}^{(v-1)}}{\prod_{q=1}^r H_k(a_{m_q}, x_{m_q}, b_{m_q})} \int_{a_{m_1}}^{b_{m_1}} ... \int_{a_{m_r}}^{b_{m_r}} f(x_1, ..., x_{m_1-1}, \sigma(s_{m_1}), x_{m_1+1}, ..., \sigma(s_{m_r}), x_{m_r+1}, ..., x_{v-1}, \sigma(s_v))$$

$$\times \prod_{q=1}^r P_{m_q, k-1}(x_{m_q}, s_{m_q}) \Delta s_{m_q} + (-1)^{v-1} \int_{a_1}^{b_1} ... \int_{a_{v-1}}^{b_{v-1}} f(\sigma(s_1), ..., \sigma(s_{v-1}), \sigma(s_v)) \prod_{j=1}^{v-1} P_{j,k-1}(x_j, s_j) \Delta s_j \}.$$

Equivalently:

$$\int_{a_1}^{b_1} ... \int_{a_v}^{b_v} \frac{\partial^v f(s_1, ..., s_v)}{\Delta s_1 ... \Delta s_v} \prod_{i=1}^v P_{i,k}(x_i, s_i) \Delta s_i = \mathfrak{L}^{(v)} f(x_1, ..., x_v)$$

$$+ \sum_{r=1}^{v-1} \sum_{1 \le m_1 < ... < m_r \le v} \frac{(-1)^r \mathfrak{L}^{(v)}}{\prod_{q=1}^r H_k(a_{m_q}, x_{m_q}, b_{m_q})} \int_{a_{m_1}}^{b_{m_1}} ... \int_{a_{m_r}}^{b_{m_r}} f(x_1, ..., x_{m_1-1}, \sigma(s_{m_1}), x_{m_1+1}, ..., \sigma(s_{m_r}), x_{m_r+1}, ..., x_v)$$

$$\times \prod_{q=1}^r P_{m_q, k-1}(x_{m_q}, s_{m_q}) \Delta s_{m_q} + (-1)^v \int_{a_1}^{b_1} ... \int_{a_v}^{b_v} f(\sigma(s_1), ..., \sigma(s_v)) \prod_{j=1}^v P_{j,k-1}(x_j, s_j) \Delta s_j,$$

showing that our statement (2) is true for $n = v$ and hence it is true for all n. This completes the proof of the lemma. $\square$

The following result is the $n-$dimensional Ostrowski type inequality.

Theorem 2.2. *Let the conditions of lemma 2.1 be satisfied, then*

$$\left| f(x_1, ..., x_n) \mathfrak{L}^{(n)} - \sum_{r=1}^{n-1} \sum_{1 \le m_1 < ... < m_r \le n} \frac{(-1)^{r+1} \mathfrak{L}^{(n)}}{\prod_{j=1}^r H_k(a_{m_j}, x_{m_j}, b_{m_j})} \right. \tag{4}$$

$$\times \left\{ \int_{a_{m_1}}^{b_{m_1}} ... \int_{a_{m_r}}^{b_{m_r}} f(x_1, ..., x_{m_1-1}, \sigma(s_{m_1}), x_{m_1+1}, ..., \sigma(s_{m_r}), x_{m_r+1} ..., x_n) \prod_{j=1}^r P_{m_j, k-1}(x_{m_j}, s_{m_j}) \Delta s_{m_j} \right\}$$

$$\left. + (-1)^n \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} f(\sigma(s_1), ..., \sigma(s_n)) \prod_{u=1}^n P_{u,k-1}(x_u, s_u) \Delta s_u \right|$$

$$\le N \prod_{i=1}^n H_{k+1}(a_i, x_i, b_i),$$

provided that

$$N := \sup_{\substack{a_i \le s_i \le b_i \\ 1 \le i \le n}} \left| \frac{\partial^n f(s_1, ..., s_n)}{\Delta s_1 ... \Delta s_n} \right|.$$

The inequality (4) *is sharp in the sense that its right hand side can not be replaced by a smaller one.*

Proof. It is observed that

$$\int_{a_1}^{b_1} ... \int_{a_n}^{b_n} \prod_{i=1}^{n} |P_{i,k}(x_i, s_i)| \Delta s_i = \prod_{i=1}^{n} H_{k+1}(a_i, x_i, b_i). \tag{5}$$

Application of lemma 2.1 yields:

$$\left| f(x_1, ..., x_n) \mathfrak{L}^{(n)} - \mathfrak{A}_1(x_1, ..., x_n) + (-1)^n \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} f(\sigma(s_1), ..., \sigma(s_n)) \prod_{u=1}^{n} P_{u,k-1}(x_u, s_u) \Delta s_u \right| \tag{6}$$

$$= \left| \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} \frac{\partial^n f(s_1, ..., s_n)}{\Delta s_1 ... \Delta s_n} \prod_{i=1}^{n} P_{i,k}(x_i, s_i) \Delta s_i \right|$$

$$\le \prod_{i=1}^{n} \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} \left| \frac{\partial^n f(s_1, ..., s_n)}{\Delta s_1 ... \Delta s_n} \right| |P_{i,k}(x_i, s_i)| \Delta s_i$$

$$\le N \prod_{i=1}^{n} H_{k+1}(a_i, x_i, b_i),$$

which is the desired result.

For $k = 1$, the sharpness is proved in [2, Theorem 3.5] and for $k > 1$, the sharpness can be proved in a similar fashion. $\square$

Remark 2.3. *For $k = 1$, theorem 2.2 coincides with [12, Theorem 2.3]*

Theorem 2.4. *Let the conditions of theorem 2.2 be satisfied. If there exist constants M_1, M_2 such that:*

$$M_1 \le \frac{\partial^n f(s_1, ..., s_n)}{\Delta s_1 ... \Delta s_n} \le M_2,$$

then

$$\left| f(x_1, ..., x_n) \mathfrak{L}^{(n)} - \sum_{r=1}^{n-1} \sum_{1 \le m_1 < ... < m_r \le n} \frac{(-1)^{r+1} \mathfrak{L}^{(n)}}{\prod_{j=1}^{r} H_k(a_{m_j}, x_{m_j}, b_{m_j})} \right. \tag{7}$$

$$\times \left\{ \int_{a_{m_1}}^{b_{m_1}} ... \int_{a_{m_r}}^{b_{m_r}} f(x_1, ..., x_{m_1-1}, \sigma(s_{m_1}), x_{m_1+1}, ..., \sigma(s_{m_r}), x_{m_r+1} ..., x_n) \prod_{j=1}^{r} P_{m_j,k-1}(x_{m_j}, s_{m_j}) \Delta s_{m_j} \right\}$$

$$\left. + (-1)^n \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} f(\sigma(s_1), ..., \sigma(s_n)) \prod_{u=1}^{n} P_{u,k-1}(x_u, s_u) \Delta s_u - \frac{M_1 + M_2}{2} \prod_{i=1}^{n} H_{k+1}(a_i, x_i, b_i) \right|$$

$$\le \frac{M_2 - M_1}{2} \prod_{i=1}^{n} H_{k+1}(a_i, x_i, b_i)$$

Proof. It may be noted that:

$$\left| \frac{\partial^n f(s_1, ..., s_n)}{\Delta s_1...\Delta s_n} - \frac{M_1 + M_2}{2} \right| \le \frac{M_2 - M_1}{2}. \tag{8}$$

A combination of (5) and (8) yield:

$$\left| \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} (\frac{\partial^n f(s_1, ..., s_n)}{\Delta s_1...\Delta s_n} - \frac{M_1 + M_2}{2}) \prod_{i=1}^{n} P_{i,k}(x_i, s_i) \Delta s_i \right| \tag{9}$$

$$= \left| \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} \frac{\partial^n f(s_1, ..., s_n)}{\Delta s_1...\Delta s_n} \prod_{i=1}^{n} P_{i,k}(x_i, s_i) \Delta s_i - \frac{M_1 + M_2}{2} \prod_{i=1}^{n} H_{k+1}(a_i, x_i, b_i) \right|$$

$$\le \prod_{i=1}^{n} \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} |\frac{\partial^n f(s_1, ..., s_n)}{\Delta s_1...\Delta s_n} - \frac{M_1 + M_2}{2}||P_{i,k}(x_i, s_i)| \Delta s_i$$

$$\le \frac{M_2 - M_1}{2} \prod_{i=1}^{n} H_{k+1}(a_i, x_i, b_i).$$

From (2) and (9) we get the desired result. $\square$

3. Applications

Corollary 3.1. *(continuous case)*
Let $\mathbb{T}_i = \mathbf{R}$, $1 \le i \le n$, then in this case (4) reduces to

$$\left| f(x_1, ..., x_n) \mathfrak{L}_c^{(n)} - \sum_{r=1}^{n-1} \sum_{1 \le m_1 < ... < m_r \le n} \prod_{i=1}^{n} \prod_{j=1}^{r} \frac{(-1)^{r+1}(x_i - a_i)^k - (x_i - b_i)^k}{(x_{m_j} - a_{m_j})^k - (x_{m_j} - b_{m_j})^k} \right.$$

$$\times \left\{ \int_{a_{m_1}}^{b_{m_1}} ... \int_{a_{m_r}}^{b_{m_r}} f(x_1, ..., x_{m_1-1}, s_{m_1}, x_{m_1+1}, ..., s_{m_r}, x_{m_r+1}..., x_n) \prod_{j=1}^{r} P_{m_j,k-1}^c(x_{m_j}, s_{m_j}) ds_{m_j} \right\}$$

$$\left. + (-1)^n \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} f(s_1, ..., s_n) \prod_{u=1}^{n} P_{u,k-1}^c(x_u, s_u) ds_u \right|$$

$$\le \frac{\mathfrak{M}_1}{(k+1)!} \prod_{i=1}^{n} \left[(x_i - a_i)^{k+1} - (x_i - b_i)^{k+1} \right]$$

Corollary 3.2. *(discrete case)*
Let $\mathbb{T}_i = \mathbb{Z}$, $1 \le i \le n$, then in this case (4) reduces to

$$\left| f(x_1, ..., x_n) \mathfrak{L}_d^{(n)} - \sum_{r=1}^{n-1} \sum_{1 \le m_1 < ... < m_r \le n} \prod_{i=1}^{n} \prod_{j=1}^{r} \frac{(-1)^{r+1}(x_i - a_i)^{(k)} - (x_i - b_i)^{(k)}}{(x_{m_j} - a_{m_j})^{(k)} - (x_{m_j} - b_{m_j})^{(k)}} \right.$$

$$\times \sum_{s_{m_1}=a_{m_1}}^{b_{m_1}-1} ... \sum_{s_{m_r}=a_{m_r}}^{b_{m_r}-1} f(x_1, ..., x_{m_1-1}, s_{m_1+1}, x_{m_1+1}, ..., s_{m_r+1}, x_{m_r+1}..., x_n) \prod_{j=1}^{r} P_{m_j,k-1}^d(x_{m_j}, s_{m_j}) s_{m_j}$$

$$\left. + (-1)^n \sum_{s_1=a_1}^{b_1-1} ... \sum_{s_n=a_n}^{b_n-1} f(s_1 + 1, ..., s_n + 1) \prod_{u=1}^{n} P_{u,k-1}^d(x_u, s_u) \Delta s_u \right| \le \frac{\mathfrak{M}_2}{(k+1)!} \prod_{i=1}^{n} [(x_i - a_i)^{(k+1)} - (x_i - b_i)^{(k+1)}],$$

where,
$\mathfrak{M}_2$ is the maximum value of the absolute of the difference $\Delta_n \Delta_{n-1}...\Delta_1 f$ over $[a_1, b_1 - 1]_{\mathbb{Z}} \times ... \times [a_n, b_n - 1]_{\mathbb{Z}}$

Corollary 3.3. *(quantum case)*
Let $\mathbb{T}_i = q_i^{\mathbb{N}_0}$, $1 \le i \le n$, *then in this case* (4) *reduces to*

$$
\left| f(x_1,...,x_n)\mathfrak{Q}_q^{(n)} - \sum_{r=1}^{n-1} \sum_{1 \le m_1 < ... < m_r \le n} \prod_{i=1}^{n} \prod_{j=1}^{r} \frac{(-1)^{r+1}[(x_i - a_i)_{q_i}^k - (x_i - b_i)_{q_i}^k]}{(x_{m_j} - a_{m_j})_{q_{m_j}}^k - (x_{m_j} - b_{m_j})_{q_{m_j}}^k} \right.
$$

$$
\times \left\{ \prod_{l=1}^{r} \prod_{p=1}^{r} a_{m_l}(q_{m_l} - 1) \sum_{s_{m_p}=0}^{\log_{q_{m_p}}^{(b_{m_p}/a_{m_p}q_{m_p})}} \prod_{t=1}^{r} P_{m_t,k-1}^q(x_{m_t}, s_{m_t}) \prod_{i=1}^{r} q_{m_i}^{s_{m_i}} \right.
$$

$$
\left. \times f(x_1,...x_{m_1-1}, a_{m_1}q_{m_1}^{s_{m_1}+1}, x_{m_1+1},..., a_{m_r}q_{m_r}^{s_{m_r}+1}, x_{m_r+1},...,x_n) \right\}
$$

$$
\left. + (-1)^n \prod_{i=1}^{n} \prod_{u=1}^{n} a_i(q_i - 1) \sum_{s_u=0}^{\log_{q_u}^{(b_u/a_uq_u)}} \prod_{v=1}^{n} q_v^{s_v} f(a_1 q_1^{s_1+1},...,a_n q_n^{s_n+1}) \right|
$$

$$
\le \mathfrak{M}_3 \prod_{i=1}^{n} \frac{(x_i - a_i)_{q_i}^{k+1} - (x_i - b_i)_{q_i}^{k+1}}{[k+1]!},
$$

where,
$\mathfrak{M}_3$ *is the maximum value of the absolute value of* $q_n...q_1$*-difference* $D_{q_n...q_1} f(x_1,...,x_n)$ *over* $[a_1, b_1/q_1]_{\mathbb{T}_1} \times [a_2, b_2/q_2]_{\mathbb{T}_2} \times ... \times [a_n, b_n/q_n]_{\mathbb{T}_n}$.

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