Coefficient Estimates for Three Generalized Classes of Meromorphic and Bi-Univalent Functions

Hai-Gen Xiao, Qing-Hua Xu

College of Mathematics and Information Science, JiangXi Normal University, NanChang 330022, China

Abstract. In this paper, we introduce and investigate three interesting subclass $\Sigma^\ast_\vartheta$($\lambda, \alpha$), $\Sigma^B_\vartheta$($\beta, \alpha$) and $\tilde{\Sigma}^\ast_\vartheta$($\alpha, \beta$) of meromorphic and bi-univalent functions on $\Delta = \{z \in \mathbb{C} : |z| > 1\}$, obtain their coefficient estimates. The results presented in this paper generalize the recent work of several earlier authors.

1. Introduction and Definitions

Let $\mathcal{A}$ be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$  

We denote by $\mathcal{S}$ the subclass of the normalized analytic function class $\mathcal{A}$ consisting of all functions in which are also univalent in $\mathbb{U}$.

It is well known that every function $f \in \mathcal{S}$ has an inverse $f^{-1}$, which is defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f^{-1}(f(w)) = w \quad (|w| < r_0(f); \ r_0(f) \geq \frac{1}{4}).$$

A function $f \in \mathcal{A}$ is said to be bi-univalent on $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let $\sigma$ denote the class of bi-univalent functions on $\mathbb{U}$ given by (1). For a brief history and interesting examples...
of functions in the class $\sigma$, see [8]. In fact, the aforecited work of Srivastava et al. [8] essentially revived the investigation of various subclasses of the bi-univalent function class $\sigma$ in recent years; it was followed by such works as those by Frasin and Aouf [2], Srivastava et al. [9-13], Xu et al. [19-21], and others (see, for example, [4-7]).

In this paper, the concept of bi-univalency is extended to the class of meromorphic functions defined on $\Delta = \{ z \in \mathbb{C} : |z| > 1 \}$. The class of functions

$$g(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n}$$

that are meromorphic and univalent in $\Delta$ is denoted by $\Sigma$ and every univalent function $g$ has an inverse $g^{-1}$ satisfying the series expansion

$$g^{-1}(w) = w + \sum_{n=1}^{\infty} \frac{B_n}{w^n},$$

where $0 < M < |w| < \infty$. Analogous to the bi-univalent analytic functions, a function $g \in \Sigma$ is said to be meromorphic and bi-univalent if $g^{-1} \in \Sigma$. The class of all meromorphic and bi-univalent functions is denoted by $\Sigma_{\varphi}$.

Estimates on the coefficients of the inverses of meromorphic univalent functions were widely investigated in the literature. For example, Schiffer[14] showed that if $g$, defined by (2), is in $\Sigma$ with $b_0 = 0$, then $|b_2| \leq 2/3$. In 1971, Duren[1] obtained the inequality $|b_n| \leq 2/(n + 1)$ for $g \in \Sigma$ with $b_k = 0$, $1 \leq k < n/2$. For $g^{-1}$ is the inverse of $g$, Springer[15] showed that

$$|B_3| \leq 1 \quad \text{and} \quad |B_3 + \frac{1}{2}B_1^2| \leq \frac{1}{2},$$

and conjectured that

$$|B_{2n-1}| \leq \frac{(2n - 2)!}{n!(n-1)!} \quad (n = 1, 2, ...).$$

In 1977, Kubota [3] proved that the Springer conjecture is true for $n = 3, 4, 5$ and subsequently Schober[16] obtained a sharp bounds for the coefficients $B_{2n-1}$, $1 \leq n \leq 7$. Recently, Kapoor and Mishra [6] found the coefficient estimates for inverses of meromorphic starlike functions of positive order $\alpha$ in $\Delta$. Samaneh G. Hamidi et al.[4] introduced the following subclasses of the meromorphic bi-univalent function and obtained non-sharp estimates on the initial coefficients $|b_0|$ and $|b_1|$.

**Definition 1.** (see[4]) The function $g$ given by (2) is said to belong to class $\tilde{\Sigma}_\varphi(\alpha)$ of bi-univalent strongly starlike meromorphic functions of order $\alpha$, $0 < \alpha \leq 1$, if

$$\left| \arg \left( \frac{zg'(z)}{g(z)} \right) \right| < \frac{\alpha \pi}{2} \quad (z \in \Delta)$$

and

$$\left| \arg \left( \frac{wh'(w)}{h(w)} \right) \right| < \frac{\alpha \pi}{2} \quad (w \in \Delta),$$

where the function $h$ is given by

$$h(w) = g^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n}. \quad (3)$$
Theorem A. (see [4]) If the function \( g \) given by (2) is in the class \( \Sigma_{p}(\alpha) \), \( 0 < \alpha \leq 1 \), then the coefficients \( b_{0} \) and \( b_{1} \) satisfy the inequalities
\[
|b_{0}| \leq 2\alpha \quad \text{and} \quad |b_{1}| \leq \sqrt{5}\alpha^{2}.
\]

**Definition 2.** (see [5]) A function \( g \) given by series expansion (2) is a meromorphic starlike bi-univalent functions of order \( \alpha \), \( 0 \leq \alpha < 1 \), if
\[
\Re \left( \frac{zg'(z)}{g(z)} \right) > \alpha \quad (z \in \Delta)
\]
and
\[
\Re \left( \frac{wh'(w)}{h(w)} \right) > \alpha \quad (w \in \Delta),
\]
where the function \( h \) is given by (3). The class of all meromorphic starlike bi-univalent functions of order \( \alpha \) is denote by \( \Sigma_{p}^{*}(\alpha) \).

**Theorem B.** (see [5]) If the function \( g \) given by (2) is a meromorphic starlike bi-univalent function of order \( \alpha \), \( 0 \leq \alpha < 1 \), then the coefficients \( b_{0} \) and \( b_{1} \) satisfy the inequalities
\[
|b_{0}| \leq 2(1 - \alpha) \quad \text{and} \quad |b_{1}| \leq (1 - \alpha) \sqrt{4\alpha^{2} - 8\alpha + 5}.
\]

Here, in our present sequel to some of the aforesaid works, we introduce the following three new subclasses of the function class \( \Sigma_{p} \) and find coefficient estimates. As a special case, we also generalize the recent work of several earlier authors.

**Definition 3.** A function \( g \in \Sigma_{\mathcal{S}} \) given by series expansion (2) is called meromorphic \( \lambda \)-convex bi-univalent functions of order \( \alpha \), \( \lambda \in \mathbb{R} \), \( 0 < \alpha \leq 1 \), if the following conditions are satisfied:
\[
\left| \arg \left( (1 - \lambda) \frac{zg'(z)}{g(z)} + \lambda (1 + \frac{zg''(z)}{g'(z)}) \right) \right| < \frac{\alpha\pi}{2} \quad (z \in \Delta)
\]
and
\[
\left| \arg \left( (1 - \lambda) \frac{wh'(w)}{h(w)} + \lambda (1 + \frac{wh''(w)}{h'(w)}) \right) \right| < \frac{\alpha\pi}{2} \quad (w \in \Delta),
\]
where the function \( h \) is given by (3). We denote by \( \Sigma_{\mathcal{S}}^{*}(\lambda, \alpha) \) the class of meromorphic \( \lambda \)-convex bi-univalent functions of order \( \alpha \).

We note that for \( \lambda = 0 \), the class \( \Sigma_{\mathcal{S}}^{*}(\lambda, \alpha) \) reduces to the class \( \Sigma_{\mathcal{S}}^{*}(\alpha) \) introduced and studied by Samaneh G. Hamidi et al. [4].

The next definition is related to a general class called of holomorphic Bazilevič functions. We denote by \( \mathcal{S}_{\beta} \) the class of all functions in \( \mathcal{S} \) which are starlike in \( \mathbb{U} \). Let us denote by \( B(\alpha, \beta, h, p) \) the class of functions \( f(z) \) which are analytic in \( \mathbb{U} \), have the form (1), and which, for some \( p(z) \in \mathcal{P}, h(z) \in \mathcal{S}_{\beta} \) and real numbers \( \alpha \) and \( \beta \) with \( \beta > 0 \), may be represented as
\[
f(z) = \left[ (\beta + ia) \int_{0}^{\infty} p(t)h(t)e^{i\alpha t} dt \right]^{1/(\beta + ia)}.
\]

For the sake of brevity we shall simply denote by \( B \) and where \( \mathcal{P} \) is the class of holomorphic functions \( p \) in \( \mathbb{U} \) such that \( p(0) = 1 \) and \( \Re p(z) > 0, z \in \mathbb{U} \). In the case when \( \alpha = 0 \), a computation shows that
\[
zf'(z) = f(z)^{1-\beta}h(z)^{\beta}p(z)
\]
or

$$\Re\left(\frac{zf'(z)}{f(z)^{1-\beta}}h(z)^\beta\right) > 0, \quad z \in \mathbb{U}. \tag{4}$$

Thomas [18] called a function satisfying the condition (4) as a Bazilevič function of type $\beta$. Furthermore, if $h(z) = z$ in (4), then the condition (4) becomes

$$\Re\left(\frac{zf'(z)}{f(z)^{1-\beta}}\right) > 0, \quad z \in \mathbb{U}. \tag{5}$$

The class of all functions $f \in A$ satisfies (5) and $\beta \geq 0$ is introduced by Singh [17] and the class of all such functions is denoted by $B_1(\beta)$. It is clear that $B_1(0) = S^*$.

**Definition 4.** A function $g \in \Sigma_\vartheta$ given by series expansion (2) is said meromorphic strongly Bazilevič bi-univalent functions of type $\beta$ and order $\alpha$, $0 \leq \alpha < 1$, $\beta \geq 0$, if the following conditions are satisfied:

$$\Re\left(\frac{g'(z)}{g(z)^{1-\beta}}\right) > \alpha \quad (z \in \Delta)$$

and

$$\Re\left(\frac{wh'(w)}{h(w)^{1-\beta}}\right) > \alpha \quad (w \in \Delta),$$

where the function $h$ is given by (3). We denote by $\Sigma_\vartheta^B(\beta, \alpha)$ the class of meromorphic strongly Bazilevič bi-univalent functions of type $\beta$ and order $\alpha$.

We note that for $\beta = 0$, the class $\Sigma_\vartheta^B(\beta, \alpha)$ reduces to the class $\Sigma^*_\vartheta(\alpha)$ introduced and studied by Samaneh G. Hamidi et al. [4].

For the function $g$ given by series expansion (2) with $b_1 = b_2 = \cdots = b_{k-1} = 0$, some estimates on the initial coefficients can be obtained. We shall give the result in section 4 as an interesting result of the following class.

**Definition 5.** A function $g(z) = z + \sum_{n=k}^{\infty} \frac{g_n}{n!} z^n$ is called weakerly meromorphic $\beta$-spirallike bi-univalent functions of order $\alpha$, $0 \leq \alpha < 1$, $|\beta| < \frac{\pi}{2}$, if the following conditions are satisfied:

$$\Re\left(\frac{e^{\beta}zg'(z)}{g(z)^{1-\beta}}\right) > \alpha \cos \beta \quad (z \in \Delta)$$

and

$$\Re\left(\frac{we^{\beta}h'(w)}{h(w)^{1-\beta}}\right) > \alpha \cos \beta \quad (w \in \Delta),$$

where the function $h$ is given by (3). We denote by $\Sigma_\vartheta^*(\alpha, \beta)$ the class of meromorphic $\beta$-spirallike bi-univalent functions of order $\alpha$.

We note that for $\beta = 0$ and $k = 1$, the class $\Sigma_\vartheta^*(\alpha, \beta)$ reduces to the class $\Sigma^*_\vartheta(\alpha)$ introduced and studied by Samaneh G. Hamidi et al. [5].

In order to derive our main result, the following elementary may be required.

**Lemma 1** (see [1]). If $p \in \mathcal{P}$ has the power series $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, $z \in \mathbb{U}$. Then $|p_n| \leq 2$ for all $n = 1, 2, \ldots$. The estimates are sharp.
2. Coefficient Estimates for the Function Class $\Sigma^*_\vartheta(\lambda, \alpha)$

For functions in the class $\Sigma^*_\vartheta(\lambda, \alpha)$, we first establish the following result.

**Theorem 1.** If $g \in \Sigma^*_\vartheta(\lambda, \alpha)$, $\lambda \in \mathbb{R}\setminus\{\frac{1}{2}, 1\}$ and $0 < \alpha \leq 1$. Then

$$|b_0| \leq \frac{2\alpha}{|1 - \lambda|} \quad \text{and} \quad |b_1| \leq \frac{\sqrt{1^2 - 2\lambda + 5}}{|1 - \lambda||2\lambda - 1|}\alpha^2.$$

**Proof.** Since $h$ is given by (3), therefore a simple computation yields that

$$w = g(h(w)) = (b_0 + B_0) + w + \frac{b_1 + B_1}{w} + \frac{b_2 - b_1B_0 + b_2}{w^2} + \frac{B_3 - b_1B_1 + b_1B_0^2 - 2b_2B_0 + b_3}{w^3} + \cdots.$$

Comparing the initial coefficients, we have the following relations:

$$b_0 + B_0 = 0,$$

$$b_1 + B_1 = 0,$$

$$B_2 - b_1B_0 + b_2 = 0$$

and

$$B_3 - b_1B_1 + b_1B_0^2 - 2b_2B_0 + b_3 = 0.$$

From (6)-(9), we readily obtain that

$$B_0 = -b_0,$$

$$B_1 = -b_1,$$

$$B_2 = -b_1b_0 - b_2$$

and

$$B_3 = -(b_1^2 + b_1B_0^2 + 2b_2b_0 + b_3).$$

Combining the above equalities, we can show that

$$h(w) = g^{-1}(w) = w - b_0 - \frac{b_1}{w} - \frac{b_1b_0 + b_2}{w^2} - \frac{b_1^2 + b_1B_0^2 + 2b_2b_0 + b_3}{w^3} + \cdots.$$  

Given $g \in \Sigma^*_\vartheta(\lambda, \alpha)$, then, by the definition of the class $\Sigma^*_\vartheta(\lambda, \alpha)$, there exist two functions $p, q$ with positive real part in $\Delta$ and have the forms

$$p(z) = 1 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots \quad (z \in \Delta)$$

and

$$q(w) = 1 + \frac{d_1}{w} + \frac{d_2}{w^2} + \frac{d_3}{w^3} + \cdots \quad (w \in \Delta).$$
such that

\[
(1 - \lambda) \frac{zg'(z)}{g(z)} + \lambda (1 + \frac{zg''(z)}{g'(z)}) = (p(z))' \tag{17}
\]

and

\[
(1 - \lambda) \frac{wh'(w)}{h(w)} + \lambda (1 + \frac{wh''(w)}{h'(w)}) = (q(w))'. \tag{18}
\]

A computation yields

\[
(p(z))^a = 1 + \frac{ac_1}{z} + \frac{1}{2} \alpha (\alpha - 1)c_1^2 + \alpha c_2 + \cdots \tag{19}
\]

and

\[
(q(w))^a = 1 + \frac{ad_1}{w} + \frac{1}{2} \alpha (\alpha - 1)d_1^2 + \alpha d_2 + \cdots. \tag{20}
\]

Using the relations (2) and (14), we have

\[
(1 - \lambda) \frac{zg'(z)}{g(z)} + \lambda (1 + \frac{zg''(z)}{g'(z)}) = 1 - \frac{(1 - \lambda)b_0}{z} + \frac{(1 - \lambda)b_1^2 + (4 \lambda - 2)b_1}{z^2} + \cdots \tag{21}
\]

and

\[
(1 - \lambda) \frac{wh'(w)}{h(w)} + \lambda (1 + \frac{wh''(w)}{h'(w)}) = 1 + \frac{(1 - \lambda)b_0}{w} + \frac{(1 - \lambda)b_1^2 + (2 - 4 \lambda)b_1}{w^2} + \cdots, \tag{22}
\]

then comparing the coefficients in (19) and (21), we easily deduce that

\[
\alpha c_1 = -(1 - \lambda)b_0 \tag{23}
\]

and

\[
\frac{1}{2} \alpha (\alpha - 1)c_1^2 + \alpha c_2 = (1 - \lambda)b_0^2 + (4 \lambda - 2)b_1, \tag{24}
\]

and comparing the coefficients in (20) and (22), we have

\[
ad_1 = (1 - \lambda)b_0 \tag{25}
\]

and

\[
\frac{1}{2} \alpha (\alpha - 1)d_1^2 + \alpha d_2 = (1 - \lambda)b_0^2 + (2 - 4 \lambda)b_1. \tag{26}
\]

The Lemma 1 implies \(|d_1| \leq 2\), it then follows from (25) that

\[
|b_0| = \frac{a|d_1|}{|1 - \lambda|} \leq \frac{2\alpha}{|1 - \lambda|}. \tag{27}
\]

This gives the bound on \(|b_0|\), as asserted.

Next, from (23) and (25), we conclude that

\[
b_0^2 = \frac{\alpha^2(c_1^2 + d_1^2)}{2(1 - \lambda)^2}. \tag{28}
\]
and in order to find the bound on \(|b_1|\), multiplying both sides of (26) by both sides of (24), respectively, and substituting \(b_0^2\) by the above equality, we get

\[-(4\lambda - 2)^2 b_0^2 = \frac{1}{4} \alpha^2 (\alpha - 1)^2 c_1^2 d_1^2 + \frac{\alpha^2}{2} (\alpha - 1)(c_1^2 d_2 + d_1^2 c_2) + \alpha^2 c_2 d_2 - \frac{\alpha^4 (c_1^2 + d_1^2)^2}{4(1 - \lambda)^2},\]

and the Lemma 1 implies \(|c_i| \leq 2, |d_i| \leq 2\) for \(i = 1, 2\), thus we conclude that

\[|b_1| \leq \alpha \frac{1}{|1 - 2\lambda|},\]

which refines the result in Theorem 1 when \(\frac{|1 - \lambda|}{\sqrt{\alpha - 2\lambda + 1}} \leq \alpha \leq 1\).

3. Coefficient Estimates for the Function Class \(\Sigma_{\beta}^B(\alpha)\)

Next, we will find the estimates on the coefficients \(b_0\) and \(b_1\) for functions in the class \(\Sigma_{\beta}^B(\alpha)\).

**Theorem 2.** Assume \(g \in \Sigma_{\beta}^B(\alpha)\), \(0 \leq \alpha < 1\) and \(\beta\) is nonnegative real number minus 1 and 2. Then

\[|b_0| \leq \frac{2(1 - \alpha)}{|1 - \beta|} \quad \text{and} \quad |b_1| \leq 2(1 - \alpha) \sqrt{\frac{(1 - \alpha)^2}{(1 - \beta)^2} + \frac{1}{(2 - \beta)^2}}.\]

**Proof.** In view of the equality (2), we have

\[\frac{zg'\left(\frac{z}{g(z)}\right)}{g(z)^{1-\beta} z^\beta} = 1 - \frac{(1 - \beta)b_0}{z} + \frac{(2 - \beta)((1 - \beta)b_0^2 - 2b_1)}{2z^2} + \cdots.\]

(27)

On the other hand, using (14) we obtain

\[\frac{wh'(w)}{h(w)^{1-\beta} w^\beta} = 1 + \frac{(1 - \beta)b_0}{w} + \frac{(2 - \beta)((1 - \beta)b_0^2 + 2b_1)}{2w^2} + \cdots.\]

(28)

Because \(g \in \Sigma_{\beta}^B(\alpha)\), there exist two functions \(p, q\) with positive real part in \(\Delta\) and have the forms

\[p(z) = 1 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots \quad (z \in \Delta)\]

(29)

and

\[q(z) = 1 + \frac{d_1}{w} + \frac{d_2}{w^2} + \frac{d_3}{w^3} + \cdots \quad (w \in \Delta)\]

(30)
such that
\[
\frac{zg'(z)}{g(z)^{1+p}} = \alpha + (1 - \alpha)p(z)
\] (31)
and
\[
\frac{wh'(w)}{h(w)^{1+q}} = \alpha + (1 - \alpha)q(w).
\] (32)

Using (29) in (31), we obtain that
\[
\frac{zg'(z)}{g(z)^{1+p}} = 1 + \frac{(1 - \alpha)c_1}{z} + \frac{(1 - \alpha)c_2}{z^2} + \frac{(1 - \alpha)c_3}{z^3} + \cdots.
\] (33)

Using (30) in (32), we have that
\[
\frac{wh'(w)}{h(w)^{1+q}} = 1 + \frac{(1 - \alpha)d_1}{w} + \frac{(1 - \alpha)d_2}{w^2} + \frac{(1 - \alpha)d_3}{w^3} + \cdots.
\] (34)

Thus, comparing the coefficients in (27) and (33), we get
\[
(1 - \alpha)c_1 = -(1 - \beta)b_0
\] (35)
and
\[
(1 - \alpha)c_2 = \frac{1}{2}(2 - \beta)((1 - \beta)b_0^2 - 2b_1).
\] (36)

Similarly, from the relations (28) and (34) we deduce that
\[
(1 - \alpha)d_1 = (1 - \beta)b_0
\] (37)
and
\[
(1 - \alpha)d_2 = \frac{1}{2}(2 - \beta)((1 - \beta)b_0^2 + 2b_1).
\] (38)

The Lemma 1 implies \(|c_i| \leq 2\), it then follows from (35) that
\[
|b_0| = \frac{1 - \alpha}{1 - \beta}|c_1| \leq \frac{2(1 - \alpha)}{1 - \beta}.
\]

This gives the bound on \(|b_0|\) as asserted.

Next, from (35) and (37), we can conclude that
\[
b_0^2 = \frac{(1 - \alpha)^2}{(1 - \beta)^2}(c_1^2 + d_1^2),
\]
and in order to find the bound on \(|b_1|\), multiplying both sides of (38) by both sides of (36), respectively, and substituting \(b_0^2\) by above equality, we get
\[
b_1^2 = \frac{1}{16}(1 - \beta)^2(\frac{(1 - \alpha)^4}{(1 - \beta)^2}(c_1^2 + d_1^2)^2 - \frac{(1 - \alpha)^2}{(2 - \beta)^2}c_2d_2),
\]
and the Lemma 1 implies \(|c_i| \leq 2, |d_i| \leq 2\) for \(i = 1, 2\), we immediately have
\[
|b_1| \leq 2(1 - \alpha)\sqrt{\frac{(1 - \alpha)^2}{(1 - \beta)^2} + \frac{1}{(2 - \beta)^2}}.
\]
This completes the proof of Theorem 2.

Putting $\beta = 0$ in Theorem 2, we have the following corollary.

**Corollary 2.** Let $g \in \Sigma_{\alpha}(\alpha)$ be defined by (2). Then

$$|b_0| \leq 2(1 - \alpha), \quad \text{and} \quad |b_1| \leq (1 - \alpha) \sqrt{4\alpha^2 - 8\alpha + 5} \quad (0 \leq \alpha < 1).$$

**Remark 3.** Thus, the Theorem 2 reduces to the estimates in Theorem B.

**Remark 4.** Choosing $b_0 = 0$ in (2), from (36) or (38) we obtain the following inequality:

$$|b_1| \leq \frac{2(1 - \alpha)}{|1 - \beta|},$$

which refines the result in Theorem 2 when $(1 - \alpha)^2 \geq \frac{3 - 2\alpha}{(1 - \beta)^2}$.

### 4. Coefficient Estimates for the Function Class $\tilde{\Sigma}_{\alpha}(\alpha, \beta)$

Finally, for functions in the class $\tilde{\Sigma}_{\alpha}(\alpha, \beta)$, we find the following result.

**Theorem 3.** Assume $g \in \tilde{\Sigma}_{\alpha}(\alpha, \beta)$, $0 \leq \alpha < 1$ and $|\beta| < \pi$. Then

$$|b_0| \leq \frac{2}{k + 1} \sqrt{1 + \alpha(\alpha - 2) \cos^2 \beta},$$

and

(a) for each positive odd integer $k$,

$$|b_k| \leq \frac{2}{k + 1} \sqrt{1 + \alpha(\alpha - 2) \cos^2 \beta} \left[1 + \frac{4(1 + \alpha(\alpha - 2) \cos^2 \beta)}{(k + 1)^2}ight].$$

(b) for each positive even integer $k$,

$$|b_k| \leq \frac{2}{k + 1} \sqrt{1 + \alpha(\alpha - 2) \cos^2 \beta} \left[1 + \frac{2(1 + \alpha(\alpha - 2) \cos^2 \beta)}{(k + 1)^2}ight].$$

**Proof.** Since $g(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n}$. Then we have

$$e^{\beta \psi} \frac{g'(z)}{g(z)} = e^{\psi \beta} - \frac{e^{\psi \beta} b_0}{z} + \cdots + \frac{(-1)^{k+1} b_{k+1} e^{\psi \beta}}{z^{k+1}} + \cdots .$$

(41)

On the other hand, since $h$ is given by (3), a computation applying $w = g(h(w))$ yields that

$$h(w) = g^{-1}(w) = w - b_0 - \frac{b_k}{w^k} = \frac{k b_{k+1} + b_k}{w^{k+1}} + \cdots .$$

(42)

From the relation (42), we get

$$e^{\beta \psi} \frac{h'(w)}{h(w)} = e^{\psi \beta} + \frac{e^{\psi \beta} b_0}{w} + \cdots + \frac{e^{\psi \beta} b_{k+1}}{w^k} + \frac{k b_{k+1} + b_k}{w^{k+1}} + \cdots .$$

(43)

Because $g \in \Sigma_{\alpha}(\alpha, \beta)$, hence there exist two functions $p, q$ with positive real part in $\Delta$ and have the forms

$$p(z) = 1 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots \quad (z \in \Delta)$$

(44)
The Lemma 1 implies

\[ q(w) = 1 + \frac{d_1}{w} + \frac{d_2}{w^2} + \frac{d_3}{w^3} + \cdots \quad (w \in \Delta) \quad (45) \]

such that

\[ e^{\beta^2} \frac{zg'(z)}{g(z)} = \alpha \cos \beta + (e^{\beta} - \alpha \cos \beta)p(z) \quad (46) \]

and

\[ e^{\beta} \frac{wh'(w)}{h(w)} = \alpha \cos \beta + (e^{\beta} - \alpha \cos \beta)q(w). \quad (47) \]

Using (44) in (46), we obtain that

\[ e^{\beta^2} \frac{zg'(z)}{g(z)} = e^{\beta} + \frac{(e^{\beta} - \alpha \cos \beta)c_1}{z} + \cdots \frac{(e^{\beta} - \alpha \cos \beta)c_k}{z^k} + \cdots \quad (48) \]

Using (45) in (47), we have that

\[ e^{\beta} \frac{wh'(w)}{h(w)} = e^{\beta} + \frac{(e^{\beta} - \alpha \cos \beta)d_1}{w} + \cdots \frac{(e^{\beta} - \alpha \cos \beta)d_k}{w^k} + \cdots \quad (49) \]

Thus, comparing the coefficients in (41) and (48), we get

\[ (e^{\beta} - \alpha \cos \beta)c_k = (-1)^k b_0\beta^{k} e^{\beta^k} \quad (50) \]

and

\[ (e^{\beta} - \alpha \cos \beta)c_{k+1} = (-k + 1) b_k + (-1)^{k+1} b_{k+1} \beta^{k}. \quad (51) \]

Also, from the relations (43) and (49), we deduce that

\[ (e^{\beta} - \alpha \cos \beta)d_k = e^{\beta} b_0^{k} \quad (52) \]

and

\[ (e^{\beta} - \alpha \cos \beta)d_{k+1} = e^{\beta} (b_0^{k+1} + (k + 1) b_k). \quad (53) \]

The Lemma 1 implies \(|c_k| \leq 2\), it then follows from (50) that

\[ |b_0| = (|e^{\beta} - \alpha \cos \beta||c_k|) \approx (2|e^{\beta} - \alpha \cos \beta|)^{\frac{1}{2}} = \sqrt{4(1 + \alpha(\alpha - 2) \cos^2 \beta)}. \]

This gives the bound on \(|b_0|\), as asserted.

Next, in order to find the bound on \(|b_1|\). For each positive odd integer \(k\), multiplying both sides of (53) by both sides of (51), respectively, we get

\[ (k + 1)^2 b_k^2 = \frac{(e^{\beta} - \alpha \cos \beta)^2 c_{k+1} d_{k+1}}{e^{2\beta}} + b_0^{2k+2}, \]

combining the Lemma 1 and considering the bound on \(b_0\), we conclude that

\[ |b_k| \leq \frac{2}{k+1} \sqrt{1 + \alpha(\alpha - 2) \cos^2 \beta} [1 + 4^k (1 + \alpha(\alpha - 2) \cos^2 \beta)^{\frac{1}{2}}]. \]
On the other hand, for every positive even integer \( k \), from (53) and combining the Lemma 1 and considering the bound on \( b_0 \), we conclude that

\[
|b_k| \leq \frac{2}{k+1} \sqrt{1 + \alpha (\alpha - 2) \cos^2 \beta (1 + \frac{1}{2} (1 + \alpha (\alpha - 2) \cos^2 \beta)^{\frac{1}{2}})}.
\]

This completes the proof of Theorem 3.

**Remark 5.** If we take \( \beta = 0 \) and \( k = 1 \) in Theorem 3, we deduce the Theorem B.

**Remark 6.** Choosing \( b_0 = 0 \) in (2), from (51) or (53) we obtain the following inequality:

\[
|b_k| \leq \frac{2}{k+1} \sqrt{1 + \alpha (\alpha - 2) \cos^2 \beta},
\]

which refines the result in Theorem 3.

References


