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Coincidence and Fixed Points for Multivalued Mappings in Incomplete Metric Spaces with Applications

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Abstract. In the present paper, firstly, we review the notion of R-complete metric spaces, where R is a binary relation (not necessarily a partial order). This notion lets us to consider some fixed point theorems for multivalued mappings in incomplete metric spaces. Secondly, as motivated by the recent work of Wei-Shih Du (On coincidence point and fixed point theorems for nonlinear multivalued maps, *Topology and its Applications* 159 (2012) 49–56), we prove the existence of coincidence points and fixed points of a general class of multivalued mappings satisfying a new generalized contractive condition in R-complete metric spaces which extends some well-known results in the literature. In addition, this article consists of several non-trivial examples which signify the motivation of such investigations. Finally, we give an application to the nonlinear fractional boundary value equations.

1. Introduction and Preliminaries

Throughout this paper, $\mathbb{N}, \mathbb{Q}$ and $\mathbb{R}$ denote, respectively, the sets of all natural numbers, rational numbers and real numbers.

Let (X, d) be a metric space. We denote by $CB(X)$ the class of all nonempty closed and bounded subsets of X, and $K(X)$ the class of all nonempty compact subsets of X.

For $A, B \in CB(X)$ and $x \in X$, define

$$D(x, A) := \inf\{d(x, a); a \in A\}$$

and

$$H(A, B) := \max\{\sup_{a\in A} D(a, B), \sup_{b\in B} D(b, A)\}.$$

The function H is a metric on $CB(X)$ and is called a Pompeiu-Hausdorff metric induced by d. It is well known that if X is a complete metric space, then so is the metric space $(CB(X), H)$.

Let $f : X \to X$ be a self-mapping and $T : X \to CB(X)$ be a multivalued map. A point $x \in X$ is a coincidence point of f and T if $fx \in Tx$. If $f = id$, the identity mapping, then $x = fx \in Tx$ and we call x a fixed point of T. The set of fixed points of T and the set of coincidence points of f and T are denoted by $F(T)$ and $COP(f, T)$, respectively.

In 1969, Nadler [15] extended the Banach contraction principle to multivalued mappings as follows.

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Theorem 1.1. *Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume that there exists $r \in [0, 1)$ such that $H(Tx, Ty) \leq rd(x, y)$ for all $x, y \in X$. Then there exists $z \in X$ such that $z \in T(z)$.*

Inspiring from the results of Nadler the fixed point theory of multivalued contraction was further developed in different directions by many authors, in particular, by Reich [18], Berinde-Berinde [7], Mizoguchi and Takahashi [14], Du [11], Daffer *et al.* [9, 10], Amini-Harandi [2], Boonsri *et al.* [8], Petrusel *et al.*[16] and many others.

Recently, Du [11] proved a generalization of Berinde-Berinde's fixed point theorem [7] as follows.

Theorem 1.2. *Let (X, d) be a complete metric space. Let $T : X \to CB(X)$ be a multivalued mapping, $f : X \to X$ be a continuous self-mapping and $\beta : [0, \infty) \to [0, 1)$ be a function such that $\limsup_{s \to t^+} \beta(s) < 1$ for each $t \geq 0$. Assume that*
(a_1) for each $x \in X$, $\{fy : y \in Tx\} \subseteq Tx$;
(a_2) there exists a function $\hat{h} : X \to [0, \infty)$ such that

$$H(Tx, Ty) \leq \beta(d(x, y)).d(x, y) + \hat{h}(fy)D(fy, Tx)$$

for each $x, y \in X$. Then $\mathrm{COP}(f, T) \bigcap \mathrm{F}(T) \neq \emptyset$.

In the following, we state Berinde-Berinde's fixed point theorem [7].

Theorem 1.3. *Let (X, d) be a complete metric space. Let $T : X \to CB(X)$ be a multivalued mapping and $\beta : [0, \infty) \to [0, 1)$ be a function such that $\limsup_{s \to t^+} \beta(s) < 1$ for each $t \geq 0$. Assume that*

$$H(Tx, Ty) \leq \beta(d(x, y)).d(x, y) + L.D(y, Tx)$$

for each $x, y \in X$, where $L \geq 0$. Then $\mathrm{F}(T) \neq \emptyset$.

Notice that, if we let $L = 0$ in above theorem, then we can obtain Mizoguchi-Takahashi's fixed point theorem [14] which is a partial answer of Problem 9 in [18]. Indeed, Reich established the following:

Theorem 1.4. *Let (X, d) be a complete metric space. Let $T : X \to K(X)$ be a multivalued mapping and $\beta : [0, \infty) \to [0, 1)$ be a function such that $\limsup_{s \to t^+} \beta(s) < 1$ for each $t \geq 0$. Assume that*

$$H(Tx, Ty) \leq \beta(d(x, y)).d(x, y)$$

for each $x, y \in X$. Then $\mathrm{F}(T) \neq \emptyset$.

Reich [18] posed the question whether above theorem is also true for a mapping $T : X \to CB(X)$. Mizoguchi and Takahashi [14] in 1989 responded to this conjecture and proved the following theorem which additionally is more general than Nadler's theorem.

Theorem 1.5. *Let (X, d) be a complete metric space. Let $T : X \to CB(X)$ be a multivalued mapping and $\beta : [0, \infty) \to [0, 1)$ be a function such that $\limsup_{s \to t^+} \beta(s) < 1$ for each $t \geq 0$. Assume that*

$$H(Tx, Ty) \leq \beta(d(x, y)).d(x, y)$$

for each $x, y \in X$. Then $\mathrm{F}(T) \neq \emptyset$.

In 2011, Amini-Harandi [2] introduced the concept of a set-valued quasi-contraction and proved the following interesting fixed point theorem.

Theorem 1.6. *Let (X, d) be a complete metric space. Let $T : X \to CB(X)$ be a multivalued mapping. Assume that*

$$H(Tx, Ty) \leq k. \max\{d(x, y), D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)\}$$

for each $x, y \in X$, where $0 < k < \frac{1}{2}$. Then $\mathrm{F}(T) \neq \emptyset$.

On the other hand, Boonsri and Saejung in [8] showed that the conclusion of Daffer and Kaneno[9] remains true without assuming the lower semicontinuity of the function $x \mapsto D(x, Tx)$. In the following, we state Boonsri-Saejung's fixed point theorem.

Theorem 1.7. *Let (X, d) be a complete metric space. Let $T : X \to CB(X)$ be a multivalued mapping. Assume that*

$$H(Tx, Ty) \le k.\max\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2}\}$$

for each $x, y \in X$, where $0 < k < 1$. Then $\mathrm{F}(T) \neq \emptyset$.

As motivated by these works, we define a new type of monotone multivalued mappings and prove some coincidence point and fixed point theorems under a new generalized contractive condition which are different from Nadler's theorem, Berinde-Berinde's theorem, Boonsri-Saejung's theorem, Mizoguchi-Takahashi's theorem, Du's theorem and Amini-Harandi's theorem for nonlinear multivalued contractive mappings. Our results compliment and extend some important fixed point theorems for multivalued contractive mappings.

2. Basic Definitions and Notations

Very recently, Eshaghi Gordji *et al.* [12] and Baghani *et al.* [4] introduced the notation of orthogonal sets and gave a real generalization of the Banach fixed point theorem in incomplete metric spaces. The notion helps them to find the solution of a integral equation in incomplete metric spaces. For more details, we refer the reader to [1, 3, 5, 6, 17].

To set up our results in the next sections, we need to introduce some definitions that play a major roles in further sections.

Let X be a nonempty set, $A, B \subseteq X$ and R be an arbitrary binary relation on X. The binary relations *strongly relation* (briefly, SR) and *weakly relation* (briefly, WR) are defined between A and B as follows.
(1) A (SR) B if $a \,\mathrm{R}\, b$, for all $a \in A$ and $b \in B$.
(2) A (WR) B if for each $a \in A$ there exists $b \in B$ such that $a \,\mathrm{R}\, b$.

It is clear that the relation SR implies the relation WR. Example 2.2 shows that the converse of the statement is not true in general. Now, we introduce a type of monotone multivalued mappings by using the relation SR.

Definition 2.1. *Let (X, d) be a metric space endowed a relation R on X and $T : X \to CB(X)$. Then T is said to be a monotone mapping of type* SR *if*

$$x, y \in X, x \quad \mathrm{R} \quad y \Rightarrow Tx \ \ (\mathrm{SR}) \ \ Ty.$$

Example 2.2. *Let $X = \{\frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2^n}, \cdots\} \cup \{0, 1\}$, $d(x, y) = |x - y|$ for all $x, y \in X$, and relation R be defined on X by*

$$x \ \mathrm{R} \ y \iff \begin{cases} \frac{y}{x} \in \mathbb{N}, \\ or\ x = y = 0. \end{cases}$$

Let $T : X \to CB(X)$ be defined by

$$Tx = \begin{cases} \{\frac{1}{2^n}, \frac{1}{2^{n+1}}\}, & if \quad x = \frac{1}{2^n}, n = 1, 2, \cdots, \\ \{0\}, & if \quad x = 0, \\ \{1, \frac{1}{2}, \frac{1}{4}\}, & if \quad x = 1. \end{cases}$$

It is easy to see that T is not monotone of type SR.

Example 2.3. *Let $X = [0,1)$ be equipped with the Euclidean metric. Define relation* R *on X by x R y iff either $x = 0$ or $y = 0$. Let $T : X \to CB(X)$ be a mapping defined by*

$$T(x) = \begin{cases} \{\frac{1}{2}x^2, x\}, & \text{if} \quad x \in \mathbb{Q} \cap X, \\ \{0\}, & \text{if} \quad x \in \mathbb{Q}^c \cap X. \end{cases}$$

It is easy to see that T is monotone of type SR.

Definition 2.4. *Let $X \neq \emptyset$ and $\mathrm{R} \subseteq X \times X$ be a relation. A sequence $\{x_n\}$ is called an* R*-sequence if*

$$(\forall n, k \in \mathbb{N} : \quad x_n \mathrm{R} x_{n+k}).$$

Definition 2.5. *Let (X, d) be a metric space and* R *be a relation on X. Then X is said to be* R*-regular if for each* R*-sequence $\{x_n\}$ with $x_n \to x$ for some $x \in X$, there exists $n_0 \in \mathbb{N}$ such that*

$$(\forall n \geq n_0 : \quad x_n \mathrm{R} x).$$

Definition 2.6. *Let (X, d) be a metric space and* R *be a relation on X. Then X is said to be* R*-complete if every Cauchy* R*-sequence is convergent (briefly, (X, d, R) is called an* R*-complete metric space).*

Example 2.7. *Consider $X = [0, \frac{1}{2}) \cup (\frac{1}{2}, 2]$ equipped with the Euclidean metric. Define relation* R *on X by $\mathrm{R} = \{(0,0), (0,1), (1,0), (1,1), (0,2)\}$. It is easy to see that (X, d, R) is an* R*-complete (not complete) metric space. We are going to show that (X, d, R) is an* R*-regular metric space. Take* R*-sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x$. Since $\{x_n\}$ is an* R*-sequence then for each $n \in \mathbb{N}$, $(x_n, x_{n+1}) \in \{(0,0), (0,1), (1,0), (1,1)\}$ which gives rise to $\{x_n\} \subseteq \{0,1\}$. As $\{0,1\}$ is closed, we have x_n R x for all $n \in \mathbb{N}$.*

Example 2.8. *Let X be a linear subspace of a Hilbert space H. For all $x, y \in X$, define x R y iff $|\langle x, y\rangle| = \|x\| \ \|y\|$. We claim that $(X, \|.\|, \mathrm{R})$ is an* R*-complete metric space which is not* R*-regular. Let $\{x_n\} \subseteq X$ be a Cauchy* R*-sequence. Then $\{x_n\}$ converges to some $x \in H$. Our aim is to show that x is an element of X. The relation* R *ensures that for all $n \in \mathbb{N}$,*

$$\exists \alpha_n \ \ s.t. \quad x_n = \alpha_n x_{n+1} \quad or \quad x_{n+1} = \alpha_n x_n. \tag{1}$$

We distinguish two cases.

Case 1. There exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} = 0$ for all k. This implies that $x = 0 \in X$.

Case 2. For all sufficiently large $n \in \mathbb{N}$, $x_n \neq 0$. Take $n_0 \in \mathbb{N}$ such that for all $n \geq n_o$, $x_n \neq 0$. It follows from (1) that for all $n \geq n_0$ there exists $\alpha_n > 0$, such that $x_n = \alpha_n x_{n_0}$. In other words,

$$|\alpha_n - \alpha_m| \ \|x_{n_0}\| = \|x_n - x_m\| \to 0 \quad as \ \ m, n \to \infty.$$

Therefore, $\{\alpha_n\}$ is a Cauchy sequence in $\mathbb{R}$. Assume that $\alpha_n \to \alpha$ as $n \to \infty$. Then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \alpha_n x_{n_0} = \alpha x_{n_0}$. This implies that $x \in X$.

Remark 2.9. *Every complete metric space is* R*-complete, but Examples 2.8 and 2.7 show that the converse is not true in general.*

Definition 2.10. *Let Λ denote the class of those functions $\phi(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \to \mathbb{R}_+$ which satisfy the following conditions*

(Λ_1) *ϕ is increasing in t_2, t_3, t_4 and t_5;*

(Λ_2) *$v < \phi(u, u, v, u + v, 0)$ implies that $v < u$, for each $u, v \in \mathbb{R}_+$;*

(Λ_3) *If $t_n, s_n \to 0$ and $u_n \to \gamma > 0$, as $n \to \infty$, then we have $\limsup_{n\to\infty} \phi(t_n, s_n, \gamma, u_n, t_{n+1}) \leq \gamma$;*

(Λ_4) *$\phi(u, u, u, 2u, 0) \leq u$ for each $u \in \mathbb{R}^+ := [0, +\infty)$.*

Many functions belong to the class Λ as shown by the following examples.

Example 2.11. (I)

$$\phi_1(t_1, t_2, t_3, t_4, t_5) = \hat{\alpha}t_1 + \hat{\beta}t_2 + \hat{\gamma}t_3 + \hat{\delta}t_4 + Lt_5,$$

where $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}, L \geq 0$, $\hat{\alpha} + \hat{\beta} + \hat{\gamma} + 2\hat{\delta} = 1$ *and* $\hat{\gamma} \neq 1$.

(II)

$$\phi_2(t_1, t_2, t_3, t_4, t_5) = \frac{1}{2}\max\{t_1, t_2, t_3, t_4, t_5\}.$$

(III)

$$\phi_3(t_1, t_2, t_3, t_4, t_5) = \max\{t_1, t_2, t_3, \frac{1}{2}(t_4 + t_5)\}.$$

Example 2.12. *Let* $\phi \in \Lambda$. *Suppose* $\tilde{\phi} : \mathbb{R}_+^5 \to \mathbb{R}_+$ *is defined by*

$$\tilde{\phi}(t_1, t_2, t_3, t_4, t_5) = \phi(t_1, t_2, t_3, t_4, t_5) + L.t_5,$$

where $L \geq 0$. *It is easy to see that* $\tilde{\phi} \in \Lambda$.

Definition 2.13. *Let* (X, d) *be a metric space and* R *be a relation on* X. *A mapping* $f : X \to X$ *is* R*-continuous at* $a \in X$ *if for each* R*-sequence* $\{a_n\}$ *in* X *if* $a_n \to a$, *then* $f(a_n) \to f(a)$. *Also,* f *is* R*-continuous on* X *if* f *is* R*-continuous at each* $a \in X$.

Example 2.14. *Let* $X = [0, 1]$ *with the Euclidean metric. Assume* x R y *is and only if* $xy = 0$. *Define* $f : X \to X$ *by*

$$f(x) = \begin{cases} 1, & \text{if} \quad x \in \mathbb{Q} \cap [0, 1], \\ x, & \text{if} \quad x \in \mathbb{Q}^c \cap [0, 1]. \end{cases}$$

Notice that f *is not continuous but we can see that* f *is* R*-continuous. If* $\{x_n\}$ *is a* R*-sequence in* X *which converges to* $x \in X$. *Applying definition* R *we obtain* $x_n = 0$. *This implies that* $1 = f(x_n) \to f(x) = 1$.

3. Main Results

In below, we state and prove the main theorem of this manuscript in R-complete metric spaces. This theorem helps us to find coincidence points and fixed points for multivalued mappings in incomplete metric spaces.

Theorem 3.1. *Let* (X, d, R) *be an* R*-complete (not necessarily complete) and* R*-regular metric space. Let* $T : X \to CB(X)$ *be a multivalued mapping,* $f : X \to X$ *be an* R*-continuous self-mapping and* $\varphi : [0, \infty) \to [0, 1)$ *be a function such that* $\limsup_{s \to t^+} \varphi(s) < 1$ *for each* $t \geq 0$. *Assume that*
(a_1) *for each* $x \in X$, $\{fy : y \in Tx\} \subseteq Tx$;
(a_2) *there exist functions* $\hat{h} : X \to [0, \infty)$ *and* $\phi \in \Lambda$ *such that*

$$H(Tx, Ty) \leq \varphi(d(x, y)).\phi(d(x, y), D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)) + \hat{h}(fy)D(fy, Tx) \tag{2}$$

for each x R y *with* $x \neq y$. *Suppose that*
(i) T *is monotone of type* SR;
(ii) *there exists* $x_0 \in X$ *such that for each* $x \in X$, $\{x_0\}$ (WR) Tx.
Then $\mathrm{COP}(f, T) \bigcap \mathrm{F}(T) \neq \emptyset$.

Proof. By (a_1), we note that, for each $x \in X$, $D(fy, Tx) = 0$ for all $y \in Tx$. Also, it is easy to see that, if $x^* \in T(x^*)$, then $x^* \in COP(f, T) \bigcap F(T)$. For this reason we suppose that T has no fixed point, i.e., $D(x, Tx) > 0$ for all $x \in X$.

By properties of functions φ, for each $t > 0$, there exist $k(t) > 0$ and $\delta(t) > 0$ such that

$$\varphi(s) \leq k(t) < 1 \text{ for all } s \in (t, t + \delta(t)). \tag{3}$$

Since $\{x_0\}$ (WR) Tx_0, there exists $x_1 \in Tx_0$ such that x_0 R x_1. If $x_0 = x_1$, then $x_0 = x_1 \in Tx_0$ and this is a contradiction. So, we may assume that $x_0 \neq x_1$. Moreover by monotonicity of T, we have Tx_0 (SR) Tx_1. Put $t_1 = D(x_1, Tx_1)$. It is clear that $D(x_1, Tx_1) \leq d(x_1, y)$ for all $y \in Tx_1$. The following cases are considered:

Case 1. $D(x_1, Tx_1) < d(x_1, y)$ for all $y \in Tx_1$. Select positive number $d(t_1)$ such that

$$d(t_1) < \min\{\delta(t_1), (\frac{1}{k(t_1)} - 1)t_1\}, \tag{4}$$

and put

$$\epsilon(x_1) = \min\{1, \frac{d(t_1)}{t_1}\}. \tag{5}$$

Then there exists $x_2 \in Tx_1$ such that x_1 R x_2 and

$$d(x_1, x_2) < D(x_1, Tx_1) + \epsilon(x_1)D(x_1, Tx_1) = (1 + \epsilon(x_1))D(x_1, Tx_1). \tag{6}$$

By the hypotheses that T no fixed point, we have $x_1 \neq x_2$. On the other hand by (2) and (Λ_1), we can write

$$\begin{aligned} D(x_2, Tx_2) &\leq H(Tx_1, Tx_2) \\ &\leq \varphi(d(x_1, x_2)).\phi(d(x_1, x_2), D(x_1, Tx_1), D(x_2, Tx_2), D(x_1, Tx_2), D(x_2, Tx_1)) \\ &\leq \varphi(d(x_1, x_2)).\phi(d(x_1, x_2), d(x_1, x_2), D(x_2, Tx_2), D(x_1, Tx_2), 0) \\ &\leq \varphi(d(x_1, x_2)).\phi(d(x_1, x_2), d(x_1, x_2), D(x_2, Tx_2), d(x_1, x_2) + D(x_2, Tx_2), 0) \\ &< \phi(d(x_1, x_2), d(x_1, x_2), D(x_2, Tx_2), d(x_1, x_2) + D(x_2, Tx_2), 0). \end{aligned} \tag{7}$$

Now by above relation, (Λ_2), (Λ_1) and (Λ_4), we conclude that

$$D(x_2, Tx_2) \leq \varphi(d(x_1, x_2)).d(x_1, x_2).$$

Therefore

$$\begin{aligned} D(x_1, Tx_1) - D(x_2, Tx_2) &\geq D(x_1, Tx_1) - \varphi(d(x_1, x_2)).d(x_1, x_2) \\ &> (\frac{1}{1 + \epsilon(x_1)} - \varphi(d(x_1, x_2))).d(x_1, x_2). \end{aligned} \tag{8}$$

By (4), (5) and (6)

$$t_1 = D(x_1, Tx_1) < d(x_1, x_2) < D(x_1, Tx_1) + \epsilon(x_1).D(x_1, Tx_1) \leq t_1 + d(t_1) < t_1 + \delta(t_1).$$

This implies by (3) that $\varphi(d(x_1, x_2)) \leq k(t_1) < 1$. Since $\epsilon(x_1) \leq \frac{d(t_1)}{t_1} < \frac{1}{k(t_1)} - 1$, we have

$$\frac{1}{1 + \epsilon(x_1)} - \varphi(d(x_1, x_2)) > 0. \tag{9}$$

It follows (8) that $D(x_2, Tx_2) < D(x_1, Tx_1)$.

Case 2. $D(x_1, Tx_1) = d(x_1, x_2)$ for some $x_2 \in Tx_1$. Since Tx_0 (SR) Tx_1, then x_1 R x_2 and also

$$D(x_1, Tx_1) - D(x_2, Tx_2) \geq (1 - \varphi(d(x_1, x_2))).d(x_1, x_2) > 0.$$

Therefore $D(x_2, Tx_2) < D(x_1, Tx_1)$.

Next, let $t_2 = D(x_2, Tx_2)$. Then $D(x_2, Tx_2) \leq d(x_2, y)$ for all $y \in Tx_2$. Again we consider the following two cases:

Case A. $D(x_2, Tx_2) < d(x_2, y)$ for all $y \in Tx_2$. For $\delta(t_2)$ and $k(t_2)$, choose $d(t_2)$ with

$$d(t_2) < \min\{\delta(t_2), (\frac{1}{k(t_2)} - 1)t_2\}$$

and set

$$\epsilon(x_2) = \min\{\frac{d(t_2)}{t_2}, \frac{1}{2}, \frac{t_1}{t_2} - 1\}.$$

By using the similar reason as above, we obtain $x_3 \in Tx_2$ such that x_2 R x_3, $x_2 \neq x_3$, $d(x_2, x_3) < (1 + \epsilon(x_2))D(x_2, Tx_2)$ and

$$D(x_2, Tx_2) - D(x_3, Tx_3) \geq (\frac{1}{1 + \epsilon(x_2)} - \varphi(d(x_1, x_2))).d(x_2, x_3) > 0.$$

Hence $D(x_3, Tx_3) < D(x_2, Tx_2)$. From $\epsilon(x_2) \leq \frac{t_1}{t_2} - 1$, it follows that

$$d(x_2, x_3) < (1 + \epsilon(x_2))D(x_2, Tx_2) \leq D(x_1, Tx_1) \leq d(x_1, x_2).$$

Case B. $D(x_2, Tx_2) = d(x_2, x_3)$ for some $x_3 \in Tx_2$. Since Tx_1 (SR) Tx_2, then x_2 R x_3 and also by using the same method as above, we can show that

$$D(x_2, Tx_2) - D(x_3, Tx_3) \geq (1 - \varphi(d(x_2, x_3))).d(x_2, x_3) > 0$$

and

$$d(x_2, x_3) = D(x_2, Tx_2) < D(x_1, Tx_1) \leq d(x_1, x_2).$$

Hence, $D(x_3, Tx_3) < D(x_2, Tx_2)$ and $d(x_2, x_3) < d(x_1, x_2)$. Repeating this process, we find that there exists an R-sequence $\{x_n\}$ with $x_{n+1} \in Tx_n$ such that $\{D(x_n, Tx_n)\}$ and $\{d(x_n, x_{n+1}\}$ are decreasing sequences of positive numbers and for each $n \in \mathbb{N}$,

$$D(x_n, Tx_n) - D(x_{n+1}, Tx_{n+1}) \geq (\frac{1}{1 + \gamma(x_n)} - \varphi(d(x_n, x_{n+1}))).d(x_n, x_{n+1}), \tag{10}$$

where $\gamma(x_n)$ is real number with $0 \leq \gamma(x_n) \leq \frac{1}{n}$. Since $\{d(x_n, x_{n+1})\}$ is decreasing sequence, there exists $t \in [0, \infty)$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = t$.
Let $a_n := \frac{1}{1+\gamma(x_n)} - \varphi(d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, then

$$\liminf_{n\to\infty} a_n \geq \lim_{n\to\infty} \frac{1}{1 + \gamma(x_n)} - \limsup_{n\to\infty} \varphi(d(x_n, x_{n+1}) > 0.$$

This implies that from (10), there exists $b > 0$ such that

$$D(x_n, Tx_n) - D(x_{n+1}, Tx_{n+1}) \geq b.d(x_n, x_{n+1})$$

for large enough n. Since $\{d(x_n, x_{n+1})\}$ is decreasing sequence, it is convergent. On the other hand, for each $n < m$, we have

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \\ &\leq \frac{1}{b} \sum_{i=n}^{m-1} \{D(x_i, Tx_i) - D(x_{i+1}, T_{i+1})\} \\ &= \frac{1}{b}\{D(x_n, Tx_n) - D(x_m, Tx_m)\} \to 0 \end{aligned}$$

as $n, m \to \infty$. Hence $\{x_n\}$ is a Cauchy R-sequence. Since X is R-complete then $\lim_{n\to\infty} x_n = x^*$, for some $x^* \in X$. Since $x_{n+1} \in Tx_n$, it follows from (a_1) that $fx_{n+1} \in Tx_n$ for each $n \in \mathbb{N}$. Since f is R-continuous and $\lim_{n\to\infty} x_n = x^*$, we have

$$\lim_{n\to\infty} fx_{n+1} = fx^*.$$

By assumption R-regularity of X, since x_n R x_{n+k} for all $n, k \in \mathbb{N}$ and $x_n \to x^*$, as $n \to \infty$, then x_n R x^* for $n \geq n_0$, for some $n_0 \in \mathbb{N}$. Thus, from (2) with $x = x_n$ and $y = x^*$, we obtain

$$\begin{aligned} &D(x_{n+1}, Tx^*) \leq H(Tx_n, Tx^*) \\ &\leq \varphi(d(x_n, x^*)).\phi(d(x_n, x^*), d(x_n, x_{n+1}), D(x^*, Tx^*), D(x_n, Tx^*), d(x^*, x_{n+1})) + \hat{h}(fx^*)d(fx^*, fx_{n+1}) \end{aligned} \tag{11}$$

for each $n \in \mathbb{N}$ with $n \geq n_0$.
Now since $x^* \notin Tx^*$ then by using (11) and (Λ_3) we have

$$\begin{aligned} &D(x^*, Tx^*) = \limsup_{n\to\infty} D(x_{n+1}, Tx^*) \\ &\leq \limsup_{n\to\infty} \Big(\varphi(d(x_n, x^*)).\phi(d(x_n, x^*), d(x_n, x_{n+1}), D(x^*, Tx^*), D(x_n, Tx^*), d(x^*, x_{n+1})) + \hat{h}(fx^*)d(fx^*, fx_{n+1})\Big) \\ &< D(x^*, Tx^*). \end{aligned}$$

Then $x^* \in Tx^*$ which is a contradiction because it is supposed that T has no fixed point. By (a_1), $fx^* \in Tx^*$. Hence $x^* \in COP(f, T)$. This completes the proof. □

4. Some Consequences

Letting

$$\phi(t_1, t_2, t_3, t_4, t_5) = \alpha t_1 + \beta t_2 + \gamma t_3 + \delta t_4 + L t_5,$$

where $\alpha, \beta, \gamma, \delta, L \geq 0$, $\alpha + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$, we get a generalization of Theorem 2.2 of [11], Theorem 4 of [7] and Theorem 5 of [14].

Corollary 4.1. *Let* (X, d, R) *be an* R*-complete* (*not necessarily complete*) *and* R*-regular metric space. Let* $T : X \to CB(X)$ *be a multivalued mapping,* $f : X \to X$ *be an* R*-continuous self-mapping and* $\varphi : [0, \infty) \to [0, 1)$ *be a function such that* $\limsup_{s\to t^+} \varphi(s) < 1$ *for each* $t \geq 0$*. Assume that*
(a_1) *for each* $x \in X$, $\{fy : y \in Tx\} \subseteq Tx$;
(a_2) *there exists a function* $\hat{h} : X \to [0, \infty)$ *such that*

$$H(Tx, Ty) \leq \varphi(d(x, y)).\Big(\alpha.d(x, y) + \beta.D(x, Tx) + \gamma.D(y, Ty) + \delta.D(x, Ty)\Big) + L.D(y, Tx) + \hat{h}(fy).D(fy, Tx)$$

for each x R y *with* $x \neq y$*, where* $\alpha, \beta, \gamma, \delta, L \geq 0$, $\alpha + \beta + \gamma + 2\delta = 1$ *and* $\gamma \neq 1$*. Suppose that*
(i) *T is monotone of type* SR*;*
(ii) *there exists* $x_0 \in X$ *such that for each* $x \in X$, $\{x_0\}$ (WR) Tx*.*
Then $\mathrm{COP}(f, T) \bigcap \mathrm{F}(T) \neq \emptyset$*.*

Proof. Define a function $\tilde{\beta}$ from $[0, \infty)$ into $[0, 1)$ by $\tilde{\beta}(t) = \frac{\varphi(t)+1}{2}$ for $t \in [0, \infty)$. Then the following hold:

1. $\limsup_{s\to t^+} \tilde{\beta}(s) < 1$ for all $t \in [0, \infty)$.
2. $\varphi(t) < \tilde{\beta}(t)$ for all $t \in [0, \infty)$.
3. $\tilde{\beta}(t) \geq \frac{1}{2}$ for all $t \in [0, \infty)$.

Now we have

$$\begin{aligned}H(Tx,Ty) &\le \varphi(d(x,y)).\Big(\alpha.d(x,y)+\beta.D(x,Tx)+\gamma.D(y,Ty)+\delta.D(x,Ty)\Big)+L.D(y,Tx)+\hat{h}(fy).D(fy,Tx)\\ &< \tilde{\beta}(d(x,y)).\Big(\alpha.d(x,y)+\beta.D(x,Tx)+\gamma.D(y,Ty)+\delta.D(x,Ty)\Big)\\ &+\frac{L.\tilde{\beta}(d(x,y)).D(y,Tx)}{\tilde{\beta}(d(x,y))}+\hat{h}(fy).D(fy,Tx)\\ &\le \tilde{\beta}(d(x,y)).\Big(\alpha.d(x,y)+\beta.D(x,Tx)+\gamma.D(y,Ty)+\delta.D(x,Ty)+2L.D(y,Tx)\Big)+\hat{h}(fy).D(fy,Tx)\end{aligned}$$

for each x R y with $x \neq y$.

Therefore by applying Theorem 2 and Example 2.11-I, we can see the results. □

Letting

$$\phi(t_1,t_2,t_3,t_4,t_5)=\frac{1}{2}\max\{t_1,t_2,t_3,t_4,t_5\},$$

we get a generalization of Theorem 2.2 of [2].

Corollary 4.2. *Let* (X,d,R) *be an* R-*complete* (*not necessarily complete*) *and* R-*regular metric space. Let* $T:X\to CB(X)$ *be a multivalued mapping,* $f:X\to X$ *be an* R-*continuous self-mapping and* $\varphi:[0,\infty)\to[0,1)$ *be a function such that* $\limsup_{s\to t^+}\varphi(s)<\frac{1}{2}$ *for each* $t\ge 0$. *Assume that*
(a_1) *for each* $x\in X$, $\{fy : y\in Tx\}\subseteq Tx$;
(a_2) *there exists a function* $\hat{h}:X\to[0,\infty)$ *such that*

$$\begin{aligned}H(Tx,Ty) &\le \varphi(d(x,y)).\Big(\max\{d(x,y),D(x,Tx),D(y,Ty),D(x,Ty),D(y,Tx)\}\Big)\\ &+L.D(y,Tx)+\hat{h}(fy)D(fy,Tx)\end{aligned}$$

for each x R y *with* $x\neq y$, *where* $L\ge 0$. *Suppose that*
(i) T *is monotone of type* SR;
(ii) *there exists* $x_0\in X$ *such that for each* $x\in X$, $\{x_0\}$ (WR) Tx.
Then $\mathrm{COP}(f,T)\bigcap \mathrm{F}(T)\neq\emptyset$.

Proof. We can prove this corollary by Example 2.11-II, Example 2.12 and the technique has been used in Corollary 4.1. □

Letting

$$\phi(t_1,t_2,t_3,t_4,t_5)=\max\{t_1,t_2,t_3,\frac{1}{2}(t_4+t_5)\},$$

we get a generalization Theorem 1 of [8], Theorem 2.2 of [11] and Theorem 4 of [7].

Corollary 4.3. *Let* (X,d,R) *be a* R-*complete* (*not necessarily complete*) *and* R-*regular metric space. Let* $T:X\to CB(X)$ *be a multivalued mapping,* $f:X\to X$ *be an* R-*continuous self-mapping and* $\varphi:[0,\infty)\to[0,1)$ *be a function such that* $\lim_{s\to t^+}\varphi(s)<1$ *for each* $t\ge 0$. *Assume that*
(a_1) *for each* $x\in X$, $\{fy : y\in Tx\}\subseteq Tx$;
(a_2) *there exists a function* $\hat{h}:X\to[0,\infty)$ *such that*

$$\begin{aligned}H(Tx,Ty) &\le \varphi(d(x,y)).\Big(\max\{d(x,y),D(x,Tx),D(y,Ty),\frac{1}{2}(D(x,Ty)+D(y,Tx))\}\Big)\\ &+L.D(y,Tx)+\hat{h}(fy)D(fy,Tx)\end{aligned}$$

for each x R y *with* $x\neq y$. *Suppose that*
(i) T *is monotone of type* SR;
(ii) *there exists* $x_0\in X$ *such that for each* $x\in X$, $\{x_0\}$ (WR) Tx.
Then $\mathrm{COP}(f,T)\bigcap \mathrm{F}(T)\neq\emptyset$.

Proof. We can prove this corollary by Example 2.11-III, Example 2.12 and the technique has been used in Corollary 4.1. □

5. Some Examples

The following simple examples show the generality of our main theorem over Theorem 1 of [8], Theorem 2.2 of [11], Theorem 4 of [7], Theorem 5 of [14] and Theorem 2.2 of [2].

Example 5.1. *Consider the sequence $\{S_n\}$ as follows:*

$$\begin{aligned}
&S_1 = 1\times 2,\\
&S_2 = 1\times 2 + 2\times 3,\\
&S_3 = 1\times 2 + 2\times 3 + 3\times 4,\\
&\dots\\
&S_n = 1\times 2 + 2\times 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}, n\in\mathbb{N}.
\end{aligned}$$

Let $X = \{S_n : n\in\mathbb{N}\}$ and $d(x,y) = |x-y|$, $x,y\in X$. For all $S_n, S_m \in X$ define S_n R S_m if and only if $(1 = n \le m)$. Hence (X, d, R) is an R*-complete and* R*-regular metric space. Define a multivalued mapping $T : X \to CB(X)$ by the formulae:*

$$Tx = \begin{cases} \{S_{n-1}, S_{n+1}\}, & \text{if} \quad x = S_n, n = 3, 4, \cdots, \\ \{S_1\}, & \text{if} \quad x = S_1, S_2. \end{cases}$$

It is easy to see that T is monotone of type SR *and $\{S_1\}$ (WR) TS_n for each $n\in\mathbb{N}$.*
Now since,

$$\lim_{n\to\infty} \frac{H(T(S_n), T(S_1))}{d(S_n, S_1)} = 1,$$

then T is not contraction.
First, observe that

$$S_n \,\mathrm{R}\, S_m\ ,\ T(S_n) \neq T(S_m) \Longleftrightarrow (1 = n, m > 2).$$

On the other hand, for every $m\in\mathbb{N}, m>2$ we have

$$H(TS_1, TS_m) \le \varphi(d(S_1, S_m))(\alpha.d(S_1, S_m)) + L.D(S_m, TS_1),$$

where $\alpha = 1$, $L = \frac{9}{2}$ and $\varphi : [0,\infty) \to [0,1)$ is defined by $\varphi(t) = \frac{1}{2}$, $t\in[0,\infty)$. Hence by Corollary 4.1, for any function $\hat{h} : X \to [0,\infty)$ and any R*-continuous self-mapping $f : X \to X$ satisfying condition (a_1) of Corollary 4.1, we conclude that $\mathrm{COP}(f,T) \bigcap \mathrm{F}(T) \neq \emptyset$.*

Notice that the mapping T does not satisfy the assumptions of Theorem 1 of [8], Theorem 2.2 of [11], Theorem 4 of [7], Theorem 5 of [14] and Theorem 2.2 of [2]. For this reason take $x = S_3$ and $y = S_4$.

Example 5.2. *Let ℓ^∞ be the Banach space consisting of all bounded real sequences with supremum norm and let $\{e_n\}$ be the canonical basis of ℓ^∞. Let $\{\tau_n\}$ be a bounded, strictly increasing sequence in $(0,\infty)$ satisfying $\tau_{n+1} < 2\tau_n$ for all $n\in\mathbb{N}$ (for example, let $\tau_n = \frac{2^n-1}{2^n}$ $n\in\mathbb{N}$). Put $x_n = \tau_n e_n$ for each $n\in\mathbb{N}$. Define a bounded, complete subset X of ℓ^∞ by $X = \{x_1, x_2, x_3, \cdots\}$ and a mapping T from X into $CB(X)$ by*

$$Tx_n = \begin{cases} \{x_{n-1}, x_{n+1}\}, & \text{if} \quad n = 2, 3, \cdots, \\ \{x_1\}, & \text{if} \quad n = 1. \end{cases}$$

For all $x_n, x_m \in X$ define x_n R x_m if and only if $(1 = n \le m)$. Hence (X, d, R) is an R*-complete and* R*-regular metric space. It is easy to see that T is monotone of type* SR *and $\{x_1\}$ (WR) Tx_n for each $n\in\mathbb{N}$. On the other hand, for every $m\in\mathbb{N}$ we have*

$$H(Tx_1, Tx_m) \le \varphi(d(x_1, x_m))(\alpha.d(x_1, x_m)) + L.D(x_m, Tx_1),$$

where $\alpha = 1$, $L = \frac{3}{2}$ *and* $\varphi : [0,\infty) \to [0,1)$ *is defined by* $\varphi(t) = \frac{1}{2}$, $t \in [0,\infty)$. *Hence by Corollary 4.1, for any function* $\hat{h} : X \to [0,\infty)$ *and any* R*-continuous self-mapping* $f : X \to X$ *satisfying condition* (a_1) *of Corollary 4.1, we conclude that* $\mathrm{COP}(f,T) \bigcap \mathrm{F}(T) \neq \emptyset$.

Notice that the mapping T does not satisfy the assumptions of Theorem 1 of [8], Theorem 2.2 of [11], Theorem 4 of [7], Theorem 5 of [14] and Theorem 2.2 of [2]. For this reason take $x = x_4$ *and* $y = x_5$.

Below we explain a simple proof of Example A and Example B of [11].

Example 5.3. *[11] Let* ℓ^∞ *be the Banach space consisting of all bounded real sequences with supremum norm and let* $\{e_n\}$ *be the canonical basis of* ℓ^∞. *Let* $\{\tau_n\}$ *be a sequence of positive real numbers satisfying* $\tau_1 = \tau_2$ *and* $\tau_{n+1} < \tau_n$ *for* $n \geq 2$ *(for example, let* $\tau_1 = \frac{1}{2}$ *and* $\tau_n = \frac{1}{n}$ *for* $n \geq 2$*). Put* $x_n = \tau_n e_n$ *for each* $n \in \mathbb{N}$. *Define a bounded, complete subset X of* ℓ^∞ *by* $X = \{x_1, x_2, x_3, \cdots\}$ *and a mapping T from X into* $CB(X)$ *by*

$$Tx_n = \begin{cases} \{x_1, x_2\}, & if \quad n = 1,2, \\ X \backslash \{x_1, x_2, \cdots, x_n, x_{n+1}\}, & if \quad n \geq 3. \end{cases}$$

For all $x_n, x_m \in X$ *define* $x_n \ \mathrm{R} \ x_m$ *if and only if* $(1 = n \leq m)$. *Hence* (X, d, R) *is an* R*-complete and* R*-regular metric space. It is easy to see that T is monotone of type* SR *and* $\{x_1\}$ (WR) Tx_n *for each* $n \in \mathbb{N}$. *On the other hand, for every* $m \in \mathbb{N}$ *we have*

$$H(Tx_1, Tx_m) \leq \varphi(d(x_1, x_m))(\alpha.d(x_1, x_m)) + L.D(x_m, Tx_1),$$

where $\alpha = 1$, $L = 3$ *and* $\varphi : [0,\infty) \to [0,1)$ *is defined by* $\varphi(t) = \frac{1}{2}$, $t \in [0,\infty)$. *Hence by Corollary 4.1, for any function* $\hat{h} : X \to [0,\infty)$ *and any* R*-continuous self-mapping* $f : X \to X$ *satisfying condition* (a_1) *of Corollary 4.1, we conclude that* $\mathrm{COP}(f,T) \bigcap \mathrm{F}(T) \neq \emptyset$. *In particular, let* $f : X \to X$ *be defined by*

$$fx_n = \begin{cases} x_2, & if \quad n = 1,2, \\ x_{n+1}, & if \quad n \geq 3, \end{cases}$$

then $\mathrm{COP}(f,T) \bigcap \mathrm{F}(T) \neq \emptyset$.

6. Application to the Nonlinear Fractional Boundary Value Equations

Let $X = \{u \in C[0,1] : u(t) \geqslant 0, \forall t \in [0,1]\}$ endowed with the metric d induced by supremum norm. Consider the following nonlinear fractional boundary value equations

$$\begin{cases} D^\alpha_{0^+} u(t) + \lambda f(t, u(t)) = 0, & 0 < t < 1, \quad 3 < \alpha \leq 4, \\ u(0) = u'(0) = u''(0) = u''(1) = 0, \end{cases} \tag{12}$$

where $0 < \lambda < 1$ is constant, $f : [0,1] \times [0,\infty) \longrightarrow [0,\infty)$ is a continuous function and $D^\alpha_{0^+}$ is the standard Riemann-Liouville fractional derivative.

Here, we consider the following hypotheses:

(C_1) For all $u, v \in X$ with $u(t)v(t') \leq \max\{v(t), v(t')\}$ for each $t, t' \in [0,1]$, we have

$$\Big(f(t,u(t))f(t',v(t')) \leq \frac{1}{\lambda} f(t, v(t)), \forall t, t' \in [0,1]\Big),$$

or

$$\Big(f(t,u(t))f(t',v(t')) \leq \frac{1}{\lambda} f(t', v(t')), \forall t, t' \in [0,1]\Big).$$

(C_2) For all $u, v \in X$ with $u(t)v(t) \leq v(t)$ for each $t \in [0,1]$, we have

$$|f(t,u(t)) - f(t,v(t))| \leq \frac{\|u-v\|}{A},$$

where $\|u\| = \max_{t\in[0,1]} u(t)$ and $A = \max_{0\leqslant t\leqslant 1} \int_0^1 k(t,s)ds$, where $k : [0,1]\times[0,1] \longrightarrow [0,1]$ denotes the Green's function for the boundary value system (12).

Note that $f : [0,1]\times[0,\infty) \longrightarrow [0,\infty)$ is not necessarily Lipschitz from the given condition (C_2) and there exist some functions satisfying in condition (C_2) but not Lipschitz.

Theorem 6.1. *Let the above conditions are satisfied. Then, the fractional boundary value problem* (12) *has a positive solution.*

Proof. We define a operator equation $T : X \to X$ as follows:

$$Tu(t) = \lambda \int_0^1 k(t,s)f(s,u(s))ds, \tag{13}$$

where

$$k(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-3} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-3}, & 0 \leq t \leq s \leq 1. \end{cases}$$

We know that the differential equation has a positive solution if and only if T has a fixed point in X (see [13, Lemma 2.3]). We consider the following relation in X:

$$u \;\mathrm{R}\; v \iff u(t)v(t^{'}) \leq \max\{v(t), v(t^{'})\}, \tag{14}$$

for all $t, t^{'} \in [0,1]$ and $u, v \in X$. Since (X,d) is a complete metric space, then (X,d,R) is an R-complete and R-regular metric space. Now, we prove the following two steps to complete the proof.

Step1: T is monotone of type SR. Let $u, v \in X$ with $u\mathrm{R}v$. We must show that

$$Tu(t)Tv(t^{'}) \leq \max\{T(v(t)), T(v(t^{'}))\}$$

for all $t, t^{'} \in [0,1]$. Applying (13), we have

$$Tu(t)Tv(t^{'}) = \lambda^2 \int_0^1 \int_0^1 k(t,s)k(t^{'},s^{'})f(s,u(s))f(s',v(s'))ds'ds.$$

Applying (C_1), we have two cases:

(1). $f(s,u(s))f(s^{'},v(s^{'})) \leqslant \frac{1}{\lambda}f(s,v(s))$ for each $s, s^{'} \in [0,1]$. Applying (13) and definition of k, we have

$$\begin{aligned} Tu(t)Tv(t^{'}) &= \lambda^2 \int_0^1 \int_0^1 k(t,s)k(t^{'},s^{'})f(s,u(s))f(s',v(s'))ds'ds \\ &\leq \lambda \int_0^1 \int_0^1 k(t,s)k(t^{'},s^{'})f(s,v(s))ds'ds \\ &\leq \lambda \int_0^1 k(t,s)f(s,v(s))ds \\ &= T(v(t)) \\ &\leq \max\{T(v(t)), T(v(t^{'}))\}. \end{aligned}$$

(2). $f(s,u(s))f(s^{'},v(s^{'})) \leqslant \frac{1}{\lambda}f(s',v(s'))$ for each $s,s^{'} \in [0,1]$. Applying (13) and definition of k, we have

$$\begin{aligned}
Tu(t)Tv(t^{'}) &= \lambda^2 \int_0^1 \int_0^1 k(t,s)k(t^{'},s^{'})f(s,u(s))f(s',v(s'))ds'ds \\
&\leq \lambda \int_0^1 \int_0^1 k(t,s)k(t^{'},s^{'})f(s',v(s'))ds'ds \\
&\leq \lambda \int_0^1 k(t',s')f(s',v(s'))ds' \\
&= T(v(t')) \\
&\leq \max\{T(v(t)), T(v(t^{'}))\}.
\end{aligned}$$

These imply that T is monotone of type SR.

Step2: Show that for each elements $u,v \in X$ with $u \ \mathrm{R} \ v$, we have

$$d(Tu,Tv) \leq \lambda d(u,v).$$

Let $u,v \in X$ with $u\mathrm{R}v$. Then for all $t \in [0,1]$, we have $u(t)v(t) \leq v(t)$. Applying (C_2), we obtain that

$$\begin{aligned}
|Tu(t) - Tv(t)| &= \Big|\lambda \int_0^1 k(t,s)f(s,u(s))ds - \lambda \int_0^1 k(t,s)f(s,v(s))ds\Big| \\
&\leq \lambda \int_0^1 k(t,s)\Big|f(s,u(s)) - f(s,v(s))\Big|ds \\
&\leq \lambda \|u-v\|
\end{aligned}$$

for all $t \in [0,1]$. Hence,

$$d(Tu,Tv) \leq \lambda d(u,v)$$

for all $u,v \in X$ with $u \ \mathrm{R} \ v$.

Applying Corollary 4.1, T has a fixed point in X which is a positive solution of the differential equation (12). □

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