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Characterization of Strong Preserver Operators of Convex Equivalent on the Space of All Real Sequences Tend to Zero

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Abstract. In this work we consider all bounded linear operators $T : \mathfrak{c}_0 \to \mathfrak{c}_0$ that preserve convex equivalent relation $\sim_c$ on $\mathfrak{c}_0$ and we denote by $\mathcal{P}_{ce}(\mathfrak{c}_0)$ the set of such operators. If T strongly preserves convex equivalent, we denote them by $\mathcal{P}_{sce}(\mathfrak{c}_0)$. Some interesting properties of $\mathcal{P}_{ce}(\mathfrak{c}_0)$ are given. For $T \in \mathcal{P}_{ce}(\mathfrak{c}_0)$, we show that all rows of T belong to ℓ^1 and for any $j \in \mathbb{N}$, we have $0 \in \mathrm{Im}(T\mathrm{e}_j)$, also there are $a, b \in \mathrm{Im}(T\mathrm{e}_j)$ such that $\mathrm{co}(T\mathrm{e}_j) = [a, b]$. It is shown that all row sums of T belong to $[a, b]$. We characterize the elements of $\mathcal{P}_{ce}(\mathfrak{c}_0)$, and some interesting results of all $T \in \mathcal{P}_{sce}(\mathfrak{c}_0)$ are given, for example we prove that $a = 0 < b$ or $a < 0 = b$. Also the elements of $\mathcal{P}_{sce}(\mathfrak{c}_0)$ are characterized. We obtain the matrix representation of $T \in \mathcal{P}_{sce}(\mathfrak{c}_0)$ does not contain any zero row. Some relevant examples are given.

1. Introduction

Throughout this work, $\mathfrak{c}_0$ is the Banach space of all real sequences converge to zero with the supremum norm. An element $x \in \mathfrak{c}_0$ can be represented by $\sum_{i\in\mathbb{N}} x(i)\mathrm{e}_i$, where $\mathrm{e}_i : \mathbb{N} \to \mathbb{R}$ is defined by $\mathrm{e}_i(j) = \delta_{ij}$, the Kronecker delta. For $x \in \mathfrak{c}_0$, we write $\mathrm{co}(x)$, instead of the convex combination of the set $\mathrm{Im}(x) = \{x(i) : i \in \mathbb{N}\}$.

Let $T : \mathfrak{c}_0 \to \mathfrak{c}_0$ be a bounded linear operator. It is easy to show that, T is represented by a matrix $(t_{ij})_{i,j\in\mathbb{N}}$ in the sense that

$$(Tx)(i) = \sum_{j\in\mathbb{N}} t_{ij}x(j), \qquad \text{for } x \in \mathfrak{c}_0 \text{ and } i \in \mathbb{N},$$

where $t_{ij} = (T\mathrm{e}_j)(i)$. To simplify, we will incorporate T to its matrix form $(t_{ij})_{i,j\in\mathbb{N}}$.

Definition 1.1. *[3] For $x, y \in \mathfrak{c}_0$, we say that x is convex majorized by y, and denoted by $x \prec_c y$, if $\mathrm{co}(x) \subseteq \mathrm{co}(y)$ and x is said to be convex equivalent to y, denoted by $x \sim_c y$, whenever $x \prec_c y \prec_c x$, i.e., $\mathrm{co}(x) = \mathrm{co}(y)$.*

The relation $\sim_c$ is an equivalent relation on $\mathfrak{c}_0$. For $x \in \mathfrak{c}_0$, if $0 \in \mathrm{co}(x)$, then $\mathrm{co}(x) = [a, b]$, for some $a, b \in \mathbb{R}$ with $a \le 0 \le b$, and if $0 \notin \mathrm{co}(x)$, then $\mathrm{co}(x)$ is equal to either $[a, 0)$, for some $a < 0$, or $(0, b]$, for some $b > 0$.

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Definition 1.2. *[5] Let $\mathcal{R}$ be a relation on c_0. The linear operator $T : c_0 \to c_0$ is said preserve $\mathcal{R}$ if for each $x, y \in c_0$,*

$$\mathcal{R}(x,y) \text{ implies } \mathcal{R}(Tx,Ty),$$

and T is called strongly preserve $\mathcal{R}$ if

$$\mathcal{R}(x,y) \text{ if and only if } \mathcal{R}(Tx,Ty).$$

The set of all bounded linear operators $T : c_0 \to c_0$ which preserve convex majorization, convex equivalent, strongly preserve convex majorization and strongly preserve convex equivalent denoted by $\mathcal{P}_{cm}(c_0)$, $\mathcal{P}_{ce}(c_0)$, $\mathcal{P}_{scm}(c_0)$ and $\mathcal{P}_{sce}(c_0)$, respectively. Obviously, $\mathcal{P}_{cm}(c_0) \subseteq \mathcal{P}_{ce}(c_0)$, $\mathcal{P}_{scm}(c_0) \subseteq \mathcal{P}_{cm}(c_0)$ and $\mathcal{P}_{sce}(c_0) \subseteq \mathcal{P}_{ce}(c_0)$.

Example 1.3. *Let $T : c_0 \to c_0$ be defined by $Tx = (ax_1, bx_1, ax_2, bx_2, \ldots)$, for $a, b \in \mathbb{R}$ and $x = (x_1, x_2, \ldots) \in c_0$. Clearly $T \in \mathcal{P}_{ce}(c_0)$.*
In general case, let (n_k) be a bounded sequence in $\mathbb{N}$. The operator $T : c_0 \to c_0$ defined by

$$Tx = (\underbrace{ax_1,\ldots,ax_1}_{n_1},\underbrace{bx_1,\ldots,bx_1}_{n_2},\underbrace{ax_2,\ldots,ax_2}_{n_3},\underbrace{bx_2,\ldots,bx_2}_{n_4},\ldots)$$

lies in $\mathcal{P}_{ce}(c_0)$, for $x = (x_1, x_2, \ldots) \in c_0$.

Example 1.4. *Let $T : c_0 \to c_0$ be a bounded linear operator defined by $Tx = (0, x_1, x_2, x_3, \ldots)$, for $x = (x_1, x_2, x_3, \ldots) \in c_0$. It is easy to show that $T \in \mathcal{P}_{ce}(c_0)$.*

Remark 1.5. *Note that for $T \in \mathcal{P}_{ce}(c_0)$ and $j_1, j_2 \in \mathbb{N}$, since $\mathrm{co}(Te_{j_1}) = \mathrm{co}(Te_{j_2})$ holds because $e_{j_1} \sim_c e_{j_2}$, the values $a := \inf Te_j$ and $b := \sup Te_j$ are constants, independent of chosen $j \in \mathbb{N}$ (similarly as in [3, Remark 2.10]). That is, for $T \in \mathcal{P}_{ce}(c_0)$, there is a bounded real interval I, such that $\mathrm{co}(Te_j) = I$, for all $j \in \mathbb{N}$. Therefore $a = \inf I$, and $b = \sup I$, for any $T \in \mathcal{P}_{ce}(c_0)$. Also, we define $I^+ = \{j \in \mathbb{N} : (Te_j)(i) > 0\}$ and $I^- = \{j \in \mathbb{N} : (Te_j)(i) < 0\}$.*

From now on a, b and I^+, I^- are as in Remark 1.5.

In [3], Bayati et al. characterized the elements of $\mathcal{P}_{cm}(c_0)$ and obtained some properties of them as follows.

Theorem 1.6. *[3, Theorem 2.8 and Corollary 2.9] For $T \in \mathcal{P}_{cm}(c_0)$, all rows of T lie in ℓ^1. Moreover for any fixed $i \in \mathbb{N}$, we have $\sum_{j\in\mathbb{N}} |(Te_j)(i)| \leq \|T\|$. Also, independent of chosen distinct $j_1, j_2 \in \mathbb{N}$, we have $\|Te_{j_1} - Te_{j_2}\| = \|T\|$.*

Theorem 1.7. *[3, Theorem 2.13 and Lemma 2.14] Let $T \in \mathcal{P}_{cm}(c_0)$. Then $\|Te_j\| = \|T\|$ and $0 \in \mathrm{Im}(Te_j)$, for all $j \in \mathbb{N}$.*

Theorem 1.8. *[3, Theorem 2.19] Let $T : c_0 \to c_0$ be a linear operator. Then $T \in \mathcal{P}_{cm}(c_0)$ if and only if*

(i) *for any $j \in \mathbb{N}$, the value of $\min_{i\in\mathbb{N}}(Te_j)(i)$ exists and independent of $j \in \mathbb{N}$ is equal to a.*

(ii) *for any $j \in \mathbb{N}$, the value of $\max_{i\in\mathbb{N}}(Te_j)(i)$ exists and independent of $j \in \mathbb{N}$ is equal to b.*

(iii) *if $a < 0 < b$, we have $\frac{1}{a}\sum_{j\in I^-}(Te_j)(i) + \frac{1}{b}\sum_{j\in I^+}(Te_j)(i) \leq 1$; if $a < 0 = b$, then we have $\sum_{j\in\mathbb{N}}(Te_j)(i) \geq a$ and if $a = 0 < b$, then it implies $\sum_{j\in\mathbb{N}}(Te_j)(i) \leq b$,*

where $((Te_j)(i))_{j\in\mathbb{N}}$ is an arbitrary row of T.

Some of the results in this work are obtained by the similar technique developed in [3, 4, 7].

We organize this paper as follows. In Section 2 we extend some of recent results of bounded linear preservers of the convex majorization on $\mathfrak{c}_0$ to the set of bounded linear operators which preserve convex equivalent on $\mathfrak{c}_0$, we denote this set by $\mathcal{P}_{ce}(\mathfrak{c}_0)$. It is shown that some of the above mentioned results are satisfied for $\mathcal{P}_{ce}(\mathfrak{c}_0)$. For $T \in \mathcal{P}_{ce}(\mathfrak{c}_0)$, we show that all rows of T belong to ℓ^1 and for any $j \in \mathbb{N}$, we have $0 \in \mathrm{Im}(T\mathrm{e}_j)$, also there are $a, b \in \mathrm{Im}(T\mathrm{e}_j)$ such that $\mathrm{co}(T\mathrm{e}_j) = [a, b]$. It is shown that any row sums of T belong to $[a, b]$. We characterize the elements of $\mathcal{P}_{ce}(\mathfrak{c}_0)$. Section 3 is devoted to study of the properties of strong preservers of convex equivalent on $\mathfrak{c}_0$, we denote this set by $\mathcal{P}_{sce}(\mathfrak{c}_0)$. We investigate some interesting properties of $T \in \mathcal{P}_{sce}(\mathfrak{c}_0)$, and obtain that $a = 0 < b$ or $a < 0 = b$ for all $T \in \mathcal{P}_{sce}(\mathfrak{c}_0)$, also we prove the matrix representation of T does not contain any zero row. At the end, we characterize the set $\mathcal{P}_{sce}(\mathfrak{c}_0)$.

2. Some properties of the operators in $\mathcal{P}_{ce}(\mathfrak{c}_0)$

The topic of linear preservers is of interest to a large group of matrix theorists. For a survey of linear preserver problems see [9], and for relative papers and book in the theory of majorization, see [1, 2, 6, 8].

In [3], Bayati et al. characterized the operators in $\mathcal{P}_{cm}(\mathfrak{c}_0)$. In this section, we prove some properties of linear preservers of convex equivalent on $\mathfrak{c}_0$ and characterize the operators in $\mathcal{P}_{ce}(\mathfrak{c}_0)$.

Remark 2.1. *Some general properties of $\mathcal{P}_{ce}(\mathfrak{c}_0)$ are as follow.*

- $0, \mathrm{id} \in \mathcal{P}_{ce}(\mathfrak{c}_0)$.
- *If $T_1, T_2 \in \mathcal{P}_{ce}(\mathfrak{c}_0)$, then $T_1 \circ T_2 \in \mathcal{P}_{ce}(\mathfrak{c}_0)$.*
- *If $T \in \mathcal{P}_{ce}(\mathfrak{c}_0)$, then $\lambda T \in \mathcal{P}_{ce}(\mathfrak{c}_0)$, for all $\lambda \in \mathbb{R}$.*
- *Any constant coefficient of a permutation on $\mathfrak{c}_0$ lies in $\mathcal{P}_{ce}(\mathfrak{c}_0)$.*

We now consider some important properties of $T \in \mathcal{P}_{ce}(\mathfrak{c}_0)$.

Theorem 2.2. *Let $T : \mathfrak{c}_0 \to \mathfrak{c}_0$ be a bounded linear operator. Then for any $i \in \mathbb{N}$, we have $\sum_{j\in\mathbb{N}} |(T\mathrm{e}_j)(i)| \leq \|T\|$, and moreover each row of T belongs to ℓ^1.*

Proof. Let $i, j, n \in \mathbb{N}$. We set $\delta_j = \mathrm{sgn}(T\mathrm{e}_j)(i)$ and $x_n = \sum_{j=1}^{n} \delta_j \mathrm{e}_j \in \mathfrak{c}_0$. Then $Tx_n = \sum_{j=1}^{n} \delta_j T\mathrm{e}_j$, and so $(Tx_n)(i) = \sum_{j=1}^{n} \delta_j (T\mathrm{e}_j)(i) = \sum_{j=1}^{n} |(T\mathrm{e}_j)(i)|$. Since $\|x_n\| \leq 1$, we have

$$(Tx_n)(i) = \sum_{j=1}^{n} |(T\mathrm{e}_j)(i)| = |(Tx_n)(i)| \leq \|Tx_n\| \leq \|T\|\|x_n\| \leq \|T\|.$$

Let n tend to infinity, so we have $\sum_{j=1}^{\infty} |(T\mathrm{e}_j)(i)| \leq \|T\|$, that is, all rows of T belong to ℓ^1. □

Remark 2.3. *Indeed, for a bounded linear operator $T : \mathfrak{c}_0 \to \mathfrak{c}_0$, we have $\|T\| = \sup_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} |(T\mathrm{e}_j)(i)|$. (see for instance [10, page 217, Theorem 4.51-c])*

Theorem 2.4. *Let $T \in \mathcal{P}_{ce}(\mathfrak{c}_0)$. Then $\|T\mathrm{e}_{j_0} - T\mathrm{e}_{j'_0}\|_\infty = \|T\mathrm{e}_j\|_\infty$ for any $j, j_0, j'_0 \in \mathbb{N}$ with $j_0 \neq j'_0$.*

Proof. If $T \equiv 0$, we are done. So suppose that $T \not\equiv 0$. Let $i_0, j, j_0, j'_0 \in \mathbb{N}$, with $j_0 \neq j'_0$. Put

$$\delta_j = \begin{cases} 1 & \text{if } T\mathrm{e}_j(i_0) \geq 0, \\ -1 & \text{if } T\mathrm{e}_j(i_0) < 0. \end{cases}$$

Then $\sum_{k=1}^{n} \delta_{j_k} \mathrm{e}_{j_k}$ is convex equivalent to either $\pm \mathrm{e}_{j_0}$ or $\mathrm{e}_{j_0} - \mathrm{e}_{j'_0}$. Since $T \in \mathcal{P}_{ce}(\mathfrak{c}_0)$, it follows from Theorem 2.2 that $\sum_{j\in\mathbb{N}} |(T\mathrm{e}_j)(i_0)| < \infty$. Thus for any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that for $j^* > n$, we have $|(T\mathrm{e}_{j^*})(i_0)| < \varepsilon$. Define

$$\delta^* = \begin{cases} -1 & \text{if } \delta_{j_1} = \cdots = \delta_{j_n} = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Since $\sum_{k=1}^{n} \delta_{j_k} \mathrm{e}_{j_k} + \delta^* \mathrm{e}_{j^*} \sim_c \mathrm{e}_{j_0} - \mathrm{e}_{j'_0}$ for $j^* \neq j_1, \dots, j_n$, it follows that

$$\sum_{k=1}^{n} \delta_{j_k} T\mathrm{e}_{j_k} + \delta^* T\mathrm{e}_{j^*} \sim_c T\mathrm{e}_{j_0} - T\mathrm{e}_{j'_0}.$$

So

$$\sum_{k=1}^{n} |(T\mathrm{e}_{j_k})(i_0)| + \delta^*(T\mathrm{e}_{j^*})(i_0) \in \mathrm{co}\left(\sum_{k=1}^{n} \delta_{j_k} T\mathrm{e}_{j_k} + \delta^* T\mathrm{e}_{j^*}\right) = \mathrm{co}\left(T\mathrm{e}_{j_0} - T\mathrm{e}_{j'_0}\right),$$

that yields

$$\mathrm{dist}\left(\sum_{k=1}^{n} |(T\mathrm{e}_{j_k})(i_0)|, \mathrm{co}\left(T\mathrm{e}_{j_0} - T\mathrm{e}_{j'_0}\right)\right) \leq |\delta^*(T\mathrm{e}_{j^*})(i_0)| = |(T\mathrm{e}_{j^*})(i_0)| < \varepsilon.$$

As ε is arbitrary, the above distance equals zero and so

$$\sum_{k=1}^{n} |(T\mathrm{e}_{j_k})(i_0)| \in \overline{\mathrm{co}\left(T\mathrm{e}_{j_0} - T\mathrm{e}_{j'_0}\right)},$$

which implies that $\sum_{k=1}^{n} |(T\mathrm{e}_{j_k})(i_0)| \leq \|T\mathrm{e}_{j_0} - T\mathrm{e}_{j'_0}\|_\infty$. Let n tend to infinity, so we have

$$\sum_{k=1}^{\infty} |(T\mathrm{e}_{j_k})(i_0)| \leq \|T\mathrm{e}_{j_0} - T\mathrm{e}_{j'_0}\|_\infty. \tag{1}$$

The inequality (1) implies that for any $i, j \in \mathbb{N}$, we have $|(T\mathrm{e}_j)(i)| \leq \|T\mathrm{e}_{j_0} - T\mathrm{e}_{j'_0}\|_\infty$, which follows that

$$\|T\mathrm{e}_j\|_\infty \leq \|T\mathrm{e}_{j_0} - T\mathrm{e}_{j'_0}\|_\infty. \tag{2}$$

It is sufficient to show that

$$\|T\mathrm{e}_{j_0} - T\mathrm{e}_{j'_0}\|_\infty \leq \|T\mathrm{e}_j\|_\infty. \tag{3}$$

Let $\varepsilon > 0$. Since $T\mathrm{e}_j \in \mathfrak{c}_0$, there is $M \in \mathbb{N}$ such that for all $i > M$, we have

$$|T\mathrm{e}_j(i)| < \frac{\varepsilon}{2}. \tag{4}$$

On the other hand (1) implies that

$$\lim_{j\to\infty} (T\mathrm{e}_j)(1) = 0, \dots, \lim_{j\to\infty} (T\mathrm{e}_j)(M) = 0.$$

Hence there is $N \in \mathbb{N}$ such that for all $j > N$,

$$|(T\mathrm{e}_j)(1)|, \ldots, |(T\mathrm{e}_j)(M)| < \frac{\varepsilon}{2}. \tag{5}$$

The relations (4) and (5) yield that if $j^* \neq j_0, 1, \ldots, N$, then for all $i \in \mathbb{N}$, we have

$$\begin{aligned}\left|(T\mathrm{e}_{j_0})(i) - (T\mathrm{e}_{j^*})(i)\right| &\leq \begin{cases} |(T\mathrm{e}_{j_0})(i)| + \frac{\varepsilon}{2} & \text{if } i \neq 1, \ldots, M, \\ \frac{\varepsilon}{2} + |(T\mathrm{e}_{j^*})(i)| & \text{if } i = 1, \ldots, M, \end{cases} \\ &\leq \|T\mathrm{e}_{j_0}\|_\infty + \varepsilon,\end{aligned}$$

which follows

$$\|T\mathrm{e}_{j_0} - T\mathrm{e}_{j_0'}\|_\infty = \|T\mathrm{e}_{j_0} - T\mathrm{e}_{j^*}\|_\infty \leq \|T\mathrm{e}_{j_0}\|_\infty + \varepsilon.$$

As ε is arbitrary, we get (3). Therefore (2) and (3) follow the assertion. □

Theorem 2.5. *For $T \in \mathcal{P}_{ce}(\mathfrak{c}_0)$ and $i \in \mathbb{N}$, we have*

$$a \leq \sum_{j \in I^-} (T\mathrm{e}_j)(i) \leq 0 \leq \sum_{j \in I^+} (T\mathrm{e}_j)(i) \leq b,$$

where $I^+ = \{j \in \mathbb{N} : (T\mathrm{e}_j)(i) > 0\}$, $I^- = \{j \in \mathbb{N} : (T\mathrm{e}_j)(i) < 0\}$.

Proof. Let $i \in \mathbb{N}$ and $F \subseteq I^-$ be a nonempty finite set. Since for $j_0 \in \mathbb{N}$, $\sum_{j \in F} \mathrm{e}_j \sim_c \mathrm{e}_{j_0}$ hence

$$\mathrm{co}\left(\sum_{j \in F} T\mathrm{e}_j\right) = \mathrm{co}(T\mathrm{e}_{j_0}).$$

It follows

$$\sum_{j \in F} (T\mathrm{e}_j)(i) \in \mathrm{Im}\left(\sum_{j \in F} T\mathrm{e}_j\right) \subseteq \mathrm{co}\left(\sum_{j \in F} T\mathrm{e}_j\right) = \mathrm{co}\left(T\mathrm{e}_{j_0}\right).$$

Hence $a = \inf_{i \in \mathbb{N}} (T\mathrm{e}_{j_0})(i) \leq \sum_{j \in F} (T\mathrm{e}_j)(i) \leq 0$. Since the latter inequality holds for all finite subsets $F \subseteq I^-$, we have $a \leq \sum_{j \in I^-} (T\mathrm{e}_j)(i) \leq 0$.

The other inequality follows by a similar argument. □

Corollary 2.6. *Let $T \in \mathcal{P}_{ce}(\mathfrak{c}_0)$. Then any row sums of T belong to $[a, b]$.*

Proof. By adding two inequalities in Theorem 2.5, we get the assertion. □

Lemma 2.7. *Let $j \in \mathbb{N}$ and $T \in \mathcal{P}_{ce}(\mathfrak{c}_0)$. Then $0 \in \mathrm{Im}(T\mathrm{e}_j)$.*

Proof. Let $j_1, j_2 \in \mathbb{N}$ be distinct. If $a = b = 0$, then $T\mathrm{e}_{j_1} = 0$ and the assertion follows. Otherwise, $a < 0$ or $b > 0$. For $j \in \mathbb{N}$, as $a = \inf_{i \in \mathbb{N}} (T\mathrm{e}_j)(i)$ and $b = \sup_{i \in \mathbb{N}} (T\mathrm{e}_j)(i)$, we have $\|T\mathrm{e}_j\|_\infty = \max\{b, -a\} > 0$.

Now if $\|T\mathrm{e}_{j_2}\|_\infty = b > 0$, then there is $i_0 \in \mathbb{N}$ such that $(T\mathrm{e}_{j_2})(i_0) = b$. Applying Theorem 2.2, we conclude that

$$b = |(T\mathrm{e}_{j_2})(i_0)| \leq \sum_{j=1}^{\infty} |(T\mathrm{e}_j)(i_0)| \leq \|T\mathrm{e}_{j_2}\|_\infty = b.$$

The latter inequalities lead to $|(T\mathrm{e}_j)(i_0)| = 0$ for any $j \neq j_2$. Therefore $(T\mathrm{e}_{j_1})(i_0) = 0$, it implies that $0 \in \mathrm{Im}(T\mathrm{e}_{j_1})$. For $\|T\mathrm{e}_{j_2}\|_\infty = -a > 0$, the assertion follows by a similar argument. □

Lemma 2.8. *Let $j \in \mathbb{N}$ and $T \in \mathcal{P}_{ce}(\mathfrak{c}_0)$. Then $a,\ b \in \mathrm{Im}(T\mathrm{e}_j)$ and $\mathrm{co}(T\mathrm{e}_j) = [a, b]$.*

Proof. According to Remark 1.5 we have $\mathrm{co}(T\mathrm{e}_j) = I$, where I is a bounded interval and $a = \inf I$ and $b = \sup I$. Since $T\mathrm{e}_j \in \mathfrak{c}_0$, it follows that zero can be at most a limit point of $\mathrm{Im}(T\mathrm{e}_j)$ and $a \leq 0 \leq b$. If $a < 0$, then a can not be a limit point of $\mathrm{Im}(T\mathrm{e}_j)$. As $a = \inf_{i\in\mathbb{N}}(T\mathrm{e}_j)(i)$, we have $a \in \mathrm{Im}(T\mathrm{e}_j)$. If $a = 0$, Lemma 2.7 implies that $a = 0 \in \mathrm{Im}(T\mathrm{e}_j)$. By a similar argument, $b \in \mathrm{Im}(T\mathrm{e}_j)$. Therefore $\mathrm{co}(T\mathrm{e}_j) = [a, b]$. □

Corollary 2.9. *For $j \in \mathbb{N}$ and $T \in \mathcal{P}_{ce}(\mathfrak{c}_0)$, we have $\min_{i\in\mathbb{N}}(T\mathrm{e}_j)(i) = a$ and $\max_{i\in\mathbb{N}}(T\mathrm{e}_j)(i) = b$.*

Theorem 2.10. *Let $T \in \mathcal{P}_{ce}(\mathfrak{c}_0)$ and $a < 0 < b$. Then for distinct $j_1, j_2 \in \mathbb{N}$, we have*

$$\max_{i\in\mathbb{N}}\left\{\frac{1}{a}(T\mathrm{e}_{j_1})(i) + \frac{1}{b}(T\mathrm{e}_{j_2})(i)\right\} = 1.$$

Proof. By Lemma 2.8, it follows that

$$\max_{i\in\mathbb{N}}\left\{\frac{1}{a}(T\mathrm{e}_{j_1})(i)\right\} = \max_{i\in\mathbb{N}}\left\{\frac{1}{b}(T\mathrm{e}_{j_2})(i)\right\} = 1. \tag{6}$$

Since $T\mathrm{e}_{j_1} \in \mathfrak{c}_0$, it follows that for arbitrary $0 < \varepsilon < 1$, there is $m \in \mathbb{N}$ such that

$$\left|\frac{1}{a}(T\mathrm{e}_{j_1})(i)\right| < \varepsilon, \qquad \text{for all } i > m. \tag{7}$$

Theorem 2.2 implies that $\sum_{j\in\mathbb{N}} |(T\mathrm{e}_j)(i)| < \infty$, for all $i \in \mathbb{N}$. So there exists $n \in \mathbb{N}$ such that

$$\left|\frac{1}{b}(T\mathrm{e}_j)(i)\right| < \varepsilon, \qquad \text{for all } i \in \{1, \ldots, m\} \text{ and } j > n. \tag{8}$$

Assume that $j_0 > n$ and $j_0 \neq j_1$, then (7) and (8) imply that for all $i \in \mathbb{N}$,

$$\frac{1}{a}(T\mathrm{e}_{j_1})(i) + \frac{1}{b}(T\mathrm{e}_{j_0})(i) \leq 1 + \varepsilon, \qquad \text{for } i \in \{1, \ldots, m\}, \tag{9}$$

$$\frac{1}{a}(T\mathrm{e}_{j_1})(i) + \frac{1}{b}(T\mathrm{e}_{j_0})(i) \leq \varepsilon + 1, \qquad \text{for } i > m. \tag{10}$$

As $\frac{1}{a}T\mathrm{e}_{j_1} + \frac{1}{b}T\mathrm{e}_{j_2} \sim_c \frac{1}{a}T\mathrm{e}_{j_1} + \frac{1}{b}T\mathrm{e}_{j_0}$, for all $\varepsilon > 0$, the relations (9) and (10) imply that

$$\sup_{i\in\mathbb{N}}\left\{\frac{1}{a}(T\mathrm{e}_{j_1})(i) + \frac{1}{b}(T\mathrm{e}_{j_2})(i)\right\} = \sup_{i\in\mathbb{N}}\left\{\frac{1}{a}(T\mathrm{e}_{j_1})(i) + \frac{1}{b}(T\mathrm{e}_{j_0})(i)\right\} \leq \varepsilon + 1.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\sup_{i\in\mathbb{N}}\left\{\frac{1}{a}(T\mathrm{e}_{j_1})(i) + \frac{1}{b}(T\mathrm{e}_{j_2})(i)\right\} = \sup_{i\in\mathbb{N}}\left\{\frac{1}{a}(T\mathrm{e}_{j_1})(i) + \frac{1}{b}(T\mathrm{e}_{j_0})(i)\right\} \leq 1. \tag{11}$$

On the other hand, (6) yields that there is $i_0 \in \mathbb{N}$ such that $\frac{1}{a}(T\mathrm{e}_{j_1})(i_0) = 1$. In (7), as $\varepsilon < 1$, we have $i_0 \in \{1, \ldots, m\}$ and so (8) concluds $\frac{1}{a}(T\mathrm{e}_{j_1})(i_0) + \frac{1}{b}(T\mathrm{e}_{j_0})(i_0) \geq 1 - \varepsilon$, thus for all $\varepsilon > 0$, we have

$$\sup_{i\in\mathbb{N}}\left\{\frac{1}{a}(T\mathrm{e}_{j_1})(i) + \frac{1}{b}(T\mathrm{e}_{j_2})(i)\right\} = \sup_{i\in\mathbb{N}}\left\{\frac{1}{a}(T\mathrm{e}_{j_1})(i) + \frac{1}{b}(T\mathrm{e}_{j_0})(i)\right\} \geq 1 - \varepsilon,$$

since $\varepsilon > 0$ is arbitrary, so

$$\sup_{i\in\mathbb{N}}\left\{\frac{1}{a}(T\mathrm{e}_{j_1})(i) + \frac{1}{b}(T\mathrm{e}_{j_2})(i)\right\} = \sup_{i\in\mathbb{N}}\left\{\frac{1}{a}(T\mathrm{e}_{j_1})(i) + \frac{1}{b}(T\mathrm{e}_{j_0})(i)\right\} \geq 1,$$

together (11) follow that $\sup_{i\in\mathbb{N}}\left\{\frac{1}{a}(Te_{j_1})(i)+\frac{1}{b}(Te_{j_2})(i)\right\}=1$. As $Te_j\in c_0$, so 1 is not a limit point of $\mathrm{Im}\left\{\frac{1}{a}Te_{j_1}+\frac{1}{b}Te_{j_2}\right\}$, so $1\in\mathrm{Im}\left\{\frac{1}{a}Te_{j_1}+\frac{1}{b}Te_{j_2}\right\}$, that is

$$\max_{i\in\mathbb{N}}\left\{\frac{1}{a}(Te_{j_1})(i)+\frac{1}{b}(Te_{j_2})(i)\right\}=1.$$

□

Theorem 2.11. *Let $T\in\mathcal{P}_{ce}(c_0)$ and $a<0<b$. Then for $i\in\mathbb{N}$, we have*

$$\frac{1}{a}\sum_{j\in I^-}(Te_j)(i)+\frac{1}{b}\sum_{j\in I^+}(Te_j)(i)\le 1,$$

Proof. By Theorem 2.5, for $I^+=\emptyset$, and $i\in\mathbb{N}$, we have $a\le\sum_{j\in I^-}(Te_j)(i)\le 0$. Multiplying the latter inequalities by $\frac{1}{a}$, we get the assertion. For $I^-=\emptyset$, the assertion follows by a similar argument.

We now suppose that I^+ and I^- are both nonempty. Let $E\subseteq I^+$ and $F\subseteq I^-$, where E and F are nonempty finite sets. For distinct $j_1,j_2\in\mathbb{N}$, as $\frac{1}{a}\sum_{j\in F}e_j+\frac{1}{b}\sum_{j\in E}e_j\sim_c\frac{1}{a}e_{j_1}+\frac{1}{b}e_{j_2}$, it follows that $\frac{1}{a}\sum_{j\in F}Te_j+\frac{1}{b}\sum_{j\in E}Te_j\sim_c\frac{1}{a}Te_{j_1}+\frac{1}{b}Te_{j_2}$. Theorem 2.10 together the latter formula follow that for $i\in\mathbb{N}$, we have

$$\frac{1}{a}\sum_{j\in F}(Te_j)(i)+\frac{1}{b}\sum_{j\in E}(Te_j)(i)\le\max_{i\in\mathbb{N}}\left\{\frac{1}{a}(Te_{j_1})(i)+\frac{1}{b}(Te_{j_2})(i)\right\}=1.$$

Since the above inequality holds for any finite subsets $F\subseteq I^-$ and $E\subseteq I^+$, we get

$$\frac{1}{a}\sum_{j\in I^-}(Te_j)(i)+\frac{1}{b}\sum_{j\in I^+}(Te_j)(i)\le 1.$$

□

Corollary 2.12. *Let $T\in\mathcal{P}_{ce}(c_0)$ and T consider in the matrix form. Then the following sentences hold.*

(i) *If $a<0$, then in any row of T which appears a, the other entries equal zero.*

(ii) *If $b>0$, then in any row of T which appears b, the other entries equal zero.*

Proof. For part (i), let $a<0$ and it appears in the row $i\in\mathbb{N}$.
If $b=0$, then $I^+=\emptyset$ and $I^-\neq\emptyset$. On the other hand, Theorem 2.5 implies that $a\le\sum_{j\in I^-}(Te_j)(i)$, where $(Te_j)(i)\le 0$, for all $j\in I^-$ and one of them is equal to a, then we have $\sum_{j\in I^-}(Te_j)(i)=a$. Let $j_0\in I^-$ be such that $(Te_{j_0})(i)=a$, it follows that $a=\sum_{\substack{j\in I^-\\ j\neq j_0}}(Te_j)(i)+a$. This concludes that $(Te_j)(i)=0$, for all $j\in\mathbb{N}$ with $j\neq j_0$.
If $b>0$, Theorem 2.11 follows that

$$\sum_{j\in I^-}\frac{(Te_j)(i)}{a}+\sum_{j\in I^+}\frac{(Te_j)(i)}{b}\le 1.$$

As all the elements of both series are nonnegative and there is $j_0\in I^-$ such that $(Te_{j_0})(i)=a$, it gives $(Te_j)(i)=0$, for all $j\in\mathbb{N}$ with $j\neq j_0$. This completes the proof of part (i).
By applying similar arguments, the assertion (ii) follows. □

The following theorem and Theorem 1.8 characterize the set $\mathcal{P}_{ce}(\mathfrak{c}_0)$.

Theorem 2.13. *We have* $\mathcal{P}_{cm}(\mathfrak{c}_0) = \mathcal{P}_{ce}(\mathfrak{c}_0)$.

Proof. Obviously, $\mathcal{P}_{cm}(\mathfrak{c}_0) \subseteq \mathcal{P}_{ce}(\mathfrak{c}_0)$. Now suppose that $T \in \mathcal{P}_{ce}(\mathfrak{c}_0)$. Corollary 2.9 implies that $\min\limits_{i\in\mathbb{N}}(T\mathrm{e}_j)(i) = a$ and $\max\limits_{i\in\mathbb{N}}(T\mathrm{e}_j)(i) = b$, for any $j \in \mathbb{N}$. If $a < 0 < b$, according to Theorems 1.8 and 2.11, we have $T \in \mathcal{P}_{cm}(\mathfrak{c}_0)$, and if $a < 0 = b$, it follows $I^+ = \emptyset$, and if $a = 0 < b$, it follows $I^- = \emptyset$, now we can use Theorems 1.8 and 2.5 to get $T \in \mathcal{P}_{cm}(\mathfrak{c}_0)$. That is $\mathcal{P}_{ce}(\mathfrak{c}_0) \subseteq \mathcal{P}_{cm}(\mathfrak{c}_0)$, which follows that $\mathcal{P}_{cm}(\mathfrak{c}_0) = \mathcal{P}_{ce}(\mathfrak{c}_0)$. □

3. Characterization of strong preservers of convex equivalent on $\mathfrak{c}_0$

As we mentioned, the set of all bounded linear operators $T : \mathfrak{c}_0 \to \mathfrak{c}_0$ which strongly preserve convex majorization is denoted by $\mathcal{P}_{scm}(\mathfrak{c}_0)$, that is $f \prec_c g$ if and only if $Tf \prec_c Tg$, for $f, g \in \mathfrak{c}_0$, and the set of all bounded linear operators $T : \mathfrak{c}_0 \to \mathfrak{c}_0$ which strongly preserve convex equivalent is denoted by $\mathcal{P}_{sce}(\mathfrak{c}_0)$, that is

$$f \sim_c g \qquad \text{if and only if} \qquad Tf \sim_c Tg.$$

The aim of this section is to study some important properties of $\mathcal{P}_{sce}(\mathfrak{c}_0)$ and characterize the elements of $\mathcal{P}_{scm}(\mathfrak{c}_0)$ and $\mathcal{P}_{sce}(\mathfrak{c}_0)$.

Obviously the following sentences are satisfied.

- $\mathcal{P}_{scm}(\mathfrak{c}_0) \subseteq \mathcal{P}_{sce}(\mathfrak{c}_0) \subseteq \mathcal{P}_{ce}(\mathfrak{c}_0)$.
- $\mathcal{P}_{scm}(\mathfrak{c}_0)$ and $\mathcal{P}_{sce}(\mathfrak{c}_0)$ are both closed under the combination and nonzero scalar multiplication.
- If $T \in \mathcal{P}_{sce}(\mathfrak{c}_0)$, then $\mathrm{Ker}(T) = \{0\}$.

Example 3.1. *In Example 1.4, we get the right shift operator on* $\mathfrak{c}_0$ *defined by*

$$Tf = (0, f_1, f_2, \ldots), \qquad \textit{for all } f \in \mathfrak{c}_0,$$

preserves convex equivalent. Now let $f = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$ *and* $g = (0, 1, \frac{1}{2}, \frac{1}{3}, \ldots)$. *Then we have* $\mathrm{co}(Tf) = \mathrm{co}(Tg) = [0, 1]$ *and so* $Tf \sim_c Tg$. *But* $f \nsim_c g$. *Therefore* $T \notin \mathcal{P}_{sce}(\mathfrak{c}_0)$. *That is* $\mathcal{P}_{sce}(\mathfrak{c}_0)$ *is a proper subset of* $\mathcal{P}_{ce}(\mathfrak{c}_0)$.

Lemma 3.2. *If* $T \in \mathcal{P}_{sce}(\mathfrak{c}_0)$, *then* $a \neq -b$.

Proof. On the contrary suppose that, $a = -b$. Then we have

$$\mathrm{co}(T\mathrm{e}_j) = \mathrm{co}(T(-\mathrm{e}_j)) = [a, b],$$

which implies that $T\mathrm{e}_j \sim_c T(-\mathrm{e}_j)$, but we have $\mathrm{e}_j \nsim_c -\mathrm{e}_j$. This is a contradiction. □

For $T \in \mathcal{P}_{sce}(\mathfrak{c}_0)$, we need some lemmas to prove that $a = 0 < b$ or $a < 0 = b$.

Lemma 3.3. *Let* $T \in \mathcal{P}_{ce}(\mathfrak{c}_0)$, $a < 0 < b$ *and* $\alpha \leq \min\left\{\frac{a}{b}, \frac{b}{a}\right\}$. *Let* $j_1, j_2 \in \mathbb{N}$ *be distinct and* $g = \alpha\mathrm{e}_{j_1} + \mathrm{e}_{j_2}$, *then we have* $\alpha b \leq \inf Tg \leq \sup Tg \leq \alpha a$.

Proof. Suppose that $0 < \varepsilon \leq \min\{-a, b\}$. Since $T\mathrm{e}_{j_1} \in \mathfrak{c}_0$, there is an $n \in \mathbb{N}$, such that for all $i > n$, we have $|(T\mathrm{e}_{j_1})(i)| < \frac{\varepsilon}{-\alpha} \leq \varepsilon$. Theorem 2.2 implies that all rows of the matrix form of T belong to ℓ^1. Hence there is $j_0 \in \mathbb{N}$, $(j_0 \neq j_1)$ such that $|(T\mathrm{e}_{j_0})(i)| < \varepsilon$, for all $i \in \{1, \ldots, n\}$. We now investigate the following two cases for $i \in \mathbb{N}$:

Case 1: Let $i \in \{1, \ldots, n\}$. As $a \leq (T\mathrm{e}_{j_1})(i) \leq b$ and $|(T\mathrm{e}_{j_0})(i)| < \varepsilon$, we have

$$\alpha b - \varepsilon \leq \alpha(T\mathrm{e}_{j_1})(i) + (T\mathrm{e}_{j_0})(i) \leq \alpha a + \varepsilon. \tag{12}$$

Case 2: Let $i \in \mathbb{N} \setminus \{1, \dots, n\}$. Since $|(Te_{j_1})(i)| < \frac{\varepsilon}{-\alpha}$ and $a \le (Te_{j_0})(i) \le b$, it follows

$$a - \varepsilon \le \alpha(Te_{j_1})(i) + (Te_{j_0})(i) \le b + \varepsilon. \tag{13}$$

Therefore (12) and (13) deduce that

$$\alpha b - \varepsilon = \min\{\alpha b - \varepsilon, a - \varepsilon\} \le \alpha(Te_{j_1})(i) + (Te_{j_0})(i) \le \max\{\alpha a + \varepsilon, b + \varepsilon\} = \alpha a + \varepsilon.$$

Since ε is arbitrary, it follows that

$$\mathrm{co}(Tg) = \mathrm{co}(\alpha Te_{j_1} + Te_{j_2}) = \mathrm{co}(\alpha Te_{j_1} + Te_{j_0}) \subseteq [\alpha b, \alpha a].$$

This gives the assertion. □

Lemma 3.4. *If $T \in \mathcal{P}_{ce}(\mathfrak{c}_0)$ and $\min Te_j = a < 0 < b = \max Te_j$, then $T \notin \mathcal{P}_{sce}(\mathfrak{c}_0)$.*

Proof. Let $\alpha = \min\left\{\frac{a}{b}, \frac{b}{a}\right\}$ and for distinct natural numbers j_1, j_2, define $f = \alpha e_{j_1}$ and $g = \alpha e_{j_1} + e_{j_2}$. Thus $Tf = \alpha Te_{j_1}$, which implies $\mathrm{co}(Tf) = \mathrm{co}(\alpha Te_{j_1}) = \alpha[a, b] = [\alpha b, \alpha a]$. Corollary 2.9 implies that there are $i_1, i_1^* \in \mathbb{N}$ such that $(Te_{j_1})(i_1) = a$ and $(Te_{j_1})(i_1^*) = b$. Also, Corollary 2.12 concludes that $(Te_{j_2})(i_1) = (Te_{j_2})(i_1^*) = 0$ and so

$$\alpha a = \alpha(Te_{j_1})(i_1) + (Te_{j_2})(i_1) \in \mathrm{co}(Tg), \tag{14}$$
$$\alpha b = \alpha(Te_{j_1})(i_1^*) + (Te_{j_2})(i_1^*) \in \mathrm{co}(Tg). \tag{15}$$

Lemma 3.3 together (14) and (15) imply that $\mathrm{co}(Tg) = [\alpha b, \alpha a] = \mathrm{co}(Tf)$. Which follows that $Tf \sim_c Tg$, although $f \nsim_c g$. This means that $T \notin \mathcal{P}_{sce}(\mathfrak{c}_0)$. □

In the following, we obtain some results of Lemma 3.4.

Theorem 3.5. *If $T \in \mathcal{P}_{sce}(\mathfrak{c}_0)$, then $a = 0 < b$ or $a < 0 = b$.*

Proof. Obviously $a \le 0 \le b$. Lemma 3.4 implies that $a = 0 \le b$ or $a \le 0 = b$. It is impossible $a = b = 0$, because it follows that $T \equiv 0$ and so T is not in $\mathcal{P}_{sce}(\mathfrak{c}_0)$. This completes the proof. □

Theorem 3.6. *If $T \in \mathcal{P}_{sce}(\mathfrak{c}_0)$, then the matrix representation of T does not contain zero row.*

Proof. Suppose, contrary to our claim, that all the entries of the i_0th row of T are equal to zero. Let $f = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots) \in \mathfrak{c}_0$. So for any $j \in \mathbb{N}$, Theorem 3.5 implies that

$$\mathrm{co}(Te_j) = [a, 0] = \mathrm{co}(Tf), \quad \text{or} \quad \mathrm{co}(Te_j) = [0, b] = \mathrm{co}(Tf),$$

which follows $Te_j \sim_c Tf$, but $f \nsim_c e_j$. That is $T \notin \mathcal{P}_{sce}(\mathfrak{c}_0)$, which is a contradiction. □

Example 3.7. *Let $T : \mathfrak{c}_0 \to \mathfrak{c}_0$ be a bounded linear operator defined by*

$$Tf = (2f_1, 2f_1, 2f_2, 2f_2, 2f_3, 2f_3, \dots), \qquad \text{for all } f \in \mathfrak{c}_0.$$

Then we have $\mathrm{co}(Tf) = 2\mathrm{co}(f)$ and obviously $T \in \mathcal{P}_{sce}(\mathfrak{c}_0)$.

In this part, we recall the generalization of convex combination.

Definition 3.8. *Let X be a normed linear space and $A \subseteq X$. The countable convex hull of A is defined as follows*

$$\mathrm{cco}(A) = \left\{ \sum_{i=1}^{\infty} \alpha_i x_i : \ x_i \in A, \ \alpha_i \ge 0, \ \sum_{i=1}^{\infty} \alpha_i = 1, \ \sum_{i=1}^{\infty} \alpha_i x_i \text{ converges} \right\}.$$

It is easy to check that for $A \subseteq \mathbb{R}$, we have $\mathrm{cco}(A) = \mathrm{co}(A)$.

Lemma 3.9. *[3, Lemma 2.6] Let* $x \in \mathfrak{c}_0$, $\alpha_i \geq 0$ *and* $0 < \sum_{i=1}^{\infty} \alpha_i \leq 1$. *Then* $\sum_{i=1}^{\infty} \alpha_i x(i) \in \mathrm{co}(x)$.

Let $\mathcal{E}$ denote the set of all bounded linear operators $T : \mathfrak{c}_0 \to \mathfrak{c}_0$ satisfy $\mathrm{co}(Tf) = \mathrm{co}(f)$, for all $f \in \mathfrak{c}_0$. In [3], Bayati et al. proved that $\mathcal{E} \subseteq \mathcal{P}_{cm}(\mathfrak{c}_0)$ and any permutation lies in $\mathcal{E}$, also proved the following theorems.

Theorem 3.10. *[3, Theorem 2.20] If* $T \in \mathcal{E}$, *then*

(i) *for all* $j \in \mathbb{N}$, $\min_{i\in\mathbb{N}}\{(Te_j)(i)\} = 0$, *and* $\max_{i\in\mathbb{N}}\{(Te_j)(i)\} = 1$.

(ii) *if* $((Te_j)(i))_{j\in\mathbb{N}}$ *is the ith row of the matrix form of T, then* $\sum_{j\in\mathbb{N}} (Te_j)(i) \leq 1$.

Theorem 3.11. *[3, Theorem 2.22 and Remark 2.23] If* $T \in \mathcal{E}$, *then the matrix form of T has no zero row and any row sum of T belongs to* $[0, 1]$.

Theorems 3.10 and 3.11 imply the following theorem.

Theorem 3.12. *Let* $T : \mathfrak{c}_0 \to \mathfrak{c}_0$ *be a bounded linear operator. Then* $T \in \mathcal{E}$ *if and only if*

(i) *for all* $j \in \mathbb{N}$, *we have* $\min_{i\in\mathbb{N}}\{(Te_j)(i)\} = 0$, *and* $\max_{i\in\mathbb{N}}\{(Te_j)(i)\} = 1$.

(ii) *any row sum of T belongs to* $(0, 1]$, *i.e.,* $0 < \sum_{j\in\mathbb{N}} (Te_j)(i) \leq 1$, *for any* $i \in \mathbb{N}$.

Proof. Let $T \in \mathcal{E}$. Theorems 3.10 and 3.11 imply (i), (ii). Now let (i), (ii) hold and $f \in \mathfrak{c}_0$. By (i), for $j_0 \in \mathbb{N}$ there is $i_0 \in \mathbb{N}$, such that $(Te_{j_0})(i_0) = 1$. Part (ii) implies that $\sum_{j\in\mathbb{N}} (Te_j)(i_0) \leq 1$ and $0 \leq (Te_j)(i_0) \leq 1$, $(Te_{j_0})(i_0) = 1$, so $(Te_j)(i_0) = 0$, for all $j \in \mathbb{N} \setminus \{j_0\}$. Therefore $(Tf)(i_0) = \sum_{j\in\mathbb{N}} (Te_j)(i_0) f(j) = f(j_0)$ and so $\mathrm{Im}(f) \subseteq \mathrm{Im}(Tf)$, and so $\mathrm{co}(f) \subseteq \mathrm{co}(Tf)$.

Now for any $i \in \mathbb{N}$, we have $0 < \sum_{j\in\mathbb{N}} (Te_j)(i) \leq 1$. According to Lemma 3.9 we have $(Tf)(i) = \sum_{j=1}^{\infty} (Te_j)(i) f(j) \in \mathrm{co}(f)$ and so $\mathrm{co}(Tf) \subseteq \mathrm{co}(f)$. Therefore $\mathrm{co}(Tf) = \mathrm{co}(f)$, i.e., $T \in \mathcal{E}$. □

In the following theorem, we characterize the elements of $\mathcal{P}_{sce}(\mathfrak{c}_0)$.

Theorem 3.13. $\mathcal{P}_{sce}(\mathfrak{c}_0) = \{\lambda T : \lambda \in \mathbb{R} \setminus \{0\}, T \in \mathcal{E}\}$.

Proof. It is easy to show that $\{\lambda T : \lambda \in \mathbb{R} \setminus \{0\}, T \in \mathcal{E}\} \subseteq \mathcal{P}_{sce}(\mathfrak{c}_0)$. Now, let $T \in \mathcal{P}_{sce}(\mathfrak{c}_0)$. The fact $\mathcal{P}_{sce}(\mathfrak{c}_0) \subseteq \mathcal{P}_{ce}(\mathfrak{c}_0)$ and Theorems 1.8, 2.13, 3.5, 3.6 and 3.12 imply that $\frac{1}{b}T \in \mathcal{E}$, whenever $a = 0 < b$ and $\frac{1}{a}T \in \mathcal{E}$, whenever $a < 0 = b$. □

As a consequence of Theorem 3.13, we obtain the next theorem.

Theorem 3.14. $\mathcal{P}_{scm}(\mathfrak{c}_0) = \mathcal{P}_{sce}(\mathfrak{c}_0)$.

Proof. It is easy to show that $\mathcal{P}_{scm}(\mathfrak{c}_0) \subseteq \mathcal{P}_{sce}(\mathfrak{c}_0)$. Now suppose that $T \in \mathcal{P}_{sce}(\mathfrak{c}_0)$. Theorem 3.13 implies that $T = \lambda T_1$, for some $\lambda \neq 0$ and $T_1 \in \mathcal{E}$. Hence $\mathrm{co}(Tf) = \mathrm{co}(\lambda T_1(f)) = \lambda \mathrm{co}(T_1(f)) = \lambda \mathrm{co}(f)$. So $f \prec_c g$ if and only if $\mathrm{co}(Tf) = \lambda \mathrm{co}(f) \subseteq \lambda \mathrm{co}(g) = \mathrm{co}(Tg)$, that is $T \in \mathcal{P}_{scm}(\mathfrak{c}_0)$. □

The above two theorem characterize the elements of $\mathcal{P}_{scm}(\mathfrak{c}_0)$.

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