Filomat 29:7 (2015), 1587–1599
DOI 10.2298/FIL1507587M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Equivalent Characterizations of Inner Product Spaces and Their Consequences

Dan Ştefan Marinescu[a], Mihai Monea[b], Mihai Opincariu[c], Marian Stroe[d]

[a]*Colegiul Naţional "Iancu De Hunedoara" Hunedoara, Romania*
[b]*Colegiul Naţional "Decebal" Deva, Romania*
[c]*Colegiul Naţional "Avram Iancu" Brad, Romania*
[d]*Colegiul Economic "Emanoil Gojdu" Hunedoara, Romania*

Abstract. In this paper, first we prove the equivalence of the Frechet and the Jordan-Neumann relations in normed spaces. Next, we will present some characterizations of the inner product spaces and we will show their equivalence. Finally, we will prove some consequences of our main results.

1. Introduction

The problem of finding necessary and sufficient conditions for a normed space to be an inner product space has been previously investigated by many mathematicians. The first results were obtained by M. Frechet [6] in 1935.

Proposition 1.1. (Frechet) *A complex normed space* $(X, \ \|\cdot\|)$ *is an inner product space if and only if*

$$\left\|x+y+z\right\|^2+\|x\|^2+\left\|y\right\|^2+\|z\|^2=\left\|x+y\right\|^2+\left\|y+z\right\|^2+\left\|z+y\right\|^2,$$

for all $x, y, z \in X$.

In the same year, Jordan and von Neumann [9] have given another characterization of the inner product spaces. This result is also known as "*the parallelogram law*".

Proposition 1.2. (Jordan, von Neumann) *A complex normed space* $(X, \|\cdot\|)$ *is an inner product space if and only if*

$$\left\|x+y\right\|^2+\left\|x-y\right\|^2=2\|x\|^2+2\left\|y\right\|^2,$$

for all $x, y \in X$.

2010 *Mathematics Subject Classification*. Primary 46C15.
Keywords. normed spaces, inner product spaces
Received: 06 January 2014; Accepted: 01 July 2014
Communicated by Hari M. Srivastava
Email addresses: marinescuds@gmail.com (Dan Ştefan Marinescu), mihaimonea@yahoo.com (Mihai Monea), opincariumihai@yahoo.com (Mihai Opincariu), maricu_stroe@yahoo.com (Marian Stroe)

These two characterizations are equivalent, as we will see in the next paragraph. More similar results are known today. A large collection of the results can be found in [2]. In this paper we want to present other characterizations of inner product spaces. The results will be detailed in the third section and they are more general than most of those mentioned in the bibliography. Finally, we will present a consistent lot of consequences.

2. The Equivalence of the First Two Characterizations

The characterizations described in the Propositions 1.1 and 1.2 are the best known at this moment. But these results are equivalent and this equivalence exists in any normed spaces, not necessarily inner product spaces.

Proposition 2.2. *Let $(X, \|\cdot\|)$ be a real or complex normed space. The next two statements are equivalent:*

(FR) $\|x+y+z\|^2 + \|x\|^2 + \|y\|^2 + \|z\|^2 = \|x+y\|^2 + \|y+z\|^2 + \|z+y\|^2$, *for all $x, y, z \in X$,*

(JN) $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$, *for all $x, y \in X$.*

Proof. (FR)⇒(JN) If we choose $z = -y$ in (FR), we observe that we obtain (JN).

(JN)⇒(FR) We apply (JN) three times and we obtain

$$\|x+y+z\|^2 + \|x\|^2 = \frac{1}{2}\|2x+y+z\|^2 + \frac{1}{2}\|y+z\|^2$$

and

$$\|y\|^2 + \|z\|^2 = \frac{1}{2}\|y+z\|^2 + \frac{1}{2}\|y-z\|^2.$$

But

$$\frac{1}{2}\|2x+y+z\|^2 + \frac{1}{2}\|y-z\|^2 = \frac{1}{2}\|(x+y)+(x+z)\|^2 + \frac{1}{2}\|(x+y)-(x+z)\|^2 =$$

$$= \|x+y\|^2 + \|x+z\|^2.$$

By adding these equalities, we obtain (FR). □

3. Main Results

In this section we present six characterizations of an inner product space. These results are presented in Theorem 3.1. Firstly, we consider $(X, \|\cdot\|)$ a real or complex normed space.

Theorem 3.1. *The following seven statements are equivalent:*

a) *X is an inner product space;*

b) *For all $n \in \mathbb{N}$, $n \geq 2$, for all $a_1, a_2, ..., a_n \in \mathbb{R}$ and for any $x, x_1, x_2, ..., x_n \in X$*

$$\left\|\left(\sum_{k=1}^{n} a_k\right)x - \sum_{k=1}^{n} a_k x_k\right\|^2 = \left(\sum_{k=1}^{n} a_k\right)\sum_{k=1}^{n} a_k \|x - x_k\|^2 - \sum_{1\leq k<l\leq n} a_k a_l \|x_l - x_k\|^2;$$

c) *For all $n \in \mathbb{N}$, $n \geq 2$, for all $a_1, a_2, ..., a_n \in \mathbb{R}$ and for any $x_1, x_2, ..., x_n \in X$*

$$\left\|\sum_{k=1}^{n} a_k x_k\right\|^2 = (a_1 + a_2 + ... + a_n)\sum_{k=1}^{n} a_k \|x_k\|^2 - \sum_{1\leq k<l\leq n} a_k a_l \|x_l - x_k\|^2;$$

d) *For all* $n \in \mathbb{N},\ n \geq 2$, *for all* $a_1, a_2, ..., a_n \in \mathbb{R}$, *with* $\sum_{k=1}^{n} a_k = 1$ *and for any* $x_1, x_2, ..., x_n \in X$

$$\left\| \sum_{k=1}^{n} a_k x_k \right\|^2 = \sum_{k=1}^{n} a_k \|x_k\|^2 - \sum_{1 \leq k < l \leq n} a_k a_l \|x_l - x_k\|^2 ;$$

e) *For all* $n \in \mathbb{N},\ n \geq 2$ *, for all* $a_1, a_2, ..., a_n \in \mathbb{R}^*$ *with* $\sum_{k=1}^{n} a_k \neq 0$ *and for any* $x_1, x_2, ..., x_n \in X$

$$\sum_{k=1}^{n} \frac{\|x_k\|^2}{a_k} - \frac{1}{\sum_{k=1}^{n} a_k} \left\| \sum_{k=1}^{n} x_k \right\|^2 = \frac{1}{\sum_{k=1}^{n} a_k} \sum_{1 \leq k < l \leq n} \frac{\left\| a_k x_l - a_j x_k \right\|^2}{a_k a_j};$$

f) *For all* $n \in N,\ n \geq 2$ *and for any* $x_1, x_2, ..., x_n \in X$

$$\frac{1}{n} \left\| \sum_{k=1}^{n} x_k \right\|^2 = \sum_{k=1}^{n} \|x_k\|^2 - \frac{1}{n} \sum_{1 \leq k < l \leq n} \|x_l - x_k\|^2 ;$$

g) (Hlawka) *For all* $n \in \mathbb{N},\ n \geq 2$ *and for any* $x_1, x_2, ..., x_n \in X$

$$\left\| \sum_{k=1}^{n} x_k \right\|^2 + (n-2) \left(\sum_{k=1}^{n} \|x_k\|^2 \right) = \sum_{1 \leq k < l \leq n} \|x_l + x_k\|^2 .$$

Proof. We will prove the next implications :

$$a) \Rightarrow b) \Rightarrow c) \Rightarrow d) \Rightarrow e) \Rightarrow f) \Rightarrow g) \Rightarrow a)$$

$a) \Rightarrow b)$ We use the identity,

$$\langle x - y, x - z \rangle + \langle x - z, x - y \rangle = \left\| x - y \right\|^2 + \|x - z\|^2 - \left\| y - z \right\|^2 ,$$

which is true for all $x, y, z \in X$. Then, we obtain

$$\left\| (a_1 + a_2 + ... + a_n)\, x - \sum_{k=1}^{n} a_k x_k \right\|^2 = \left\| \sum_{k=1}^{n} a_k (x - x_k) \right\|^2$$

$$= \sum_{k=1}^{n} a_k^2 \|x - x_k\|^2 + \sum_{1 \leq k < l \leq n} a_k a_l \langle x - x_k, x - x_l \rangle + \sum_{1 \leq k < l \leq n} a_k a_l \langle x - x_l, x - x_k \rangle$$

$$= \sum_{k=1}^{n} a_k^2 \|x - x_k\|^2 + \sum_{1 \leq k < l \leq n} a_k a_l \left(\langle x - x_k, x - x_l \rangle + \langle x - x_l, x - x_k \rangle \right)$$

$$= \sum_{k=1}^{n} a_k^2 \|x - x_k\|^2 + \sum_{1 \leq k < l \leq n} a_k a_l \left(\|x - x_k\|^2 + \|x - x_l\|^2 - \|x_k - x_l\|^2 \right)$$

$$= \left(\sum_{k=1}^{n} a_k^2 + \sum_{1 \leq k < l \leq n} a_k a_l + \sum_{1 \leq l < k \leq n} a_k a_l \right) \|x - x_k\|^2 - \sum_{1 \leq k < l \leq n} a_k a_l \|x_k - x_l\|^2$$

$$= \left(\sum_{k=1}^{n} a_k\right) \sum_{k=1}^{n} a_k \|x - x_k\|^2 - \sum_{1 \le k < l \le n} a_k a_l \|x_l - x_k\|^2 .$$

b) ⇒ *c*) We choose $x = 0$.
c) ⇒ *d*) It is obvious from the statement *c)*.
d) ⇒ *e*) Firstly, we make the substitution

$$a_k \to \frac{a_k}{\sum\limits_{k=1}^{n} a_k}.$$

The relation from d) becomes

$$\frac{1}{\left(\sum\limits_{k=1}^{n} a_k\right)^2} \left\| \sum_{k=1}^{n} a_k x_k \right\|^2 = \frac{1}{\sum\limits_{k=1}^{n} a_k} \sum_{k=1}^{n} a_k \|x_k\|^2 - \frac{1}{\left(\sum\limits_{k=1}^{n} a_k\right)^2} \sum_{1 \le k < l \le n} a_k a_l \|x_l - x_k\|^2$$

$$\Leftrightarrow \sum_{k=1}^{n} a_k \|x_k\|^2 - \frac{1}{\sum\limits_{k=1}^{n} a_k} \left\| \sum_{k=1}^{n} a_k x_k \right\|^2 = \frac{1}{\sum\limits_{k=1}^{n} a_k} \sum_{1 \le k < l \le n} a_k a_l \|x_l - x_k\|^2 .$$

Now, we replace x_k with $\frac{x_k}{a_k}$ and the conclusion follows immediately.
e) ⇒ *f*) We apply the relations from *d)* in the case $a_1 = a_2 = ... = a_n = \frac{1}{n}$.
f) ⇒ *g*) Firstly, we apply f) for any x_k and x_l, with $n = 2$. We obtain

$$\frac{1}{2} \|x_k + x_l\|^2 = \|x_k\|^2 + \|x_l\|^2 - \frac{1}{2} \|x_l - x_k\|^2 .$$

Then, the relation from f) becomes

$$\left\| \sum_{k=1}^{n} x_k \right\|^2 = n \sum_{k=1}^{n} \|x_k\|^2 - \sum_{1 \le k < l \le n} \left(2 \|x_k\|^2 + 2 \|x_l\|^2 - \|x_k + x_l\|^2\right)$$

$$= n \sum_{k=1}^{n} \|x_k\|^2 - 2 \sum_{1 \le k < l \le n} \left(\|x_k\|^2 + \|x_l\|^2\right) + \sum_{1 \le k < l \le n} \|x_k + x_l\|^2$$

$$= n \sum_{k=1}^{n} \|x_k\|^2 - 2(n-1) \sum_{k=1}^{n} \|x_k\|^2 + \sum_{1 \le k < l \le n} \|x_k + x_l\|^2$$

and we obtain the conclusion.
g) ⇒ *a*) If we choose $n = 3$, we obtain

$$\|x_1 + x_2 + x_3\|^2 + \|x_1\|^2 + \|x_2\|^2 + \|x_3\|^2 = \|x_1 + x_2\|^2 + \|x_2 + x_3\|^2 + \|x_1 + x_3\|^2$$

for all $x_1, x_2, x_3 \in X$. Proposition 1.1 concludes our proof.
□

Remarks.

1. The characterizations from *b)*, *c)* or *d)* are more general than other characterizations of the same type because the coefficients $a_1, a_2, .., a_n$ do not have to be nonnegative numbers. For example, the characterizations from *b)*, but in more restrictive conditions, represents the main result from [14] (see Lemma 2.3.)

2. The characterizations from *e)* represents a transformation for a normed space of the Theorem 3 from [15].

3. The characterizations from *f)* or *g)* are known but are very useful for our chain of implications.

4. Consequences

In this final section we want to show the usefulness of the results from the previous section. We will present some results from our references, but with solutions that use the Theorem 3.1. We will start with Corollary 2.2 from [12] and we will give it in the next improved form:

Corollary 4.1. *A real or complex normed space* $(X, \|\cdot\|)$ *is an inner product space if and only if*

$$\|ax+by\|^2 + \|bx-ay\|^2 = (a^2+b^2)(\|x\|^2+\|y\|^2),$$

for any $x, y \in X$ *and* $a, b \in \mathbb{R}$.

Proof. If the normed space X is an inner product space we have, from *c)*, Theorem 3.1, the next identity

$$\|ax+by\|^2 = (a+b)(a\|x\|^2 + b\|y\|^2) - ab\|x-y\|^2, \quad (1)$$

for any $x, y \in X$ and $a, b \in \mathbb{R}$. This is equivalent to

$$\|ax+by\|^2 = a^2\|x\|^2 + b^2\|y\|^2 + ab\left(\|x\|^2+\|y\|^2-\|x-y\|^2\right). \quad (2)$$

If we replace in (1), y with $-y$ and reverse a with b we obtain

$$\|bx-ay\|^2 = (b+a)(b\|x\|^2 + a\|y\|^2) - ab\|x+y\|^2,$$

$$\Rightarrow \|bx-ay\|^2 = b^2\|x\|^2 + a^2\|y\|^2 + ab\left(\|x\|^2+\|y\|^2-\|x+y\|^2\right). \quad (3)$$

If we choose $a=b=1$ in *(1)* we obtain "*the parallelogram law*". Now, the conclusion follows if we add the relations *(2)* and *(3)*.

The second part of the proof is obvious, if we choose $a=b=1$ in the hypothesis. □

The next results are represented by Corollary 2.2. from [13]. We recall its content here, but we will present a different solution.

Corollary 4.2. (Moslehian, Rassias) *A real or complex normed space* $(X, \|\cdot\|)$ *is an inner product space if and only if*

$$\sum_{a_i\in\{1,-1\}} \|x_1 + \sum_{i=2}^{n} a_i x_i\|^2 = \sum_{a_i\in\{1,-1\}} \left(\|x_1\| + \sum_{i=2}^{n} a_i\|x_i\|\right)^2,$$

for any $n \in \mathbb{N}, n \geq 2$ *and for all* $x_1, x_2, ..., x_n \in X$.

Proof. Firstly, it is easy to observe that

$$\sum_{a_i\in\{1,-1\}} \left(\|x_1\| + \sum_{i=2}^{n} a_i\|x_i\|\right)^2 = 2^{n-1}\sum_{i=1}^{n}\|x_i\|^2. \quad (*)$$

Now we apply *c)* from Theorem 3.1. and we obtain

$$\|x_1 + \sum_{i=2}^{n} a_i x_i\|^2 = \left(1+\sum_{i=2}^{n} a_i\right)\left(\|x_1\|^2 + \sum_{i=2}^{n} a_i\|x_i\|^2\right) -$$

$$-\sum_{i=2}^{n} a_i\|x_1-x_i\|^2 - \sum_{2\leq i<j\leq n} a_i a_j\|x_i-x_j\|^2.$$

We do the summation for $a_i \in \{1, -1\}$ and we obtain

$$\|x_1 + \sum_{i=2}^{n} a_i x_i\|^2 = 2^{n-1} \sum_{i=1}^{n} \|x_i\|^2.$$

We use (*) and we obtain the conclusion for the first part.

The second part is obtained if we choose $n = 2$. □

The results from Proposition 4.3 and 4.4. can be considered as an extension for "*the distance formula*" in the inner product space.

Proposition 4.3. *Let X be an inner product space and* $a_1, a_2, ..., a_n \in \mathbb{R}, b_1, b_2, ..., b_n \in \mathbb{R}$ *with* $\sum_{i=1}^{n} a_i = 1$ *and* $\sum_{i=1}^{n} b_i = 1$. *Let be* $x_1, x_2, ..., x_n \in X$. *If* $u, v \in X$ *so that* $u = \sum_{i=1}^{n} a_i x_i$ *and* $v = \sum_{i=1}^{n} b_i x_i$ *then*

$$\|u - v\|^2 = -\sum_{1 \le i < j \le n} \left(b_j - a_j\right)(b_i - a_i) \left\|x_j - x_i\right\|^2$$

Proof. Using various properties we obtain the following:

$$\|u - v\|^2 = \left\|\sum_{i=1}^{n} b_i u - \sum_{i=1}^{n} b_i x_i\right\|^2$$

$$= \sum_{i=1}^{n} b_i \|u - x_i\|^2 - \sum_{1 \le i < j \le n} b_j b_i \left\|x_j - x_i\right\|^2$$

$$= \sum_{i=1}^{n} b_i \left\|\sum_{j=1}^{n} a_j x_j - x_i\right\|^2 - \sum_{1 \le i < j \le n} b_j b_i \left\|x_j - x_i\right\|^2$$

$$= \sum_{i=1}^{n} b_i \left\|\sum_{i=1}^{n} a_j x_j - \sum_{j=1}^{n} a_j x_i\right\|^2 - \sum_{1 \le i < j \le n} b_j b_i \left\|x_j - x_i\right\|^2$$

$$= \sum_{i=1}^{n} b_i \left(\sum_{j=1}^{n} a_j \left\|x_j - x_i\right\|^2 - \sum_{1 \le i < j \le n} a_i a_j \left\|x_j - x_i\right\|^2\right) - \sum_{1 \le i < j \le n} b_j b_i \left\|x_j - x_i\right\|^2$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} b_i a_j \left\|x_j - x_i\right\|^2 - \sum_{1 \le i < j \le n} a_i a_j \left\|x_j - x_i\right\|^2 - \sum_{1 \le i < j \le n} b_j b_i \left\|x_j - x_i\right\|^2$$

$$= -\sum_{1 \le i < j \le n} \left(b_j - a_j\right)(b_i - a_i) \left\|x_j - x_i\right\|^2,$$

q.e.d. □

Proposition 4.4. *Let X be an inner product space and $a_1, a_2, ..., a_n \in \mathbb{R}$ with $\sum_{i=1}^{n} a_i \neq 0$. Let $x_1, x_2, ..., x_n \in X$ and $y_1, y_2, ..., y_n \in X$. If $u, v \in X$ such that $u = \sum_{i=1}^{n} a_i x_i$ and $v = \sum_{i=1}^{n} a_i y_i$ then*

$$\|u - v\|^2 = \sum_{i,j=1}^{n} a_i a_j \left\|x_i - y_j\right\|^2 - \sum_{1 \le i < j \le n} a_i a_j \left(\left\|x_i - x_j\right\|^2 + \left\|y_i - y_j\right\|^2\right).$$

Proof. Firstly, we will prove the result in the case $\sum_{i=1}^{n} a_i = 1$. Then

$$\|u - v\|^2 = \left\|\sum_{i=1}^{n} a_i u - \sum_{i=1}^{n} a_i y_i\right\|^2$$

$$= \sum_{j=1}^{n} a_j \left\|u - y_j\right\|^2 - \sum_{1 \le i < j \le n} a_i a_j \left\|y_i - y_j\right\|^2$$

$$= \sum_{j=1}^{n} a_j \left\|\sum_{i=1}^{n} a_i x_i - y_j\right\|^2 - \sum_{1 \le i < j \le n} a_i a_j \left\|y_i - y_j\right\|^2$$

$$= \sum_{j=1}^{n} a_j \left\|\sum_{i=1}^{n} a_i x_i - \sum_{i=1}^{n} a_i y_j\right\|^2 - \sum_{1 \le i < j \le n} a_i a_j \left\|y_i - y_j\right\|^2$$

$$= \sum_{j=1}^{n} a_j \left(\sum_{i=1}^{n} a_i \left\|x_i - y_j\right\|^2 - \sum_{1 \le i < j \le n} a_i a_j \left\|x_i - x_j\right\|^2\right) - \sum_{1 \le i < j \le n} a_i a_j \left\|y_i - y_j\right\|^2$$

$$= \sum_{i,j=1}^{n} a_j a_i \left\|x_i - y_j\right\|^2 - \sum_{1 \le i < j \le n} a_i a_j \left(\left\|x_i - x_j\right\|^2 + \left\|y_i - y_j\right\|^2\right)$$

Next, we consider the general case, so $\sum_{i=1}^{n} a_i \neq 0$. We assume now $b_1, b_2, ..., b_n \in \mathbb{R}$ so that $b_i = \frac{a_i}{\sum_{i=1}^{n} a_i}$ for all $i \in \{1, 2, .., n\}$. It is easy to observe that $\sum_{i=1}^{n} b_i = 1$. Now, we consider $u', v' \in X$ so that $u' = \sum_{i=1}^{n} b_i x_i$ and $v' = \sum_{i=1}^{n} b_i y_i$. Then $u' = \frac{1}{\sum_{i=1}^{n} a_i} u$ and $v' = \frac{1}{\sum_{i=1}^{n} a_i} v$. By using first part of the proof, we obtain

$$\|u' - v'\|^2 = \sum_{i,j=1}^{n} b_j b_i \left\|x_i - y_j\right\|^2 - \sum_{1 \le i < j \le n} b_i b_j \left(\left\|x_i - x_j\right\|^2 + \left\|y_i - y_j\right\|^2\right).$$

Then

$$\|u - v\|^2 = \left(\sum_{i=1}^{n} a_i\right)^2 \|u' - v'\|^2$$

$$= \left(\sum_{i=1}^{n} a_i\right)^2 \left(\sum_{i,j=1}^{n} \frac{a_j}{\sum_{i=1}^{n} a_i} \frac{a_i}{\sum_{i=1}^{n} a_i} \left\|x_i - y_j\right\|^2 - \sum_{1 \le i < j \le n} \frac{a_j}{\sum_{i=1}^{n} a_i} \frac{a_i}{\sum_{i=1}^{n} a_i} \left(\left\|x_i - x_j\right\|^2 + \left\|y_i - y_j\right\|^2\right)\right)$$

$$= \sum_{i,j=1}^{n} a_i a_j \left\|x_i - y_j\right\|^2 - \sum_{1\le i<j\le n} a_i a_j \left(\left\|x_i - x_j\right\|^2 + \left\|y_i - y_j\right\|^2\right).$$

□

As a consequence of Proposition 4.4., we obtain a very short proof for the next corollary. This result represents another characterization of the inner product spaces and it is known as a version of "*the generalized parallelogram law*" (see [11], pag. 3).

Corolarry 4.5. (Moslehian) *Let* $(X, \|\cdot\|)$ *be a real or complex normed space. Then the norm is provided from an inner product space if and only if*

$$\sum_{i,j=1}^{n} \left\|x_i - x_j\right\|^2 + \sum_{i,j=1}^{n} \left\|y_i - y_j\right\|^2 = 2\sum_{i,j=1}^{n} \left\|x_i - y_j\right\|^2 - 2\left\|\sum_{i=1}^{n} (x_i - y_i)\right\|^2,$$

for any $x_1, x_2, ..., x_n \in X$ *and* $y_1, y_2, ..., y_n \in X$.

Proof. Let $u = \sum_{i=1}^{n} x_i$ and $v = \sum_{i=1}^{n} x_i$. Then

$$\left\|\sum_{i=1}^{n} (x_i - y_i)\right\|^2 = \|u - v\|^2$$

$$= \sum_{i,j=1}^{n} \left\|x_i - y_j\right\|^2 - \sum_{1\le i<j\le n} \left(\left\|x_i - x_j\right\|^2 + \left\|y_i - y_j\right\|^2\right)$$

$$= \sum_{i,j=1}^{n} \left\|x_i - y_j\right\|^2 - \frac{1}{2}\sum_{i,j=1}^{n} \left(\left\|x_i - x_j\right\|^2 + \left\|y_i - y_j\right\|^2\right)$$

$$\Rightarrow \sum_{i,j=1}^{n} \left\|x_i - x_j\right\|^2 + \sum_{i,j=1}^{n} \left\|y_i - y_j\right\|^2 = 2\sum_{i,j=1}^{n} \left\|x_i - y_j\right\|^2 - 2\left\|\sum_{i=1}^{n} (x_i - y_i)\right\|^2.$$

For the reverse implication , we choose $n = 2$ and $y_1 = y_2 = 0$ and we obtain

$$\|x_1 - x_2\|^2 + \|x_2 - x_1\|^2 = 4\|x_1\|^2 + 4\|x_2\|^2 - 2\|x_1 + x_2\|^2.$$

Now, our proof is ready due to Proposition 1.2. □

Another version of "*the generalized parallelogram law*" could be considered the Corollary 2.7. from [11]. In this paper we will give another proof for this result.

Corollary 4.6.(Moslehian) *The real or complex normed space* $(X, \|\cdot\|)$ *is an inner product space if and only if for any* $n \in \mathbb{N}, n \ge 2$ *we have*

$$\sum_{1\le i<j\le n} \left\|\sqrt{\frac{r_i}{r_j}}x_i - \sqrt{\frac{r_j}{r_i}}x_j\right\|^2 + \sum_{1\le i<j\le n} \left\|\sqrt{\frac{r_i}{r_j}}y_i - \sqrt{\frac{r_j}{r_i}}y_j\right\|^2$$

$$= \sum_{i,j=1}^{n} \left\|\sqrt{\frac{r_i}{r_j}}x_i - \sqrt{\frac{r_j}{r_i}}y_j\right\|^2 - \left\|\sum_{i=1}^{n} (x_i - y_i)\right\|^2,$$

for all $x_1, x_2, ..., x_n \in X$, $y_1, y_2, ..., y_n \in X$ *and* $r_1, r_2, .., r_n \in (0, \infty)$.

Proof. We apply Proposition 4.4. and for $i = \overline{1,n}$, we replace $a_i = \frac{1}{r_i}$, x_i with $r_i x_i$ and y_i with $r_i y_i$. We obtain

$$\sum_{1\le i<j\le n} \frac{1}{r_i r_j} \left\|r_i x_i - r_j x_j\right\|^2 + \sum_{1\le i<j\le n} \frac{1}{r_i r_j} \left\|r_i y_i - r_j y_j\right\|^2 =$$

$$= \sum_{i,j=1}^{n} \frac{1}{r_i r_j} \left\|r_i x_i - r_j y_j\right\|^2 - \left\|\sum_{i=1}^{n} \frac{1}{r_i}(r_i x_i - r_i y_i)\right\|^2,$$

which is equivalent to

$$\sum_{1\le i<j\le n} \frac{1}{r_i r_j} \left\| \sqrt{r_i r_j}\left(\sqrt{\frac{r_i}{r_j}}x_i - \sqrt{\frac{r_j}{r_i}}x_j\right)\right\|^2 + \sum_{1\le i<j\le n} \frac{1}{r_i r_j} \left\| \sqrt{r_i r_j}\left(\sqrt{\frac{r_i}{r_j}}y_i - \sqrt{\frac{r_j}{r_i}}y_j\right)\right\|^2 =$$

$$= \sum_{i,j=1}^{n} \frac{1}{r_i r_j} \left\| \sqrt{r_i r_j}\left(\sqrt{\frac{r_i}{r_j}}x_i - \sqrt{\frac{r_j}{r_i}}y_j\right)\right\|^2 - \left\|\sum_{i=1}^{n}(x_i - y_i)\right\|^2.$$

We perform the simplification in the left term an we obtain the conclusion.

Second part of the proof follows the same idea from the proof of the previous proposition. We choose $n = 2$, $y_1 = y_2 = 0$ and $r_1 = r_2 = 0$. □

Remark. In fact, the Corollary 4.6. represents a more general result than the previous. If we choose $r_1 = r_2 = ... = r_n = 1$ we obtain

$$\sum_{1\le i<j\le n} \left\|x_i - x_j\right\|^2 + \sum_{1\le i<j\le n} \left\|y_i - y_j\right\|^2 = \sum_{i,j=1}^{n} \left\|x_i - y_j\right\|^2 - \left\|\sum_{i=1}^{n} (x_i - y_i)\right\|^2,$$

which is equivalent to the identity from Corollary 4.5.

A very interesting applications of the main result is Corrolary 4.8. We will give a version for an inner product space for an inequality of G. Dospinescu (see pag. 31 from [3]). Firstly, we will prove the next useful lemma:

Lemma 4.7. *Let $(X, \|\cdot\|)$ a real or complex normed space and $n \in \mathbb{N}, n \ge 2$. For any $a_1, a_2, ..., a_n \in (0, \infty)$, $x_1, x_2, ..., x_n \in X$ and $p \in [1, \infty)$ we put*

$$s(k) = \sum_{i=1}^{k} a_i \|x_i\|^p - \frac{1}{(a_1 + a_2 + ... + a_k)^{p-1}} \left\|\sum_{i=1}^{k} a_i x_i\right\|^p.$$

Then

$$s(k+1) \ge s(k),$$

for all $k \in \{1, 2, 3, .., n-1\}$.

Proof. We evaluate the diference $s(k+1) - s(k)$ and we obtain

$$s(k+1) - s(k) =$$

$$= a_{k+1}\|x_{k+1}\|^p - \frac{1}{(a_1 + a_2 + ... + a_{k+1})^{p-1}} \left\|\sum_{i=1}^{k+1} a_i x_i\right\|^p + \frac{1}{(a_1 + a_2 + ... + a_k)^{p-1}} \left\|\sum_{i=1}^{k} a_i x_i\right\|^p.$$

Because the function

$$x \longmapsto \|\cdot\|^p$$

is convex, we obtain

$$\frac{1}{(a_1+a_2+...+a_{k+1})^{p-1}}\left\|\sum_{i=1}^{k+1} a_i x_i\right\|^p = \left(\sum_{i=1}^{k+1} a_i\right)\left\|\frac{\sum_{i=1}^{k+1} a_i x_i}{\sum_{i=1}^{k+1} a_i}\right\|^p$$

$$= \left(\sum_{i=1}^{k+1} a_i\right)\left\|\frac{\sum_{i=1}^{k} a_i x_i}{\sum_{i=1}^{k+1} a_i} + \frac{a_{k+1}x_{k+1}}{\sum_{i=1}^{k+1} a_i}\right\|^p$$

$$= \left(\sum_{i=1}^{k+1} a_i\right)\left\|\frac{\sum_{i=1}^{k} a_i}{\sum_{i=1}^{k+1} a_i}\cdot\frac{\sum_{i=1}^{k} a_i x_i}{\sum_{i=1}^{k} a_i} + \frac{a_{k+1}}{\sum_{i=1}^{k+1} a_i}\cdot x_{k+1}\right\|^p$$

$$\leq \left(\sum_{i=1}^{k+1} a_i\right)\left(\frac{\sum_{i=1}^{k} a_i}{\sum_{i=1}^{k+1} a_i}\left\|\frac{\sum_{i=1}^{k} a_i x_i}{\sum_{i=1}^{k} a_i}\right\|^p + \frac{a_{n+1}}{\sum_{i=1}^{k+1} a_i}\|x_{n+1}\|^p\right)$$

$$= \left(\sum_{i=1}^{k} a_i\right)\left\|\frac{\sum_{i=1}^{k} a_i x_i}{\sum_{i=1}^{k} a_i}\right\|^p + a_{n+1}\|x_{n+1}\|^p$$

$$= \frac{1}{(a_1+a_2+...+a_k)^{p-1}}\left\|\sum_{i=1}^{k} a_i x_i\right\|^p + a_{k+1}\|x_{k+1}\|^p$$

and the conclusion follows. □

Corollary 4.8. *Let X an inner product space and $n, k \in \mathbb{N}$ so that $1 \leq k < n$. Then*

$$\sum_{1\leq i<j\leq n}\|x_i - x_j\|^2 \geq \frac{n}{k}\sum_{1\leq i<j\leq k}\|x_i - x_j\|^2,$$

for any $x_1, x_2, ..., x_n \in X$. Moreover, the number $\frac{n}{k}$ is the best possible.

Proof. First, let $\alpha \in \mathbb{R}$ so that

$$\sum_{1\leq i<j\leq n}\|x_i - x_j\|^2 \geq \alpha\sum_{1\leq i<j\leq k}\|x_i - x_j\|^2,$$

for any $x_1, x_2, ..., x_n \in X$. If we choose $x_1 \neq 0$ and $x_2 = x_3 = ... = x_n = 0$ we obtain that $\alpha \leq \frac{n}{k}$. Now, we want to prove the validity of the inequality for $\alpha = \frac{n}{k}$.

We apply the point c) of Theorem 3.1. and we obtain

$$\sum_{1\leq i<j\leq n}\|x_i - x_j\|^2 = n\sum_{i=1}^{n}\|x_i\|^2 - \left\|\sum_{i=1}^{n} x_i\right\|^2.$$

Then, the inequality which we want to prove is equivalent to

$$n\sum_{i=1}^{n}\|x_i\|^2 - \left\|\sum_{i=1}^{n} x_i\right\|^2 \geq \frac{n}{k}\left(k\sum_{i=1}^{k}\|x_i\|^2 - \left\|\sum_{i=1}^{k} x_i\right\|^2\right)$$

$$\Leftrightarrow \sum_{i=1}^{n}\|x_i\|^2 - \frac{1}{n}\left\|\sum_{i=1}^{n} x_i\right\|^2 \geq \sum_{i=1}^{k}\|x_i\|^2 - \frac{1}{k}\left\|\sum_{i=1}^{k} x_i\right\|^2.$$

From previous Lemma, the last inequality is true because is equivalent to $s(n) \geq s(k)$ in the case $p = 2$ and $a_1 = a_2 = ... = a_n = 1$. □

The result from the next proposition represents a version for an inner product space of a Hayashi's inequality. (see [7] or [10], pag. 297)

Proposition 4.9. *Let X be an inner product space. Then*

$$\sum_{cyclic} \|z - z_1\| \cdot \|z - z_2\| \cdot \|z_1 - z_2\| \geq \|z_1 - z_2\| \cdot \|z_1 - z_3\| \cdot \|z_2 - z_3\|,$$

for all $z, z_1, z_2, z_3 \in X$. The equality holds if $z \in \{z_1, z_2, z_3\}$ or

$$\frac{\|z_2 - z_3\|}{\|z - z_1\|}(z - z_1) + \frac{\|z_1 - z_3\|}{\|z - z_2\|}(z - z_2) + \frac{\|z_1 - z_2\|}{\|z - z_3\|}(z - z_3) = 0.$$

Proof. The case of equality for $z \in \{z_1, z_2, z_3\}$ is evident. From this consideration, we choose $z, z_1, z_2, z_3 \in X$, but $z \notin \{z_1, z_2, z_3\}$ and we apply *b)* from Theorem 3.1. for $n = 3$. We obtain

$$\|\sum_{k=1}^{3} a_k (z - z_k)\|^2 = \left(\sum_{k=1}^{3} a_k\right)\left(\sum_{k=1}^{3} a_k \|z - z_k\|^2\right) - \sum_{1 \leq k < j \leq 3} a_k a_j \|z_k - z_l\|^2,$$

for any $a_1, a_2, a_3 \in \mathbb{R}$. Then

$$\left(\sum_{k=1}^{3} a_k\right)\left(\sum_{k=1}^{3} a_k \|z - z_k\|^2\right) \geq \sum_{1 \leq k < l \leq 3} a_k a_j \|z_k - z_l\|^2.$$

With $a_1 = \frac{\|z_2 - z_3\|}{\|z - z_1\|}$, $a_2 = \frac{\|z_1 - z_3\|}{\|z - z_2\|}$ and $a_3 = \frac{\|z_1 - z_2\|}{\|z - z_3\|}$, we have

$$\sum \frac{\|z_j - z_l\|}{\|z - z_k\|} \cdot \sum \|z_j - z_l\| \cdot \|z - z_k\| \geq \sum \frac{\|z_j - z_l\|}{\|z - z_k\|} \cdot \frac{\|z_j - z_k\|}{\|z - z_l\|} \cdot \|z_k - z_l\|^2$$

$$\Leftrightarrow \sum \frac{\|z_j - z_l\|}{\|z - z_k\|} \cdot \sum \|z_j - z_l\| \cdot \|z - z_k\|$$

$$\geq \frac{\|z_1 - z_2\| \cdot \|z_1 - z_3\| \cdot \|z_2 - z_3\|}{\|z - z_1\| \cdot \|z - z_2\| \cdot \|z - z_3\|} \cdot \sum \|z_j - z_l\| \cdot \|z - z_k\|$$

$$\Leftrightarrow \sum \frac{\|z_j - z_l\|}{\|z - z_k\|} \geq \frac{\|z_1 - z_2\| \cdot \|z_1 - z_3\| \cdot \|z_2 - z_3\|}{\|z - z_1\| \cdot \|z - z_2\| \cdot \|z - z_3\|}.,$$

which is equivalent to conclusion. From the proof we also obtain the equality in the case

$$\sum_{k=1}^{3} a_k (z - z_k) = 0,$$

which is equivalent to

$$\frac{\|z_2 - z_3\|}{\|z - z_1\|}(z - z_1) + \frac{\|z_1 - z_3\|}{\|z - z_2\|}(z - z_2) + \frac{\|z_1 - z_2\|}{\|z - z_3\|}(z - z_3) = 0.$$

□

Finally, we will use the Theorem 3.1. to give a short proof for the next result. It represents the version for the inner product space of Zarantonello's inequality and it can be found in [17].

Proposition 4.10. (Zarantonello) *Let X be a Hilbert space and $f : X \to X$ a function such that $\|f(x) - f(y)\| \leq \|x - y\|$ for all $x, y \in X$. Let be $n \in \mathbb{N}, n \geq 2$. Then, for all $a_1, a_2, ..., a_n \geq 0$ with $\sum_{k=1}^{n} a_k = 1$ and for any $x_1, x_2, ..., x_n \in X$ we have:*

$$\left\| f\left(\sum_{j=1}^{n} a_j x_j\right) - \sum_{i=1}^{n} a_i f(x_i) \right\|^2 \leq \sum_{1\leq i<j\leq 1}^{n} a_i a_j \left(\|x_i - x_j\|^2 - \left\| f(x_i) - f\left(x_j\right) \right\|^2 \right).$$

Proof. We have

$$\left\| f\left(\sum_{j=1}^{n} a_j x_j\right) - \sum_{i=1}^{n} a_i f(x_i) \right\|^2 =$$

$$= \left\| \sum_{i=1}^{n} a_i f\left(\sum_{j=1}^{n} a_j x_j\right) - \sum_{i=1}^{n} a_i f(x_i) \right\|^2 = \left\| \sum_{i=1}^{n} a_i \left(f\left(\sum_{j=1}^{n} a_j x_j\right) - f(x_i) \right) \right\|^2 =$$

$$= \sum_{i=1}^{n} a_i \left\| f\left(\sum_{j=1}^{n} a_j x_j\right) - f(x_i) \right\|^2 - \sum_{1\leq i<j\leq n} a_i a_j \left\| f(x_i) - f(x_j) \right\|^2.$$

Now it is sufficient to prove that

$$\left\| f\left(\sum_{j=1}^{n} a_j x_j\right) - f(x_i) \right\|^2 \leq \sum_{j=1}^{n} a_j \left\| x_j - x_i \right\|^2.$$

But

$$\left\| f\left(\sum_{j=1}^{n} a_j x_j\right) - f(x_i) \right\|^2 \leq \left\| \sum_{j=1}^{n} a_j x_j - x_i \right\|^2 =$$

$$= \left\| \sum_{j=1}^{n} a_j x_j - \sum_{j=1}^{n} a_j x_i \right\|^2 = \left\| \sum_{j=1}^{n} a_j \left(x_j - x_i\right) \right\|^2.$$

Using the statement *d)* of Theorem 3.1. we obtain

$$\left\| \sum_{j=1}^{n} a_j \left(x_j - x_i\right) \right\|^2 \leq \sum_{j=1}^{n} a_j \left\| x_j - x_i \right\|^2.$$

□

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