Filomat 29:7 (2015), 1601–1612
DOI 10.2298/FIL1507601X

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Coefficient Estimates for Three Generalized Classes of Meromorphic and Bi-Univalent Functions

Hai-Gen Xiao[a], Qing-Hua Xu[a]

[a] *College of Mathematics and Information Science, JiangXi Normal University, NanChang 330022, China*

Abstract. In this paper, we introduce and investigate three interesting subclass $\Sigma^{*}_{\vartheta}(\lambda,\alpha)$, $\Sigma^{B}_{\vartheta}(\beta,\alpha)$ and $\tilde{\Sigma}^{*}_{\vartheta}(\alpha,\beta)$ of meromorphic and bi-univalent functions on $\Delta = \{z \in \mathbb{C} : |z| > 1\}$, obtain their coefficient estimates. The results presented in this paper generalize the recent work of several earlier authors.

1. Introduction and Definitions

Let $\mathcal{A}$ be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

We denote by $\mathcal{S}$ the subclass of the normalized analytic function class $\mathcal{A}$ consisting of all functions in which are also univalent in $\mathbb{U}$.

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f^{-1}(f(w)) = w \quad (|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}).$$

A function $f \in \mathcal{A}$ is said to be bi-univalent on $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let σ denote the class of bi-univalent functions on $\mathbb{U}$ given by (1). For a brief history and interesting examples

2010 *Mathematics Subject Classification*. Primary 30C45; Secondary 30C50

Keywords. Meromorphic functions; Bi-univalent functions; β-spirallike functions of order α; Bazilevič functions of type β and order α; λ-convex functions of order α.

Received: 06 January 2014; Accepted: 23 May 2014

Communicated by Hari M. Srivastava

Corresponding author: Qing-Hua Xu

This work was supported by NNSF of China (Grant Nos.11261022, 11061015), the Jiangxi Provincial Natural Science Foundation of China (Grant No. 20132BAB201004), Natural Science Foundation of Department of Education of Jiangxi Province and China (Grant No. GJJ12177).

Email addresses: xiaohgedu@yahoo.cn (Hai-Gen Xiao), xuqh@mail.ustc.edu.cn (Qing-Hua Xu)

of functions in the class σ, see [8]. In fact, the aforecited work of Srivastava *et al.* [8] essentially revived the investigation of various subclasses of the bi-univalent function class σ in recent years; it was followed by such works as those by Frasin and Aouf [2], Srivastava *et al.* [9-13], Xu *et al.* [19-21], and others (see, for example, [4-7]).

In this paper, the concept of bi-univalency is extended to the class of meromorphic functions defined on $\Delta = \{z \in \mathbb{C} : |z| > 1\}$. The class of functions

$$g(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n} \tag{2}$$

that are meromorphic and univalent in Δ is denoted by Σ and every univalent function g has an inverse g^{-1} satisfying the series expansion

$$g^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n},$$

where $0 < M < |w| < \infty$. Analogous to the bi-univalent analytic functions, a function $g \in \Sigma$ is said to be meromorphic and bi-univalent if $g^{-1} \in \Sigma$. The class of all meromorphic and bi-univalent functions is denoted by Σ_{ϑ}.

Estimates on the coefficients of the inverses of meromorphic univalent functions were widely investigated in the literature. For example, Schiffer[14] showed that if g, defined by (2), is in Σ with $b_0 = 0$, then $|b_2| \leq 2/3$. In 1971, Duren[1]obtained the inequality $|b_n| \leq 2/(n+1)$ for $g \in \Sigma$ with $b_k = 0$, $1 \leq k < n/2$. For g^{-1} is the inverse of g, Springer[15]showed that

$$|B_3| \leq 1 \quad \textit{and} \quad |B_3 + \frac{1}{2}B_1^2| \leq \frac{1}{2},$$

and conjectured that

$$|B_{2n-1}| \leq \frac{(2n-2)!}{n!(n-1)!} \quad (n = 1, 2, ...).$$

In 1977, Kubota [3] proved that the Springer conjecture is ture for $n = 3, 4, 5$ and subsequently Schober[16]obtained a sharp bounds for the coefficients B_{2n-1}, $1 \leq n \leq 7$. Recently, Kapoor and Mishra [6]found the coefficient estimates for inverses of meromorphic starlike functions of positive order α in Δ. Samaneh G. Hamidi *et al.*[4] introduced the following subclasses of the meromorphic bi-univalent function and obtained non-sharp estimates on the initial coefficients $|b_0|$ and $|b_1|$.

Definition 1.(see[4]) The function g given by (2) is said to belong to class $\tilde{\Sigma}^*_{\mathcal{B}}(\alpha)$ of bi-univalent strongly starlike meromorphic functions of order α, $0 < \alpha \leq 1$, if

$$\left|\arg\left(\frac{zg'(z)}{g(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (z \in \Delta)$$

and

$$\left|\arg\left(\frac{wh'(w)}{h(w)}\right)\right| < \frac{\alpha\pi}{2} \quad (w \in \Delta),$$

where the function h is given by

$$h(w) = g^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n}. \tag{3}$$

Theorem A. (see [4]) If the function g given by (2) is in the class $\tilde{\Sigma}^*_{\mathcal{B}}(\alpha)$, $0 < \alpha \le 1$, then the coefficients b_0 and b_1 satisfy the inequalities

$$|b_0| \le 2\alpha \quad \textit{and} \quad |b_1| \le \sqrt{5}\alpha^2.$$

Definition 2. (see [5]) A function g given by series expansion (2) is a meromorphic starlike bi-univalent functions of order α, $0 \le \alpha < 1$, if

$$\Re\left(\frac{zg'(z)}{g(z)}\right) > \alpha \quad (z \in \Delta)$$

and

$$\Re\left(\frac{wh'(w)}{h(w)}\right) > \alpha \quad (w \in \Delta),$$

where the function h is given by (3). The class of all meromorphic starlike bi-univalent functions of order α is denote by $\Sigma^*_{\mathcal{B}}(\alpha)$.

Theorem B. (see [5]) If the function g given by (2) is a meromorphic starllike bi-univalent function of order α, $0 \le \alpha < 1$, then the coefficients b_0 and b_1 satisfy the inequalities

$$|b_0| \le 2(1-\alpha) \quad \textit{and} \quad |b_1| \le (1-\alpha)\sqrt{4\alpha^2 - 8\alpha + 5}.$$

Here, in our present sequel to some of the aforecited works, we introduce the following three new subclasses of the function class Σ_ϑ and find coefficient estimates. As a special case, we also generalize the recent work of several earlier authors.

Definition 3. A function $g \in \Sigma_\vartheta$ given by series expansion (2) is called meromorphic λ-convex bi-univalent functions of order α, $\lambda \in \mathbb{R}$, $0 < \alpha \le 1$, if the following conditions are satisfied:

$$\left|\arg\left((1-\lambda)\frac{zg'(z)}{g(z)} + \lambda(1 + \frac{zg''(z)}{g'(z)})\right)\right| < \frac{\alpha\pi}{2} \quad (z \in \Delta)$$

and

$$\left|\arg\left((1-\lambda)\frac{wh'(w)}{h(w)} + \lambda(1 + \frac{wh''(w)}{h'(w)})\right)\right| < \frac{\alpha\pi}{2} \quad (w \in \Delta),$$

where the function h is given by (3). We denote by $\Sigma^*_\vartheta(\lambda, \alpha)$ the class of meromorphic λ-convex bi-univalent functions of order α.

We note that for $\lambda = 0$, the class $\Sigma^*_\vartheta(\lambda, \alpha)$ reduces to the class $\tilde{\Sigma}^*_{\mathcal{B}}(\alpha)$ introduced and studied by Samaneh G. Hamidi *et al.* [4].

The next definition is related to a general class called of holomorphic Bazilevič functions. We denote by $\mathcal{S}^*$ the class of all functions in $\mathcal{S}$ which are starlike in $\mathbb{U}$. Let us denote by $B(\alpha, \beta, h, p)$ the class of functions $f(z)$ which are analytic in $\mathbb{U}$, have the form (1), and which, for some $p(z) \in \mathcal{P}$, $h(z) \in \mathcal{S}^*$ and real numbers α and β with $\beta > 0$, may be represented as

$$f(z) = \left[(\beta + i\alpha)\int_0^z p(t)h(t)^\beta t^{i\alpha-1}dt\right]^{1/(\beta+i\alpha)}.$$

For the sake of brevity we shall simply denote by B and where $\mathcal{P}$ is the class of holomorphic functions p in $\mathbb{U}$ such that $p(0) = 1$ and $\Re p(z) > 0$, $z \in \mathbb{U}$. In the case when $\alpha = 0$, a computation shows that

$$zf'(z) = f(z)^{1-\beta}h(z)^\beta p(z)$$

or

$$\Re\left(\frac{zf'(z)}{f(z)^{1-\beta}h(z)^{\beta}}\right) > 0, \quad z \in \mathbb{U}. \tag{4}$$

Thomas [18] called a function satisfying the condition (4) as a Bazilevič function of type β. Furthermore, if $h(z) = z$ in (4), then the condition (4) becomes

$$\Re\left(\frac{zf'(z)}{f(z)^{1-\beta}z^{\beta}}\right) > 0, \quad z \in \mathbb{U}. \tag{5}$$

The class of all functions $f \in \mathcal{A}$ satisfies (5) and $\beta \geq 0$ is introduced by Singh [17]and the class of all such functions is denoted by $\mathrm{B}_1(\beta)$. It is clear that $\mathrm{B}_1(0) = \mathcal{S}^*$.

Definition 4. A function $g \in \Sigma_{\vartheta}$ given by series expansion (2) is said meromorphic strongly Bazilevič bi-univalent functions of type β and order α, $0 \leq \alpha < 1$, $\beta \geq 0$, if the following conditions are satisfied:

$$\Re\left(\frac{zg'(z)}{g(z)^{1-\beta}z^{\beta}}\right) > \alpha \quad (z \in \Delta)$$

and

$$\Re\left(\frac{wh'(w)}{h(w)^{1-\beta}w^{\beta}}\right) > \alpha \quad (w \in \Delta),$$

where the function h is given by (3). We denote by $\Sigma_{\vartheta}^{B}(\beta, \alpha)$ the class of meromorphic strongly Bazilevič bi-univalent functions of type β and order α.

We note that for $\beta = 0$, the class $\Sigma_{\vartheta}^{B}(\beta, \alpha)$ reduces to the class $\Sigma_{\mathcal{B}}^{*}(\alpha)$ introduced and studied by Samaneh G. Hamidi *et al.* [4].

For the function g given by series expansion (2) with $b_1 = b_2 = \cdots = b_{k-1} = 0$, some estimates on the initial coefficients can be obtained. we shall give the result in section 4 as an interesting result of the following class.

Definition 5. A function $g(z) = z + \sum\limits_{n=k}^{\infty} \frac{b_n}{z^n}$ is called weakerly meromorphic β-spirallike bi-univalent functions of order α, $0 \leq \alpha < 1$, $|\beta| < \frac{\pi}{2}$, if the following conditions are satisfied:

$$\Re\left(e^{i\beta}\frac{zg'(z)}{g(z)}\right) > \alpha\cos\beta \quad (z \in \Delta)$$

and

$$\Re\left(e^{i\beta}\frac{wh'(w)}{h(w)}\right) > \alpha\cos\beta \quad (w \in \Delta),$$

where the function h is given by (3). We denote by $\tilde{\Sigma}_{\vartheta}^{*}(\alpha, \beta)$ the class of meromorphic β-spirallike bi-univalent functions of order α.

We note that for $\beta = 0$ and $k = 1$, the class $\tilde{\Sigma}_{\vartheta}^{*}(\alpha, \beta)$ reduces to the class $\Sigma_{\mathcal{B}}^{*}(\alpha)$ introduced and studied by Samaneh G. Hamidi *et al.* [5].

In order to derive our main result, the following elementary may be required.

Lemma 1(see[1]). If $p \in \mathcal{P}$ has the power series $p(z) = 1 + \sum\limits_{n=1}^{\infty} p_n z^n$, $z \in \mathbb{U}$. Then $|p_n| \leq 2$ for all $n = 1, 2, \cdots$. The estimates are sharp.

2. Coefficient Estimates for the Function Class $\Sigma_{\vartheta}^{*}(\lambda, \alpha)$

For functions in the class $\Sigma_{\vartheta}^{*}(\lambda, \alpha)$, we first establish the following result.

Theorem 1. If $g \in \Sigma_{\vartheta}^{*}(\lambda, \alpha)$, $\lambda \in \mathbb{R}\backslash\{\frac{1}{2}, 1\}$ and $0 < \alpha \leq 1$. Then

$$|b_0| \leq \frac{2\alpha}{|1-\lambda|} \quad \textit{and} \quad |b_1| \leq \frac{\sqrt{\lambda^2 - 2\lambda + 5}}{|1-\lambda||2\lambda - 1|}\alpha^2.$$

Proof. Since h is given by (3), therefore a simple computation yields that

$$w = g(h(w)) = (b_0 + B_0) + w + \frac{b_1 + B_1}{w} + \frac{B_2 - b_1B_0 + b_2}{w^2} + \frac{B_3 - b_1B_1 + b_1B_0^2 - 2b_2B_0 + b_3}{w^3} + \cdots .$$

Comparing the initial coefficients, we have the following relations:

$$b_0 + B_0 = 0, \tag{6}$$

$$b_1 + B_1 = 0, \tag{7}$$

$$B_2 - b_1B_0 + b_2 = 0 \tag{8}$$

and

$$B_3 - b_1B_1 + b_1B_0^2 - 2b_2B_0 + b_3 = 0. \tag{9}$$

From (6)-(9), we readily obtain that

$$B_0 = -b_0, \tag{10}$$

$$B_1 = -b_1, \tag{11}$$

$$B_2 = -b_1b_0 - b_2 \tag{12}$$

and

$$B_3 = -(b_1^2 + b_1b_0^2 + 2b_2b_0 + b_3). \tag{13}$$

Combining the above equalities, we can show that

$$h(w) = g^{-1}(w) = w - b_0 - \frac{b_1}{w} - \frac{b_1b_0 + b_2}{w^2} - \frac{b_1^2 + b_1b_0^2 + 2b_2b_0 + b_3}{w^3} + \cdots . \tag{14}$$

Given $g \in \Sigma_{\vartheta}^{*}(\lambda, \alpha)$, Then, by the definition of the class $\Sigma_{\vartheta}^{*}(\lambda, \alpha)$, there exist two functions p, q with positive real part in Δ and have the forms

$$p(z) = 1 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots \quad (z \in \Delta) \tag{15}$$

and

$$q(w) = 1 + \frac{d_1}{w} + \frac{d_2}{w^2} + \frac{d_3}{w^3} + \cdots \quad (w \in \Delta) \tag{16}$$

such that

$$(1-\lambda)\frac{zg'(z)}{g(z)} + \lambda(1+\frac{zg''(z)}{g'(z)}) = (p(z))^{\alpha} \tag{17}$$

and

$$(1-\lambda)\frac{wh'(w)}{h(w)} + \lambda(1+\frac{wh''(w)}{h'(w)}) = (q(w))^{\alpha}. \tag{18}$$

A computation yields

$$(p(z))^{\alpha} = 1 + \frac{\alpha c_1}{z} + \frac{\frac{1}{2}\alpha(\alpha-1)c_1^2 + \alpha c_2}{z^2} + \cdots \tag{19}$$

and

$$(q(w))^{\alpha} = 1 + \frac{\alpha d_1}{w} + \frac{\frac{1}{2}\alpha(\alpha-1)d_1^2 + \alpha d_2}{w^2} + \cdots . \tag{20}$$

Using the relations (2) and (14), we have

$$(1-\lambda)\frac{zg'(z)}{g(z)} + \lambda(1+\frac{zg''(z)}{g'(z)}) = 1 - \frac{(1-\lambda)b_0}{z} + \frac{(1-\lambda)b_0^2 + (4\lambda-2)b_1}{z^2} + \cdots \tag{21}$$

and

$$(1-\lambda)\frac{wh'(w)}{h(w)} + \lambda(1+\frac{wh''(w)}{h'(w)}) = 1 + \frac{(1-\lambda)b_0}{w} + \frac{(1-\lambda)b_0^2 + (2-4\lambda)b_1}{w^2} + \cdots , \tag{22}$$

then comparing the coefficients in (19) and (21), we easily deduce that

$$\alpha c_1 = -(1-\lambda)b_0 \tag{23}$$

and

$$\frac{1}{2}\alpha(\alpha-1)c_1^2 + \alpha c_2 = (1-\lambda)b_0^2 + (4\lambda-2)b_1, \tag{24}$$

and comparing the coefficients in (20) and (22), we have

$$\alpha d_1 = (1-\lambda)b_0 \tag{25}$$

and

$$\frac{1}{2}\alpha(\alpha-1)d_1^2 + \alpha d_2 = (1-\lambda)b_0^2 + (2-4\lambda)b_1. \tag{26}$$

The Lemma 1 implies $|d_1| \leq 2$, it then follows from (25) that

$$|b_0| = \frac{\alpha|d_1|}{|1-\lambda|} \leq \frac{2\alpha}{|1-\lambda|}.$$

This gives the bound on $|b_0|$, as asserted.

Next, from (23) and (25), we conclude that

$$b_0^2 = \frac{\alpha^2(c_1^2+d_1^2)}{2(1-\lambda)^2},$$

and in order to find the bound on $|b_1|$, multiplying both sides of (26) by both sides of (24), respectively, and substituting b_0^2 by the above equality, we get

$$-(4\lambda-2)^2 b_1^2 = \frac{1}{4}\alpha^2(\alpha-1)^2 c_1^2 d_1^2 + \frac{\alpha^2}{2}(\alpha-1)(c_1^2 d_2 + d_1^2 c_2) + \alpha^2 c_2 d_2 - \frac{\alpha^4 (c_1^2 + d_1^2)^2}{4(1-\lambda)^2},$$

and the Lemma 1 implies $|c_i| \le 2$, $|d_i| \le 2$ for $i = 1, 2$, thus we conclude that

$$|b_1| \le \frac{\sqrt{\lambda^2 - 2\lambda + 5}}{|1-\lambda||1-2\lambda|}\alpha^2,$$

as desired.

Putting $\lambda = 0$ in Theorem 1, we have the following corollary.

Corollary 1. Let $0 < \alpha \le 1$, if $g \in \tilde{\Sigma}^*_{\mathcal{B}}(\alpha)$ be defined by (2). Then

$$|b_0| \le 2\alpha \quad \text{and} \quad |b_1| \le \sqrt{5}\alpha^2.$$

Remark 1. Thus, the Theorem 1 reduces to the estimates in Theorem A.

Remark 2. Choosing $b_0 = 0$ in (2), from (24) or (26) we obtain the following inequality:

$$|b_1| \le \frac{\alpha}{|1-2\lambda|},$$

which refines the result in Theorem 1 when $\frac{|1-\lambda|}{\sqrt{\lambda^2-2\lambda+5}} \le \alpha \le 1$.

3. Coefficient Estimates for the Function Class $\Sigma^B_\vartheta(\beta,\alpha)$

Next, we will find the estimates on the coefficients b_0 and b_1 for functions in the class $\Sigma^B_\vartheta(\beta,\alpha)$.

Theorem 2. Assume $g \in \Sigma^B_\vartheta(\beta,\alpha)$, $0 \le \alpha < 1$ and β is nonnegative real number minus 1 and 2. Then

$$|b_0| \le \frac{2(1-\alpha)}{|1-\beta|} \quad \text{and} \quad |b_1| \le 2(1-\alpha)\sqrt{\frac{(1-\alpha)^2}{(1-\beta)^2} + \frac{1}{(2-\beta)^2}}.$$

Proof. In view of the equality (2), we have

$$\frac{zg'(z)}{g(z)^{1-\beta}z^\beta} = 1 - \frac{(1-\beta)b_0}{z} + \frac{(2-\beta)((1-\beta)b_0^2 - 2b_1)}{2z^2} + \cdots . \tag{27}$$

On the other hand, using (14) we obtain

$$\frac{wh'(w)}{h(w)^{1-\beta}w^\beta} = 1 + \frac{(1-\beta)b_0}{w} + \frac{(2-\beta)((1-\beta)b_0^2 + 2b_1)}{2w^2} + \cdots . \tag{28}$$

Because $g \in \Sigma^B_\vartheta(\beta,\alpha)$, there exist two functions p, q with positive real part in Δ and have the forms

$$p(z) = 1 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots \quad (z \in \Delta) \tag{29}$$

and

$$q(z) = 1 + \frac{d_1}{w} + \frac{d_2}{w^2} + \frac{d_3}{w^3} + \cdots \quad (w \in \Delta) \tag{30}$$

such that

$$\frac{zg'(z)}{g(z)^{1-\beta}z^{\beta}} = \alpha + (1-\alpha)p(z) \tag{31}$$

and

$$\frac{wh'(w)}{h(w)^{1-\beta}w^{\beta}} = \alpha + (1-\alpha)q(w). \tag{32}$$

Using (29) in (31), we obtain that

$$\frac{zg'(z)}{g(z)^{1-\beta}z^{\beta}} = 1 + \frac{(1-\alpha)c_1}{z} + \frac{(1-\alpha)c_2}{z^2} + \frac{(1-\alpha)c_3}{z^3} + \cdots . \tag{33}$$

Using (30) in (32), we have that

$$\frac{wh'(w)}{h(w)^{1-\beta}w^{\beta}} = 1 + \frac{(1-\alpha)d_1}{w} + \frac{(1-\alpha)d_2}{w^2} + \frac{(1-\alpha)d_3}{w^3} + \cdots . \tag{34}$$

Thus, comparing the coefficients in (27) and (33), we get

$$(1-\alpha)c_1 = -(1-\beta)b_0 \tag{35}$$

and

$$(1-\alpha)c_2 = \frac{1}{2}(2-\beta)((1-\beta)b_0^2 - 2b_1). \tag{36}$$

Similarly, from the relations (28) and (34) we deduce that

$$(1-\alpha)d_1 = (1-\beta)b_0 \tag{37}$$

and

$$(1-\alpha)d_2 = \frac{1}{2}(2-\beta)((1-\beta)b_0^2 + 2b_1). \tag{38}$$

The Lemma 1 implies $|c_1| \leq 2$, it then follows from (35) that

$$|b_0| = \frac{1-\alpha}{|1-\beta|}|c_1| \leq \frac{2(1-\alpha)}{|1-\beta|}.$$

This gives the bound on $|b_0|$ as asserted.

Next, from (35) and (37), we can conclude that

$$b_0^2 = \frac{(1-\alpha)^2}{2(1-\beta)^2}(c_1^2 + d_1^2),$$

and in order to find the bound on $|b_1|$, multiplying both sides of (38) by both sides of (36), respectively, and substituting b_0^2 by above equality, we get

$$b_1^2 = \frac{1}{16}\frac{(1-\alpha)^4}{(1-\beta)^2}(c_1^2 + d_1^2)^2 - \frac{(1-\alpha)^2}{(2-\beta)^2}c_2 d_2,$$

and the Lemma 1 implies $|c_i| \leq 2$, $|d_i| \leq 2$ for $i = 1, 2$, we immediately have

$$|b_1| \leq 2(1-\alpha)\sqrt{\frac{(1-\alpha)^2}{(1-\beta)^2} + \frac{1}{(2-\beta)^2}}.$$

This completes the proof of Theorem 2.

Putting $\beta = 0$ in Theorem 2, we have the following corollary.

Corollary 2. Let $g \in \Sigma^*_{\mathcal{B}}(\alpha)$ be defined by (2). Then

$$|b_0| \leq 2(1-\alpha), \quad \textit{and} \quad |b_1| \leq (1-\alpha)\sqrt{4\alpha^2 - 8\alpha + 5} \quad (0 \leq \alpha < 1).$$

Remark 3. Thus, the Theorem 2 reduces to the estimates in Theorem B.

Remark 4. Choosing $b_0 = 0$ in (2), from (36) or (38) we obtain the following inequality:

$$|b_1| \leq \frac{2(1-\alpha)}{|1-\beta|}.$$

which refines the result in Theorem 2 when $(1-\alpha)^2 \geq \frac{3-2\beta}{(2-\beta)^2}$.

4. Coefficient Estimates for the Function Class $\tilde{\Sigma}^*_{\vartheta}(\alpha, \beta)$

Finally, for functions in the class $\tilde{\Sigma}^*_{\vartheta}(\alpha, \beta)$, we find the following result.

Theorem 3. Assume $g \in \tilde{\Sigma}^*_{\vartheta}(\alpha, \beta)$, $0 \leq \alpha < 1$ and $|\beta| < \frac{\pi}{2}$. Then

$$|b_0| \leq \sqrt[2k]{4(1 + \alpha(\alpha-2)\cos^2\beta)}$$

and

(a) for each positive odd integer k,

$$|b_k| \leq \frac{2}{k+1}\sqrt{[1 + \alpha(\alpha-2)\cos^2\beta][1 + 4^{\frac{1}{k}}(1 + \alpha(\alpha-2)\cos^2\beta)^{\frac{1}{k}}]}; \tag{39}$$

(b) for each positive even integer k,

$$|b_k| \leq \frac{2}{k+1}\sqrt{1 + \alpha(\alpha-2)\cos^2\beta}(1 + 2^{\frac{1}{k}}(1 + \alpha(\alpha-2)\cos^2\beta)^{\frac{1}{2k}}). \tag{40}$$

Proof. Since $g(z) = z + \sum\limits_{n=k}^{\infty} \frac{b_n}{z^n}$. Then we have

$$e^{i\beta}\frac{zg'(z)}{g(z)} = e^{i\beta} - \frac{e^{i\beta}b_0}{z} + \cdots + \frac{(-1)^k b_0^k e^{i\beta}}{z^k} + \frac{(-(k+1)b_k + (-1)^{k+1}b_0^{k+1})e^{i\beta}}{z^{k+1}} + \cdots. \tag{41}$$

On the other hand, since h is given by (3), a computation applying $w = g(h(w))$ yields that

$$h(w) = g^{-1}(w) = w - b_0 - \frac{b_k}{w^k} - \frac{kb_0b_k + b_{k+1}}{w^{k+1}} - \cdots. \tag{42}$$

From the relation (42), we get

$$e^{i\beta}\frac{wh'(w)}{h(w)} = e^{i\beta} + \frac{e^{i\beta}b_0}{w} + \cdots + \frac{e^{i\beta}b_0^k}{w^k} + \frac{b_0^{k+1} + (k+1)b_k}{w^{k+1}} + \cdots. \tag{43}$$

Because $g \in \tilde{\Sigma}^*_{\vartheta}(\alpha, \beta)$, hence there exist two functions p, q with positive real part in Δ and have the forms

$$p(z) = 1 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots \quad (z \in \Delta) \tag{44}$$

and

$$q(w) = 1 + \frac{d_1}{w} + \frac{d_2}{w^2} + \frac{d_3}{w^3} + \cdots \quad (w \in \Delta) \tag{45}$$

such that

$$e^{i\beta}\frac{zg'(z)}{g(z)} = \alpha\cos\beta + (e^{i\beta} - \alpha\cos\beta)p(z) \tag{46}$$

and

$$e^{i\beta}\frac{wh'(w)}{h(w)} = \alpha\cos\beta + (e^{i\beta} - \alpha\cos\beta)q(w). \tag{47}$$

Using (44) in (46), we obtain that

$$e^{i\beta}\frac{zg'(z)}{g(z)} = e^{i\beta} + \frac{(e^{i\beta} - \alpha\cos\beta)c_1}{z} + \cdots + \frac{(e^{i\beta} - \alpha\cos\beta)c_k}{z^k} + \frac{(e^{i\beta} - \alpha\cos\beta)c_{k+1}}{z^{k+1}} + \cdots. \tag{48}$$

Using (45) in (47), we have that

$$e^{i\beta}\frac{wh'(w)}{h(w)} = e^{i\beta} + \frac{(e^{i\beta} - \alpha\cos\beta)d_1}{w} + \cdots + \frac{(e^{i\beta} - \alpha\cos\beta)d_k}{w^k} + \frac{(e^{i\beta} - \alpha\cos\beta)d_{k+1}}{w^{k+1}} + \cdots. \tag{49}$$

Thus, comparing the coefficients in (41) and (48), we get

$$(e^{i\beta} - \alpha\cos\beta)c_k = (-1)^k b_0^k e^{i\beta} \tag{50}$$

and

$$(e^{i\beta} - \alpha\cos\beta)c_{k+1} = (-(k+1)b_k + (-1)^{k+1}b_0^{k+1})e^{i\beta}. \tag{51}$$

Also, from the relations (43) and (49), we deduce that

$$(e^{i\beta} - \alpha\cos\beta)d_k = e^{i\beta}b_0^k \tag{52}$$

and

$$(e^{i\beta} - \alpha\cos\beta)d_{k+1} = e^{i\beta}(b_0^{k+1} + (k+1)b_k). \tag{53}$$

The Lemma 1 implies $|c_k| \leq 2$, it then follows from (50) that

$$|b_0| = (|e^{i\beta} - \alpha\cos\beta||c_k|)^{\frac{1}{k}} \leq (2|e^{i\beta} - \alpha\cos\beta|)^{\frac{1}{k}} = \sqrt[2k]{4(1 + \alpha(\alpha - 2)\cos^2\beta)}.$$

This gives the bound on $|b_0|$, as asserted.

Next, in order to find the bound on $|b_1|$. For each positive odd integer k, multiplying both sides of (53) by both sides of (51), respectively, we get

$$(k+1)^2 b_k^2 = -\frac{(e^{i\beta} - \alpha\cos\beta)^2 c_{k+1}d_{k+1}}{e^{i2\beta}} + b_0^{2k+2},$$

combining the Lemma 1 and considering the bound on b_0, we conclude that

$$|b_k| \leq \frac{2}{k+1}\sqrt{[1 + \alpha(\alpha - 2)\cos^2\beta][1 + 4^{\frac{1}{k}}(1 + \alpha(\alpha - 2)\cos^2\beta)^{\frac{1}{k}}]}.$$

On the other hand, for every positive even integer k, from (53) and combining the Lemma 1 and considering the bound on b_0, we conclude that

$$|b_k| \leq \frac{2}{k+1}\sqrt{1+\alpha(\alpha-2)\cos^2\beta}(1+2^{\frac{1}{k}}(1+\alpha(\alpha-2)\cos^2\beta)^{\frac{1}{2k}}).$$

This completes the proof of Theorem 3.

Remark 5. If we take $\beta = 0$ and $k = 1$ in Theorem 3, we deduce the Theorem B.

Remark 6. Choosing $b_0 = 0$ in (2), from (51) or (53) we obtain the following inequality:

$$|b_k| \leq \frac{2}{k+1}\sqrt{1+\alpha(\alpha-2)\cos^2\beta},$$

which refines the result in Theorem 3.

References

[1] P.L. Duren, Coefficients of meromorphic schlicht functions, *Proc.Amer.Math.Soc.* **28** (1971), 169-172.

[2] B.A. Frasin and M.K. Aouf, New subclasses of bi-univalent functions, *Applied Mathematics Letters*. **24** (2011), 1569-1573.

[3] Y.Kubota, Coefficients of meromorphic univalent functions, *Kōdai Math. Sem. Rep.* **28** (1976/77), 253-261.

[4] Samaneh G. Hamidi, Suzeini Abd Halim and Jay M. Jahangiri, Coefficient estimates for bi-univalent strongly starlike and Bazilevič functions, *International journal of mathematics research.* **5** (2013), 87-96.

[5] Samaneh G. Hamidi, T. Janani, G. Murugusundaramoorthy and Jay M. Jahangiri, Coefficient estimates for certain classes of meromorphic functions, *C.R.Acad. Sci. Paris, Ser. I.* **352** (2014), 277-282.

[6] G.P. Kapoor and A.K. Mishra, Coefficient estimates for inverses of starlike functions of positive order, *J. Math. Anal. Appl.* **329** (2007), 922-934.

[7] M.Lewin, On a coefficient problem for bi-univalent functions, *Proc. Amer. Math. Soc.* **18** (1967), 63-68.

[8] H.M. Srivastava, A.K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, *Appl. Math. Lett.* **23** (2010), 1188-1192.

[9] H. M. Srivastava, Some inequalities and other results associated with certain subclasses of univalent and bi-univalent analytic functions, in: *Nonlinear Analysis*: *Stability*, *Approximation*, *and Inequalities* (Panos M. Pardalos, Pando G. Georgiev and Hari M. Srivastava, Editors.), Springer Series on Optimization and Its Applications, Vol. **68**, Springer-Verlag, Berlin, Heidelberg and New York, 2012, pp. 607–630.

[10] H. M. Srivastava, S. Bulut, M. Çaglar and N. Yağmur, Coefficient estimates for a general subclass of analytic and bi-univalent functions, *Filomat* **27** (2013), 831–842.

[11] H. M. Srivastava, N. Magesh and J. Yamini, Initial coefficient estimates for bi-lambda-convex and bi-mu-starlike functions connected with arithmetic and geometric means, *Electron. J. Math. Anal. Appl.* **2** (2014), 152–162 (electronic).

[12] H. M. Srivastava, G. Murugusundaramoorthy and N. Magesh, Certain subclasses of bi-univalent functions associated with the Hohlov operator, *Global J. Math. Anal.* **1** (2) (2013), 67–73.

[13] H. M. Srivastava, G. Murugusundaramoorthy and K. Vijaya, Coefficient estimates for some families of bi-Bazilevič functions of the Ma-Minda type involving the Hohlov operator, *J. Class. Anal.* **2** (2013), 167-181.

[14] M.Schiffer, Sur un problème déxtrémum de la représentation conforme, *Bull. Soc. Math.*France **66** (1938), 48-55.

[15] G.Springer, The coefficient problem for schlicht mappings of the exterior of the unit circle, *Trans. Amer. Math. Soc.* **70** (1951), 421-450.

[16] G.Schober: Coefficients of inverses of meromorphic univalent functions. Proc. Amer. Math. Soc. **67** (1977), no.1, 111-116.

[17] R.Singh, On Bazilevič functions, *Proc. Amer. Math. Soc.* **38** (1973), 261-271.

[18] D.K. Thomas, On Bazilevič functions, *Trans. Amer. Math. Soc.* **132** (1968), 353-361.

[19] Q.-H. Xu, H. M. Srivastava and Z. Li, A certain subclass of analytic and close-to-convex functions, *Appl. Math. Lett.* **24** (2011), 396–401.

[20] Q.-H. Xu, Y.-C. Gui and H. M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, *Appl. Math. Lett.* **25** (2012), 990–994.

[21] Q.-H. Xu, H.-G. Xiao and H. M. Srivastava, A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems, *Appl. Math. Comput.* **218** (2012), 11461–11465.

[22] W. Rudin, Real and Complex Analysis, (3rd edition), McGraw-Hill, New York, 1986.

[23] J. A. Goguen, L-fuzzy sets, Journal of Mathematical Analysis and Applications 18 (1967) 145–174.
[24] P. Erdös, S. Shelah, Separability properties of almost-disjoint families of sets, Israel Journal of Mathematics 12 (1972) 207–214.