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A Generalization of Divided Differences and Applications

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Abstract. In this paper, we introduce a generalization of divided differences and apply it for constructing a new class of interpolation formulae. By studying the existence and uniqueness of the introduced class, we also consider some special cases of it and investigate the interpolating function corresponding to this class at coincident points leading to an extension of Taylor interpolation. Some numerical examples are also included.

1. Introduction

Let $\Psi(x; a_0, \ldots, a_n)$ be a family of functions of a single variable x with $n+1$ free parameters $\{a_j\}_{j=0}^n$. The interpolation problem for Ψ consists of determining $\{a_j\}_{j=0}^n$ so that for $n+1$ given real or complex pairs of distinct numbers $\{(x_j, f_j)\}_{j=0}^n$ we have

$$\Psi(x_j; a_0, \ldots, a_n) = f_j. \tag{1}$$

Relation (1) leads to a linear interpolation problem if Ψ depends linearly on the parameters a_i, i.e., [22]

$$\Psi(x; a_0, \ldots, a_n) \equiv a_0\Psi_0(x) + \cdots + a_n\Psi_n(x).$$

Interpolation problems are basic tools for approximating functions and quadrature rules (cf. [14], [13]). For history of interpolation problems see e.g. [3].

Newton's formula for constructing an interpolating polynomial is well-known [4], i.e.,

$$P_n(x) = \sum_{j=0}^{n} [x_0, \ldots, x_j]f \prod_{k=0}^{j-1}(x - x_k), \qquad x \in [a, b], \tag{2}$$

where $\prod_{k=0}^{-1}(\cdot) = 1$ and $[x_0, \ldots, x_j]f$ is jth order divided difference associated with f.

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Divided differences play a fundamental role in interpolation and approximation by polynomials and spline theory; for a recent survey see [2].

By using the uniqueness of an interpolating polynomial, it can be proved that [10]

$$[x_0,\dots,x_n]f = \frac{[x_1,\dots,x_n]f - [x_0,\dots,x_{n-1}]f}{x_n - x_0} = \sum_{j=0}^{n} \frac{f(x_j)}{w_n'(x_j)}, \tag{3}$$

where $w_n(x) = (x-x_0)\dots(x-x_n)$.

Also, it is clear that [19] we have

$$[x_0,\dots,x_k]f = \frac{V\begin{pmatrix} p_0\,, & \dots\,, & p_{k-1}\,, & f \\ x_0\,, & \dots\,, & x_{k-1}\,, & x_k \end{pmatrix}}{V\begin{pmatrix} p_0\,, & \dots\,, & p_{k-1}\,, & p_k \\ x_0\,, & \dots\,, & x_{k-1}\,, & x_k \end{pmatrix}}, \tag{4}$$

where

$$V\begin{pmatrix} f_0\,, & \dots\,, & f_k \\ x_0\,, & \dots\,, & x_k \end{pmatrix} = \det f_i(x_j) = \begin{vmatrix} f_0(x_0) & \cdots & f_0(x_k) \\ \vdots & & \vdots \\ f_k(x_0) & \cdots & f_k(x_k) \end{vmatrix},$$

and $p_j(x) = x^j$.

Newton's formula (2) was generalized by Mülbach using a linear family of functions forming a complete Chebyshev system [19]. In fact, the base of any ordinary divided differences is the classical complete Chebyshev system. This means that the generalized divided differences of a function f can be derived by replacing the aforesaid system with an arbitrary Chebyshev-system $(\phi_0,\dots,\phi_k)$ via (4) as follows

$$\left[\begin{matrix} \phi_0\,, & \dots\,, & \phi_k \\ x_0\,, & \dots\,, & x_k \end{matrix}\,\middle|\, f\right] = \frac{V\begin{pmatrix} \phi_0\,, & \dots\,, & \phi_{k-1}\,, & f \\ x_0\,, & \dots\,, & x_{k-1}\,, & x_k \end{pmatrix}}{V\begin{pmatrix} \phi_0\,, & \dots\,, & \phi_{k-1}\,, & \phi_k \\ x_0\,, & \dots\,, & x_{k-1}\,, & x_k \end{pmatrix}}. \tag{5}$$

Mülbach proved that the divided differences (5) satisfy a recurrence relation [19] similar to (3) as

$$\left[\begin{matrix} \phi_0\,, & \dots\,, & \phi_k \\ x_0\,, & \dots\,, & x_k \end{matrix}\,\middle|\, f\right] = \frac{\left[\begin{matrix} \phi_0\,, & \dots\,, & \phi_{k-1} \\ x_1\,, & \dots\,, & x_k \end{matrix}\,\middle|\, f\right] - \left[\begin{matrix} \phi_0\,, & \dots\,, & \phi_{k-1} \\ x_0\,, & \dots\,, & x_{k-1} \end{matrix}\,\middle|\, f\right]}{\left[\begin{matrix} \phi_0\,, & \dots\,, & \phi_{k-1} \\ x_1\,, & \dots\,, & x_k \end{matrix}\,\middle|\, \phi_k\right] - \left[\begin{matrix} \phi_0\,, & \dots\,, & \phi_{k-1} \\ x_0\,, & \dots\,, & x_{k-1} \end{matrix}\,\middle|\, \phi_k\right]},$$

where

$$\left[\begin{matrix} \phi_0 \\ x_0 \end{matrix}\,\middle|\, f\right] = \frac{f(x_0)}{\phi_0(x_0)} \qquad \text{and} \qquad \left[\begin{matrix} \phi_0 \\ x_0 \end{matrix}\,\middle|\, \phi_k\right] = \frac{\phi_k(x_0)}{\phi_0(x_0)}.$$

The general interpolation problem, proposed by Davis [7], is the most general case including all above-mentioned cases, because it is concerned with reconstructing functions on a basis of certain functional information, which are linear in many cases. Hence, many new interpolation formulae can be constructed using linear operators [12]. See also [9, 11, 18] in this regard.

Let Π be a linear space of dimension $n+1$ and $L_0, L_1, \dots, L_n$ be $n+1$ given linear functionals defined on Π, which are independent in Π^* (the algebraic conjugate space of Π). For a given set of values $w_0, w_1, \dots, w_n$, we can find an element $f \in \Pi$ such that

$$L_j(f) = w_j \qquad j = 0, 1, \dots, n. \tag{6}$$

According to Theorem 2.5.1 of [7, p. 35], there exist uniquely $n+1$ independent elements of the space Π $p_0^*, \ldots, p_n^*$, such that $L_i(p_j^*) = \delta_{i,j}$. Thus, for any $p \in \Pi$

$$p = \sum_{j=0}^{n} w_j p_j^*,$$

is the unique solution of the problem (6) in the space Π.

The aim of this paper is to introduce a new class of interpolation formulae extending Newton's formula. In the next section, we define the generalized divided difference operator. In Section 3, a certain space of functions, which is the base of constructing new class of interpolation formulae is introduced. In this direction, existence and uniqueness of the interpolating function, as well as the remainder formula, are considered for the introduced interpolation class. Section 3.1 is devoted to some special cases. In Section 4, we study our problem at coincident points, which leads to a generalization of Taylor interpolation. Finally, some numerical examples are given in Section 5.

2. Generalized divided difference operator

Let $\Lambda = \left\{\lambda_1, \lambda_2, \ldots\right\}$ be a system of monotonous continuous functions defined on $[a, b]$ and let

$$\Lambda_0 = \emptyset \quad \text{and} \quad \Lambda_n = \left\{\lambda_1, \ldots, \lambda_n\right\}, \ n \in \mathbb{N}.$$

For such systems of functions we will say that they are *regular* systems.

Also, for a fixed $n \in \mathbb{N}$, let X_n be a set of different points in $[a, b]$, i.e., $X_n = \{x_0, x_1, \ldots, x_n\}$.

We introduce now the so-called *generalized divided difference operator of order k with respect to the system* Λ:

Definition 2.1. The generalized divided difference operator of order k (≥ 1) with respect to the system Λ at the points $x_0, \ldots, x_{k-1}, x$ ($x \in [a, b]$ and different from other points) is given by

$$[x_0, \ldots, x_{k-1}, x]_{\Lambda_k} f = \frac{[x_0, \ldots, x_{k-2}, x]_{\Lambda_{k-1}} f - [x_0, \ldots, x_{k-1}]_{\Lambda_{k-1}} f}{\lambda_k(x) - \lambda_k(x_{k-1})}, \tag{7}$$

where $[x]_\emptyset f = f(x)$.

The formula (7) can be used recursively for computing the generalized divided differences. For example, for $n = 4$ the computation is arranged in Table 1, where $[x_i]_\emptyset f = f(x_i)$, $i = 0, 1, 2, 3, 4$.

Table 1: The generalized divided differences for $n = 4$

x_0	$f(x_0)$	$[x_0, x_1]_{\Lambda_1} f$	$[x_0, x_1, x_2]_{\Lambda_2} f$	$[x_0, x_1, x_2, x_3]_{\Lambda_3} f$	$[x_0, x_1, x_2, x_3, x_4]_{\Lambda_4} f$
x_1	$f(x_1)$	$[x_0, x_2]_{\Lambda_1} f$	$[x_0, x_1, x_3]_{\Lambda_2} f$	$[x_0, x_1, x_2, x_4]_{\Lambda_3} f$	
x_2	$f(x_2)$	$[x_0, x_3]_{\Lambda_1} f$	$[x_0, x_1, x_4]_{\Lambda_2} f$		
x_3	$f(x_3)$	$[x_0, x_4]_{\Lambda_1} f$			
x_4	$f(x_4)$				

Remark 2.2. According to (3) the standard divided differences are symmetric functions of their arguments. However, for the generalized divided differences it holds only for ones of order one, i.e.,

$$[x_i, x_j]_{\Lambda_1} f = \frac{f(x_j) - f(x_i)}{\lambda_1(x_j) - \lambda_1(x_i)} = \frac{f(x_i) - f(x_j)}{\lambda_1(x_i) - \lambda_1(x_j)} = [x_j, x_i]_{\Lambda_1} f.$$

In general, the the generalized divided differences of order k are symmetric only with respect to two last arguments, i.e.,

$$[x_i, x_{i+1}, \ldots, x_{i+k-2}, x_{i+k-1}, x_{i+k}]_{\Lambda_k} f = [x_i, x_{i+1}, \ldots, x_{i+k-2}, x_{i+k}, x_{i+k-1}]_{\Lambda_k} f.$$

3. Generalized interpolation formula

First we introduce the so-called *generalized Newton functions* N_k (w.r.t. the system Λ), $k = 0, 1, \ldots, n$, as

$$N_0(x) = 1, \quad N_k(x) = \prod_{\nu=1}^{k} \Big(\lambda_\nu(x) - \lambda_\nu(x_{\nu-1})\Big), \quad 1 \le k \le n, \tag{8}$$

which are linearly independent. Note that $N_{k+1}(x) = N_k(x)\Big(\lambda_{k+1}(x) - \lambda_{k+1}(x_k)\Big)$, as well as that for $k \ge 1$ we have

$$N_k(x_j) = 0 \quad \text{for } j \le k-1 \quad \text{and} \quad N_k(x_j) \ne 0 \quad \text{for } j \ge k.$$

Using these generalized Newton functions as a basis $\{N_k(x)\}_{k=0}^n$, we generate an $(n+1)$-dimensional linear subspace $\Pi_n = \text{span}\{N_0, N_1, \ldots, N_n\}$ of $C[a, b]$ and consider its element

$$G_n(x; f; \Lambda) = \sum_{k=0}^{n} d_k N_k(x), \tag{9}$$

for which the following *interpolation conditions* at $n+1$ given points (*interpolation nodes*),

$$G_n(x_\nu; f; \Lambda) = f(x_\nu), \quad \nu = 0, 1, \ldots, n, \tag{10}$$

are satisfied.

Introducing $(n+1)$-dimensional vectors

$$\mathbf{d} = [d_0 \;\; d_1 \;\; \cdots \;\; d_n]^T \quad \text{and} \quad \mathbf{f} = [f(x_0) \;\; f(x_1) \;\; \cdots \;\; f(x_n)]^T,$$

as well as the matrix

$$\mathbf{L} = \begin{bmatrix} N_0(x_0) & 0 & \cdots & 0 \\ N_0(x_1) & N_1(x_1) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ N_0(x_n) & N_1(x_n) & \cdots & N_n(x_n) \end{bmatrix},$$

of order $n+1$, it is easy to see that *the interpolation function* (9) *exists uniquely*, because of

$$\det \mathbf{L} = \prod_{k=0}^{n} N_k(x_k) = \prod_{k=1}^{n} \prod_{\nu=1}^{k} \Big(\lambda_\nu(x_k) - \lambda_\nu(x_{\nu-1})\Big) \ne 0.$$

Denote by

$$G_n^*(x; f; \Lambda) = \sum_{k=0}^{n} d_k^* N_k(x), \tag{11}$$

the unique interpolation function for the conditions (10). Evidently,

$$[d_0^* \ d_1^* \ \cdots \ d_n^*]^T = \mathbf{L}^{-1}\mathbf{f}.$$

In order to find an appropriate representation of (11) we suppose that an expansion of the form

$$f(x) = \sum_{k=0}^{n} d_k^* N_k(x) + A_{n+1}(x) N_{n+1}(x), \tag{12}$$

holds, where $N_{n+1}(x)$ is defined in the same way as in (8), i.e.,

$$N_{n+1}(x) = \big(\lambda_1(x) - \lambda_1(x_0)\big)\big(\lambda_2(x) - \lambda_2(x_1)\big)\cdots\big(\lambda_{n+1}(x) - \lambda_{n+1}(x_n)\big),$$

and it reduces to zero at the interpolation nodes $x_\nu \in X_n$. Because of the interpolation conditions (10), the term $A_{n+1}(x)$ in (12) is the coefficient of the remainder term of our interpolation formula (11) at an arbitrary point $x \in [a, b]$. It is clear that for $f \in \Pi_n$ the formula (12) holds with $A_{n+1}(x) \equiv 0$. In the sequel, we show that (12) holds in general.

Setting $x = x_0$ in (12), evidently we have $d_0^* = f(x_0)$, so that (12) becomes

$$\begin{aligned}[x_0, x]_{\Lambda_1} f = d_1^* + d_2^*\big(\lambda_2(x) - \lambda_2(x_1)\big) + \cdots + d_n^*\big(\lambda_2(x) - \lambda_2(x_1)\big)\cdots\big(\lambda_n(x) - \lambda_n(x_{n-1})\big)\\ + A_{n+1}(x)\big(\lambda_2(x) - \lambda_2(x_1)\big)\cdots\big(\lambda_{n+1}(x) - \lambda_{n+1}(x_n)\big).\end{aligned}$$

Now, for $x = x_1$ we get $d_1^* = [x_0, x_1]_{\Lambda_1} f$.

Continuing this process we obtain $d_2^* = [x_0, x_1, x_2]_{\Lambda_2} f$. In general, by induction, we can prove that the coefficients d_k^* in (11) can be expressed in terms of generalized divided differences,

$$d_k^* = [x_0, x_1, \ldots, x_k]_{\Lambda_k} f, \qquad k = 0, 1, \ldots, n.$$

Finally, for the coefficient of the remainder term we obtain

$$A_{n+1}(x) = [x_0, x_1, \ldots, x_n, x]_{\Lambda_{n+1}} f.$$

Therefore, for the interpolation formula (11) we get the following form

$$G_n^*(x; f; \Lambda) = \sum_{k=0}^{n} N_k(x)\, [x_0, x_1, \ldots, x_k]_{\Lambda_k} f. \tag{13}$$

Thus, we have proved the following theorem.

Theorem 3.1. *Given a regular system of (monotonous continuous) functions $\Lambda = \{\lambda_\nu(x)\}_{\nu=1}^n$ on $[a, b]$ and $n + 1$ distinct points $\{x_k\}_{k=0}^n$ in $[a, b]$. Suppose that $G_n^*(x; f; \Lambda)$ is the interpolating function which interpolates a given function f at $\{x_k\}_{k=0}^n$. If x is an arbitrary point in $[a, b]$, then*

$$f(x) - G_n^*(x; f; \Lambda) = N_{n+1}(x)\, [x_0, x_1, \ldots, x_n, x]_{\Lambda_{n+1}} f. \tag{14}$$

In the sequel, we consider the remainder term (14), given by Theorem 3.1. It is well known that in the case of standard Newton (polynomial) interpolation for functions $f \in C^{n+1}[a, b]$, the remainder term can be expressed in the form

$$\frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{k=0}^{n} (x - x_k), \quad \xi \in (a, b), \tag{15}$$

because of (cf. [6, p. 361], [8, p. 97])

$$[x_0, x_1, \dots, x_n, x]f = \frac{f^{(n+1)}(\xi)}{(n+1)!}, \tag{16}$$

where $\xi = \xi(x)$ is strictly between the smallest and the largest of these points $x_0, x_1, \dots, x_n, x$. Unfortunately, such a simple expression for (14) is not possible. Following the well known approach due to Cauchy, we introduce an auxiliary function in a new variable z as

$$Q(z) = f(z) - G_n^*(z; f; \Lambda) - \gamma(x) \prod_{k=0}^{n} (z - x_k),$$

where $\gamma(x)$ is defined in such a way that $Q(z) = 0$ for an arbitrary fixed point $z = x \neq \{x_k\}_{k=0}^n$ in $[a, b]$. In this case, we have

$$\gamma(x) = \frac{f(x) - G_n^*(x; f; \Lambda)}{\prod\limits_{k=0}^{n} (x - x_k)}. \tag{17}$$

Beside this zero at $z = x$, according to the interpolation conditions (10), it is clear that $Q(z) = 0$ also for different points (interpolation nodes) $z = x_0, x_1, \dots, x_n$. Thus, for $f \in C^{n+1}[a, b]$, the function $z \mapsto Q(z)$ ($Q \in C^{n+1}[a, b]$) has at least $n + 2$ zeros in $[a, b]$, and after applying Rolle's theorem successively $n + 1$ times we can conclude that there exists a point $\xi = \xi(x) \in (a, b)$ such that $Q^{(n+1)}(\xi) = 0$, i.e.,

$$f^{(n+1)}(\xi) - \frac{\mathrm{d}^{n+1}}{\mathrm{d}z^{n+1}} G_n^*(z; f; \Lambda) \Big|_{z=\xi} = \gamma(x)(n+1)!\,.$$

Regarding (17), this means that for any $x \in [a, b]$

$$f(x) - G_n^*(x; f; \Lambda) = \frac{1}{(n+1)!} \left[f^{(n+1)}(\xi) - G_n^{*\,(n+1)}(\xi; f; \Lambda) \right] \prod_{k=0}^{n} (x - x_k).$$

Also, according to (14) we have

$$[x_0, x_1, \dots, x_n, x]_{\Lambda_{n+1}} f = \frac{1}{(n+1)!} \left[f^{(n+1)}(\xi) - G_n^{*\,(n+1)}(\xi; f; \Lambda) \right] \prod_{k=0}^{n} \frac{x - x_k}{\lambda_{k+1}(x) - \lambda_{k+1}(x_k)}\,. \tag{18}$$

Remark 3.2. Let $\lambda_\nu(x) = \lambda(x)$ for any $\nu = 1, \dots, n$, where $x \mapsto \lambda(x)$ is a monotonous continuous function on $[a, b]$. Then (13) interpolates f at distinct points $\{x_k\}_{k=0}^n$, where

$$N_0(x) = 1, \quad N_k(x) = \prod_{\nu=1}^{k} \big(\lambda(x) - \lambda(x_{\nu-1}) \big), \quad 1 \leq k \leq n.$$

In this case

$$\operatorname{span}\Big\{N_k(x)\Big\}_{k=0}^n \equiv \operatorname{span}\Big\{1, \lambda(x), \lambda(x)^2, \dots, \lambda(x)^n\Big\},$$

and $G_n^*(x; f; \Lambda)$ can be considered as

$$G_n^*(x; f; \Lambda) = A_0 + A_1 \lambda(x) + A_2 \lambda(x)^2 + \cdots + A_n \lambda(x)^n. \tag{19}$$

For example, if $\lambda(x) = \mathrm{e}^x$ we have the so-called *exponential interpolation* (cf. [15])

$$G_n^*(x; f; \Lambda) = A_0 + A_1 \mathrm{e}^x + A_2 \mathrm{e}^{2x} + \cdots + A_n \mathrm{e}^{nx}.$$

The coefficients $\{A_k\}_{k=0}^n$ in (19) are determined by the given interpolation conditions. Also, the interpolating function can be represented as

$$G_n^*(x;f;\Lambda)=\sum_{k=0}^n f(x_k)\Phi_n(x;x_k;\Lambda), \tag{20}$$

in which

$$\Phi_n(x;x_k;\Lambda)=\prod_{\substack{j=0\\ j\neq k}}^n \frac{\lambda(x)-\lambda(x_j)}{\lambda(x_k)-\lambda(x_j)},$$

satisfies the biorthogonality relation

$$\Phi_n(x_j;x_k;\Lambda)=\delta_{j,k}=\begin{cases}1 & (k=j),\\ 0 & (k\neq j).\end{cases}$$

In fact, (20) is the generalized Lagrange representation for interpolating function (19) and (13) is its generalized Newton representation.

Also, if $\lambda_\nu(x)=\lambda(x)$ for any $\nu=1,\ldots,n$, then according to (18) we obtain

$$[x_0,x_1,\ldots,x_n,x]_{\Lambda_{n+1}}f=\frac{1}{(n+1)!}\Big[f^{(n+1)}(\xi_x)-\sum_{k=0}^n f(x_k)\,\Phi_n^{(n+1)}(\xi_x;x_k;\Lambda)\Big]\prod_{k=0}^n\frac{x-x_k}{\lambda(x)-\lambda(x_k)},$$

and if $\lambda(x)=x$, then $G_n^*(x;f;\Lambda)$ is a polynomial of degree at most n interpolating f at distinct points $\{x_k\}_{k=0}^n$ and we obtain (16) and the remainder takes the form (15).

Remark 3.3. Since $N_k(x)$ depends only on $x_0,\ldots,x_{k-1}$ and the sequence of functions Λ_k, if another interpolatory point is added, we can achieve the interpolating function by adding only one more term, i.e., $d_{k+1}N_{k+1}(x)$.

Remark 3.4. Let $\Lambda_{n,1}=\{\lambda_\nu(x)\}_{\nu=1}^n$ be a system of monotonous continuous functions defined on $[a,b]$ and $\Lambda_{n,2}=\{a_\nu\lambda_\nu(x)+b_\nu\}_{\nu=1}^n$, where a_ν $(\neq 0)$ and b_ν are real numbers for every ν. It can be verified that for a given system of nodes on $[a,b]$, these systems of functions, $\Lambda_{n,1}$ and $\Lambda_{n,2}$, lead to the same interpolation function.

Remark 3.5. In a special case, for given fixed system of distinct points $\{x_k\}_{k=0}^n\in[a,b]$, Theorem 3.1 holds under certain weaker conditions. Namely, it is not necessary for $\{\lambda_\nu(x)\}$ to be just a regular system of functions, one can choose any arbitrary set of continuous functions $\Lambda_n=\{\lambda_\nu(x)\}_{\nu=1}^n$ provided that $\lambda_\nu(x_i)\neq\lambda_\nu(x_j)$ for each $i\neq j$.

3.1. Some particular examples of formula (13)

Example 3.6. Let

$$\lambda_l(x)=\frac{1}{x-\beta_l}\qquad\text{for}\qquad l=1,\ldots,n,$$

where $\beta_1<\beta_2<\cdots<\beta_n$ are real values. For given distinct points $\{x_k\}_{k=0}^n$ in (β_n,∞) first we have

$$N_k(x)=\prod_{j=1}^k\left(\frac{1}{x-\beta_j}-\frac{1}{x_{j-1}-\beta_j}\right)=\prod_{j=1}^k\frac{1}{\beta_j-x_{j-1}}\cdot\frac{x-x_{j-1}}{x-\beta_j},\qquad\text{with}\qquad N_0(x)=1.$$

Therefore, the unique solution of the interpolation problem

$$f(x_k) = w_k \qquad \text{for} \qquad k = 0, 1, \ldots, n, \tag{21}$$

is

$$G_n^*\left(x; f; \left\{\frac{1}{x-\beta_l}\right\}_{l=1}^{n}\right) = \sum_{k=0}^{n} N_k(x)[x_0, \ldots, x_k]_{\Lambda_k} f.$$

Example 3.7. Given $n+1$ distinct points $-\pi/2 \le x_0 < \cdots < x_n \le \pi/2$, assume that

$$\lambda_\nu(x) = \sin\frac{x}{\nu} \qquad \text{for} \qquad \nu = 1, \ldots, n,$$

and then construct the functions

$$N_k(x) = \prod_{\nu=1}^{k}\left(\sin\frac{x}{\nu} - \sin\frac{x_{\nu-1}}{\nu}\right) \qquad \text{with} \qquad N_0(x) = 1.$$

The functions $\lambda_\nu(x)$, $\nu = 1, \ldots, n$, for $n = 4$ are displayed in Fig. 1.

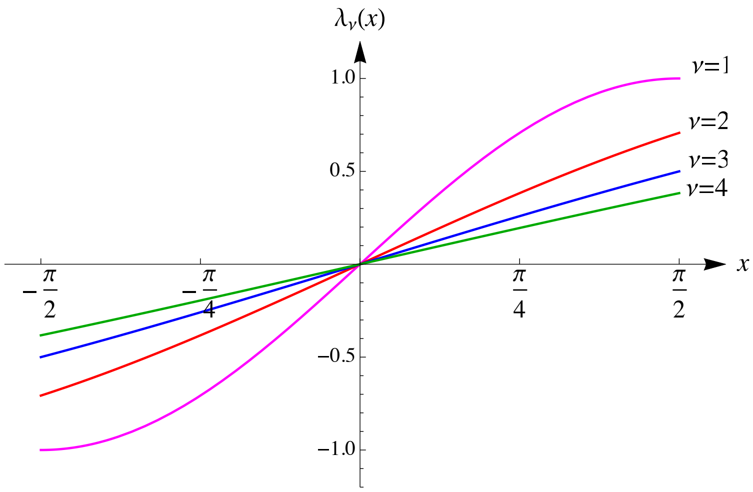

Figure 1: The functions $\lambda_\nu(x)$, $\nu = 1, \ldots, n$, in Example 3.7 for $n = 4$

Thus, the final interpolating function is given by

$$G_n^*\left(x; f; \left\{\sin\frac{x}{\nu}\right\}_{\nu=1}^{n}\right) = \sum_{k=0}^{n} N_k(x)[x_0, \ldots, x_k]_{\Lambda_k} f.$$

Example 3.8. Given $n+1$ distinct points $0 \le x_0 < \cdots < x_n < \pi$, assume that $\Lambda_n = \{\lambda_\nu(x) = \cos x\}_{\nu=1}^{n}$ and construct the functions

$$N_k(x) = \prod_{\nu=1}^{k}(\cos x - \cos x_{\nu-1}) \qquad \text{with} \qquad N_0(x) = 1.$$

The sequence $N_k(x)$ is a cosine polynomial of order k, which can be represented as $\sum_{j=0}^{k} \alpha_{j,k} \cos jx$. Now, given $n+1$ distinct values $\{w_k\}_{k=0}^{n}$, there exists a cosine polynomial of order $\le n$, which is the unique solution of the interpolation problem (21) and can be given in the form

$$G_n^*\Big(x; f, \{\cos x\}_{l=1}^n\Big) = \sum_{k=0}^n [x_0, \ldots, x_k]_{\Lambda_k} f \sum_{j=0}^k \alpha_{j,k} \cos jx = \sum_{j=0}^n \beta_j \cos jx,$$

where

$$\beta_0 = f(x_0) + \sum_{l=1}^n \alpha_{0,l}\, [x_0, \ldots, x_l]_{\Lambda_l} f \quad \text{and} \quad \beta_j = \sum_{l=j}^n \alpha_{j,l}\, [x_0, \ldots, x_l]_{\Lambda_l} f, \quad j = 1, \ldots, n.$$

4. Interpolation formula (13) at coincident points

In formulating the interpolation (13), we assumed that $\{x_k\}_{k=0}^n$ are distinct. Now, assume that $\{x_j\}_{j=1}^n$ coincide with x_0. The interpolating function in (13) therefore changes to

$$G_n^*(x; f; \Lambda) = f(x_0) + \sum_{k=1}^n \Big(\underbrace{[x_0, \ldots, x_0]}_{k+1}{}_{\Lambda_k} f \prod_{\nu=1}^k \big(\lambda_\nu(x) - \lambda_\nu(x_0)\big)\Big). \tag{22}$$

Relation (22) shows that we should evaluate the generalized divided differences in which all arguments are identical. There are different ways for this purpose. For instance, in [1] ordinary divided differences with repetitions were studied by a recursive definition, in [21] by a specific definition with determinants and in [5] by simply extending the definition of divided differences in the case of distinct arguments.

If we follow the approach given in [5], according to the definition of generalized divided difference of order one, we obtain

$$\lim_{x \to x_0} [x_0, x]_{\Lambda_1} f = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{\lambda_1(x) - \lambda_1(x_0)} = \frac{f'(x_0)}{\lambda_1'(x_0)}.$$

Also, for divided differences of order two and three we respectively obtain

$$\begin{aligned} \lim_{x \to x_0} [x_0, x_0, x]_{\Lambda_2} f &= \lim_{x \to x_0} \frac{[x_0, x]_{\Lambda_1} f - [x_0, x_0]_{\Lambda_1} f}{\lambda_2(x) - \lambda_2(x_0)} \\ &= \frac{1}{\lambda_2'(x_0)} \lim_{x \to x_0} \frac{\mathrm{d}}{\mathrm{d}x} [x_0, x]_{\Lambda_1} f \\ &= \frac{1}{2\lambda_1'(x_0)\lambda_2'(x_0)} \left(f''(x_0) - \frac{\lambda_1''(x_0)}{\lambda_1'(x_0)} f'(x_0) \right), \end{aligned}$$

and

$$\begin{aligned} \lim_{x \to x_0} [x_0, x_0, x_0, x]_{\Lambda_3} f &= \lim_{x \to x_0} \frac{[x_0, x_0, x]_{\Lambda_2} f - [x_0, x_0, x_0]_{\Lambda_2} f}{\lambda_3(x) - \lambda_3(x_0)} \\ &= \frac{1}{\lambda_3'(x_0)} \lim_{x \to x_0} \frac{\mathrm{d}}{\mathrm{d}x} [x_0, x_0, x]_{\Lambda_2} f \\ &= \frac{1}{\lambda_3'(x_0)} \lim_{x \to x_0} \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{[x_0, x]_{\Lambda_1} f - [x_0, x_0]_{\Lambda_1} f}{\lambda_2(x) - \lambda_2(x_0)} \right) \\ &= \frac{1}{6\lambda_1'(x_0)\lambda_2'(x_0)\lambda_3'(x_0)} \left\{ f'''(x_0) - \frac{3}{2} \left(\frac{\lambda_1''(x_0)}{\lambda_1'(x_0)} + \frac{\lambda_2''(x_0)}{\lambda_2'(x_0)} \right) f''(x_0) \right. \\ &\qquad \left. + \left(\frac{3}{2} \frac{(\lambda_1''(x_0))^2}{(\lambda_1'(x_0))^2} + \frac{3}{2} \frac{\lambda_1''(x_0)\lambda_2''(x_0)}{\lambda_1'(x_0)\lambda_2'(x_0)} - \frac{\lambda_1'''(x_0)}{\lambda_1'(x_0)} \right) f'(x_0) \right\}, \end{aligned}$$

provided that $\lambda'_\nu(x_0) \neq 0$ for $\nu = 1, 2, 3$.

In general, $[\underbrace{x_0, \ldots, x_0}_{n+1}]_{\Lambda_n} f$ can be explicitly computed according to the following theorem.

Theorem 4.1. *Given a regular system of functions $\Lambda = \{\lambda_\nu(x)\}_{\nu=1}^n$ on $[a,b]$ and the point $x_0 \in [a,b]$ with corresponding values $\{f^{(k)}(x_0)\}_{k=0}^n$, where $\lambda_\nu(x) \neq \lambda_\nu(x_0)$, $\forall\, x \neq x_0$. Let $f, \lambda_1, \ldots, \lambda_n \in C^n[a,b]$ and $f^{(n+1)}, \lambda_1^{(n+1)}, \ldots, \lambda_n^{(n+1)}$ exist on (a,b) and $\lambda_\nu^{(j)}(x_0) \neq 0$ for any ν, $j = 1, \ldots, n$. Consider the functions $\{N_k(x)\}_{k=0}^n$ when all the points coincide with x_0 as*

$$N_k(x) = \prod_{\nu=1}^{k} \Big(\lambda_\nu(x) - \lambda_\nu(x_0)\Big) \quad \textit{for any} \quad k = 1, 2, \ldots, n \quad \textit{and} \quad N_0(x) = 1.$$

Then, there exists a unique function $T_n(x; f; \Lambda) \in \operatorname{span}\{N_k(x)\}_{k=0}^n$ such that

$$T_n^{(k)}(x_0; f; \Lambda) = f^{(k)}(x_0) \qquad k = 0, 1, \ldots, n.$$

This function can be explicitly represented as follows

$$T_n(x; f; \Lambda) = f(x_0) + \sum_{m=1}^{n} (-1)^{m-1} \frac{D_{(m-1,m)}}{G_{(m)}} N_m(x),$$

where $D_{(0,k)} = f^{(k)}(x_0)$, $k \geq 1$, and

$$D_{(m,k)} = \begin{vmatrix} f'(x_0) & N_1'(x_0) & 0 & 0 & \ldots & 0 \\ f''(x_0) & N_1''(x_0) & N_2''(x_0) & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ f^{(m-1)}(x_0) & N_1^{(m-1)}(x_0) & N_2^{(m-1)}(x_0) & N_3^{(m-1)}(x_0) & \ldots & 0 \\ f^{(m)}(x_0) & N_1^{(m)}(x_0) & N_2^{(m)}(x_0) & N_3^{(m)}(x_0) & \ldots & N_m^{(m)}(x_0) \\ f^{(k)}(x_0) & N_1^{(k)}(x_0) & N_2^{(k)}(x_0) & N_3^{(k)}(x_0) & \ldots & N_m^{(k)}(x_0) \end{vmatrix}, \quad k > m \geq 1, \tag{23}$$

and finally

$$G_{(m)} = \prod_{\nu=1}^{m} N_\nu^{(\nu)}(x_0) \quad \textit{for} \quad m \geq 1.$$

Proof. Consider the interpolation problem (6) in the space $\Pi = \operatorname{span}\{N_k(x)\}_{k=0}^n$, where we have defined

$$L_k(f) = f^{(k)}(x_0), \qquad k = 0, 1, \ldots, n. \tag{24}$$

We observe that

$$\big|L_i(N_j)\big|_{i,j=0}^n = \begin{vmatrix} N_0(x_0) & 0 & 0 & \ldots & 0 \\ N_0'(x_0) & N_1'(x_0) & 0 & \ldots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ N_0^{(n-1)}(x_0) & N_1^{(n-1)}(x_0) & N_2^{(n-1)}(x_0) & \ldots & 0 \\ N_0^{(n)}(x_0) & N_1^{(n)}(x_0) & N_2^{(n)}(x_0) & \ldots & N_n^{(n)}(x_0) \end{vmatrix} = \prod_{k=0}^{n} N_k^{(k)}(x_0) \neq 0,$$

because

$$N_k^{(k)}(x_0) = k! \prod_{\nu=1}^{k} \lambda'_\nu(x_0) \neq 0 \quad \text{for} \quad k = 1, \dots, n \quad \text{and} \quad N_0(x_0) = 1.$$

Therefore, according to Theorem 2.2.2 of [7, p. 26], the interpolation problem (24) possesses a unique solution in Π as

$$T_n(x; f; \Lambda) = -\frac{1}{\left|L_i(N_j)\right|_{i,j=0}^{n}} \begin{vmatrix} 0 & N_0(x) & N_1(x) & \dots & N_n(x) \\ f(x_0) & N_0(x_0) & N_1(x_0) & \dots & N_n(x_0) \\ f'(x_0) & N'_0(x_0) & N'_1(x_0) & \dots & N'_n(x_0) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f^{(n)}(x_0) & N_0^{(n)}(x_0) & N_1^{(n)}(x_0) & \dots & N_n^{(n)}(x_0) \end{vmatrix}. \tag{25}$$

Since $N_0(x_0) = 1$ and $N_j(x_0) = 0$ for $j = 1, \dots, n$, we have

$$N_j^{(k)}(x_0) = 0 \qquad \text{for} \qquad j = k+1, k+2, \dots, n.$$

Therefore, (25) is simplified as

$$T_n(x; f; \Lambda) = -\frac{1}{\left|L_i(N_j)\right|_{i,j=0}^{n}} \begin{vmatrix} 0 & N_0(x) & N_1(x) & \dots & N_n(x) \\ f(x_0) & 1 & 0 & \dots & 0 \\ f'(x_0) & 0 & N'_1(x_0) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f^{(n)}(x_0) & 0 & N_1^{(n)}(x_0) & \dots & N_n^{(n)}(x_0) \end{vmatrix}. \tag{26}$$

Now, as a direct result of uniqueness of the interpolation solution, we must have

$$T_n(x; f; \Lambda) = f(x_0) + \sum_{k=1}^{n} N_k(x) [\underbrace{x_0, \dots, x_0}_{k+1}]_{\Lambda_k} f, \tag{27}$$

which is deduced from (22). On the other hand, it can be verified that the coefficient of $\prod_{\nu=1}^{n} \lambda_\nu(x)$ in (27) is $[\underbrace{x_0, \dots, x_0}_{n+1}]_{\Lambda_n} f$ and in (26) appeares as the coefficient of $N_n(x)$ when one expands the deteminant by minors of the first row. Hence, the coefficient is equal to

$$-\frac{1}{\left|L_i(N_j)\right|_{i,j=0}^{n}} (-1)^{n-1} \begin{vmatrix} f(x_0) & 1 & 0 & \dots & 0 \\ f'(x_0) & 0 & N'_1(x_0) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f^{(n-1)}(x_0) & 0 & N_1^{(n-1)}(x_0) & \dots & N_{n-1}^{(n-1)}(x_0) \\ f^{(n)}(x_0) & 0 & N_1^{(n)}(x_0) & \dots & N_{n-1}^{(n)}(x_0) \end{vmatrix},$$

or equivalently

$$\frac{(-1)^{n-1}}{\left|L_i(N_j)\right|_{i,j=0}^{n}} \begin{vmatrix} f'(x_0) & N'_1(x_0) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f^{(n-1)}(x_0) & N_1^{(n-1)}(x_0) & \dots & N_{n-1}^{(n-1)}(x_0) \\ f^{(n)}(x_0) & N_1^{(n)}(x_0) & \dots & N_{n-1}^{(n)}(x_0) \end{vmatrix}.$$

Also notice that $\left|L_i(N_j)\right|_{i,j=0}^{n}$ equals to $G_{(n)}$. Consequently, the above relations yield

$$\underbrace{[x_0,\ldots,x_0]}_{n+1}{}_{\Lambda_n} f = \frac{(-1)^{n-1}}{G_{(n)}} D_{(n-1,n)},$$

where $D_{(n-1,n)}$ is defined in (23). $\square$

In what follows, there are some useful remarks in order to compute $D_{(m,k)}$ and $N_j^{(k)}(x_0)$ to obtain the generalized Taylor interpolating function $T_n(x;f;\Lambda)$.

Remark 4.2. Expanding $D_{(m,k)}$ by minors of the last column yields a recursive relation as

$$D_{(m,k)} = N_m^{(k)}(x_0)D_{(m-1,m)} - N_m^{(m)}(x_0)D_{(m-1,k)}, \qquad m \geq 1, \tag{28}$$

and for $k = m + 1$ we get

$$D_{(m,m+1)} = N_m^{(m+1)}(x_0)D_{(m-1,m)} - N_m^{(m)}(x_0)D_{(m-1,m+1)}, \qquad m \geq 1. \tag{29}$$

Remark 4.3. One needs to compute $N_m^{(m)}(x_0)$ and $N_m^{(m+1)}(x_0)$ to obtain $D_{(m,m+1)}$, which can be respectively obtained by the following relations

$$N_m^{(m)}(x_0) = m! \prod_{\nu=1}^{m} \lambda_\nu'(x_0),$$

and

$$N_m^{(m+1)}(x) = \Big(N_{m-1}(x)\big(\lambda_m(x) - \lambda_m(x_0)\big)\Big)^{(m+1)} = \sum_{k=0}^{m+1} \binom{m+1}{k} N_{m-1}^{(k)}(x)\lambda_m^{(m+1-k)}(x).$$

Now, setting $x = x_0$ in the above relation gives

$$\begin{aligned} N_m^{(m+1)}(x_0) &= \binom{m+1}{m-1} N_{m-1}^{(m-1)}(x_0)\lambda_m''(x_0) + \binom{m+1}{m} N_{m-1}^{(m)}(x_0)\lambda_m'(x_0) \\ &= \frac{m(m+1)}{2}(m-1)! \prod_{j=1}^{m-1} \lambda_j'(x_0)\lambda_m''(x_0) + (m+1)N_{m-1}^{(m)}(x_0)\lambda_m'(x_0) \\ &= \frac{1}{2}(m+1)!\, \lambda_m''(x_0) \prod_{j=1}^{m-1} \lambda_j'(x_0) + (m+1)\lambda_m'(x_0)N_{m-1}^{(m)}(x_0), \qquad m \geq 1, \end{aligned}$$

where we have supposed $\prod_{j=1}^{0}(\,\cdot\,) = 1$. For instance, we have

$$N_1''(x_0) = \lambda_1''(x_0),$$

$$N_2'''(x_0) = 3\big(\lambda_1'(x_0)\lambda_2''(x_0) + \lambda_1''(x_0)\lambda_2'(x_0)\big),$$

and

$$N_3^{(4)}(x_0) = 12\big(\lambda_1'(x_0)\lambda_2'(x_0)\lambda_3''(x_0) + \lambda_1'(x_0)\lambda_2''(x_0)\lambda_3'(x_0) + \lambda_1''(x_0)\lambda_2'(x_0)\lambda_3'(x_0)\big).$$

Remark 4.4. Applying the recursive relation (28) j times ($j \geq 2$) for $D_{(m,k)}$ leads to the following result

$$
\begin{aligned}
D_{(m,k)} &= (-1)^j \prod_{l=m-j+1}^{m} N_l^{(l)}(x_0)\, D_{(m-j,k)} + (-1)^{j+1} N_m^{(k)}(x_0) \prod_{l=m-j+1}^{m-1} N_l^{(l)}(x_0)\, D_{(m-j,m)} \\
&\quad + \sum_{r=1}^{j-1} \left((-1)^{j-r+1} M_{r+1} \prod_{l=r}^{j-2} N_{m-l-1}^{(m-l-1)}(x_0) \right) D_{(m-j,m-r)},
\end{aligned}
$$

in which we have supposed $\prod_{l=r}^{j-2}(\,\cdot\,) = 1$ when $r < j-2$ and M_r is the determinant of the $r \times r$ down and right submatrix of $D_{(m,k)}$.

Subsequently, for $j = m$ and $k = m+1$ we get

$$
\begin{aligned}
D_{(m,m+1)} &= (-1)^m \prod_{l=1}^{m} N_l^{(l)}(x_0)\, D_{(0,m+1)} + (-1)^{m+1} N_m^{(m+1)}(x_0) \prod_{l=1}^{m-1} N_l^{(l)}(x_0)\, D_{(0,m)} \\
&\quad + \sum_{r=1}^{m-1} \left((-1)^{m-r+1} M_{r+1} \prod_{l=r}^{m-2} N_{m-l-1}^{(m-l-1)}(x_0) \right) D_{(0,m-r)} \\
&= (-1)^m G_{(m)} f^{(m+1)}(x_0) + (-1)^{m+1} N_m^{(m+1)}(x_0) G_{(m-1)} f^{(m)}(x_0) \\
&\quad + \sum_{r=1}^{m-1} \left((-1)^{m-r+1} M_{r+1} \prod_{l=r}^{m-2} N_{m-l-1}^{(m-l-1)}(x_0) \right) f^{(m-r)}(x_0), \qquad m \geq 2.
\end{aligned}
$$

The above result is another representation for $D_{(m,m+1)}$ which shows that the coefficient of basis functions $N_k(x)$ in $T_n(x; f; \Lambda)$ is a linear combination of $\{f^{(k)}\}_{k=0}^{n}$, as it is for the well-known polynomial Taylor interpolation.

Example 4.5. Let

$$
\lambda_\nu(x) = \log \frac{x}{\nu}, \quad x > 0, \qquad \text{for} \quad \nu = 1, \ldots, n.
$$

For $n = 4$, these functions are presented in Fig. 2.

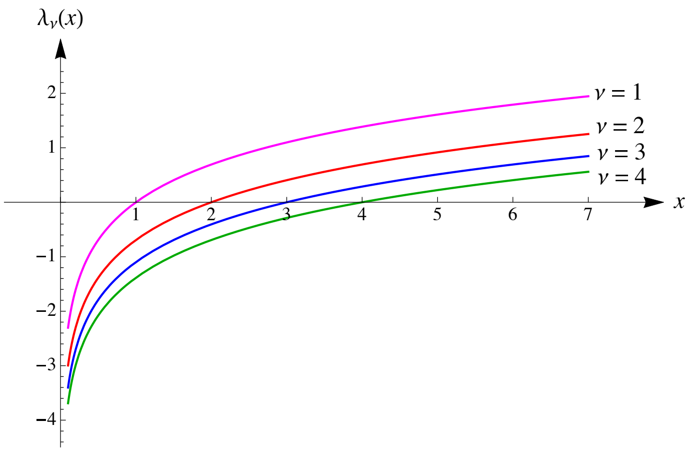

Figure 2: The functions $\lambda_\nu(x)$, $\nu = 1, \ldots, n$, in Example 4.5 for $n = 4$

For a given point $x_0 \in (0, \infty)$ we have

$$N_k(x) = \prod_{\nu=1}^{k} \Big(\lambda_\nu(x) - \lambda_\nu(x_0)\Big) = \left(\log \frac{x}{x_0}\right)^k \qquad \text{and} \qquad N_0(x) = 1.$$

Therefore, there exists a unique function in $\operatorname{span}\Big\{N_k(x)\Big\}_{k=0}^{n}$ which is the solution of the Taylor interpolation problem

$$f^{(k)}(x_0) = w_k \qquad\qquad k = 0, 1, \ldots, n,$$

and can be represented as

$$T_n\left(x; f; \left\{\log \frac{x}{\nu}\right\}_{\nu=1}^{n}\right) = f(x_0) + \sum_{k=1}^{n} (-1)^{k-1} \frac{D_{(k-1,k)}}{G_{(k)}} \left(\log \frac{x}{x_0}\right)^k,$$

in which

$$G_{(k)} = \prod_{\nu=1}^{k} N_\nu^{(\nu)}(x_0) = (x_0)^{-k(k+1)/2} \prod_{\nu=1}^{k} \nu!\,,$$

and $D_{(0,1)} = f'(x_0)$ and for $k \geq 2$, $D_{(k-1,k)}$ is computed by (29), where $N_{k-1}^{(k)}(x_0)$ and $N_{k-1}^{(k-1)}(x_0)$ are given respectively by

$$N_{k-1}^{(k)}(x_0) = -\frac{1}{2} k!\, x_0^{-k} + k\, x_0^{-1} N_{k-2}^{(k-1)}(x_0),$$

and

$$N_{k-1}^{(k-1)}(x_0) = (k-1)!\, x_0^{1-k}.$$

Remark 4.6. For linear functions $\lambda_\nu(x) = a_\nu x + b_\nu$, with $a_\nu \neq 0$, $\nu = 1, \ldots, n$, it is easy to prove that

$$T_n\Big(x; f; \{a_\nu x + b_\nu\}_{\nu=1}^{n}\Big) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k,$$

which is, as expected, the same as nth degree Taylor polynomial of f at x_0.

5. Numerical examples

In this section we give two examples.

Example 5.1. Let us choose the following sequence of functions on $[0, 1]$

$$\Lambda_8 = \left\{x^3 + x^2 + \frac{37x}{100},\ x^2 + 2x,\ x^2,\ x,\ \sqrt{x},\ 5x^2 + 3x,\ 2x^{2/5} + \sqrt{x},\ x^{9/4}\right\},$$

and interpolating nodes as Chebyshev points transformed to $[0, 1]$, i.e.,

$$\begin{aligned} \Big\{x_k\Big\}_{k=0}^{8} &= \left\{\frac{1}{2}\Big(1 + \cos \frac{(17-2k)\pi}{18}\Big)\right\}_{k=0}^{8} \\ &\cong \Big\{0.0076,\ 0.0670,\ 0.1786,\ 0.3290,\ 0.5000,\ 0.6710,\ 0.8214,\ 0.9330,\ 0.9924\Big\}. \end{aligned}$$

According to (13), the function

$$f(x) = \frac{\log(1+x)}{(1+x^2)^6} e^{x^2}, \quad x \in [0,1],$$

taken from [20] can be approximated by

$$f(x) \approx G_8^*(x; f; \Lambda_8) = \sum_{k=0}^{8} d_k^* N_k(x),$$

in which

$$N_k(x) = \prod_{\nu=1}^{k} \Big(\lambda_\nu(x) - \lambda_\nu(x_{\nu-1}) \Big) \quad \text{for} \quad k = 1, \ldots, 8, \quad \text{with} \quad N_0(x) = 1,$$

and $\left\{d_k^*\right\}_{k=0}^{8}$ are computed by the following generalized divided differences table:

0.00756	2.09100	−3.08052	9.70826	−19.03225	32.79670	−1.50797	0.87021	−0.07607
0.06340	1.31886	−2.33945	6.45355	−15.35753	30.42419	−1.37197	0.86051	
0.14052	0.62240	−1.67300	4.45567	−12.97172	28.83540	−1.30560		
0.17107	0.23138	−1.21647	3.32093	−11.56909	28.04790			
0.13647	0.07917	−0.94586	2.72026	−10.92394				
0.08656	0.02993	−0.79927	2.46116					
0.05336	0.01441	−0.73511						
0.03674	0.00972							
0.03018								

Indeed, the elements of the first row in the above table give $\{d_k^*\}_{k=0}^{8}$, respectively.

In this sense, the error norm

$$E_n = \max_{t_j} \left| f(t_j) - G_n^*(t_j; f; \Lambda) \right|,$$

is estimated as $E_8 = 3.71 \times 10^{-5}$ in which $\{t_j\}$ have been considered as 1000 equidistant points of $[0,1]$.

Alternatively, using the standard Newton interpolation $\left(\hat{\Lambda}_8 = \left\{\hat{\lambda}_\nu(x) = x\right\}_{\nu=1}^{8}\right)$, we obtain the following interpolation polynomial

$$\begin{aligned} P_8(x) &= -4.46538x^8 + 16.5961x^7 - 20.9518x^6 + 5.74091x^5 + 8.79967x^4 - 6.37325x^3 - 0.309325x^2 \\ &\quad + 0.992389x + 0.000047542, \end{aligned}$$

for which the norm of the corresponding error $e(x) = |f(x) - P_8(x)|$ can be estimated as

$$\overline{E}_8 = \max_{t_j} \left| f(t_j) - P_8(t_j) \right| = 6.45 \times 10^{-5},$$

taking again one thousand equidistant points in $[0,1]$.

According to these errors, we see that the approximation of $f(x)$ is better by $G_8^*(x; f; \Lambda_8)$ than one by $P_8(x)$. It is much clear from Fig. 3, where we present the absolute errors in the both of these cases.

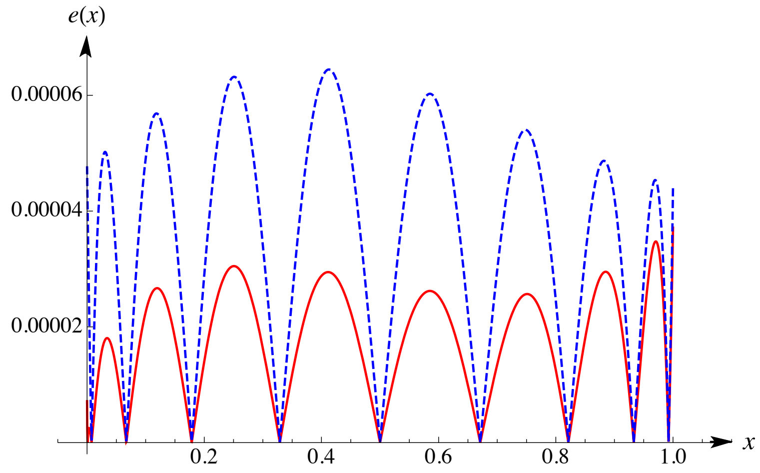

Figure 3: Absolute errors in approximation of $f(x)$ by $G_8^*(x; f; \Lambda_8)$ (red line) and $P_8(x)$ (blue line)

Quadrature rules with respect to Müntz polynomials have been considered in [16] (see also [17]).

Example 5.2. Let

$$\Lambda_4 = \{\lambda_\nu(x)\}_{\nu=1}^4 = \left\{x^3 - 3x\,,\ x+1\,,\ x^3 + 2\,x\,,\ x^3 - 6x - 3\right\}, \tag{30}$$

be a given regular system of functions on $[0, 1]$ and $n = 4$.

For four different systems of nodes in $[0, 1]$:

(a) $\left\{x_k\right\}_{k=0}^4 = \left\{\dfrac{1}{4}k\right\}_{k=0}^4 = \left\{0,\ 0.25,\ 0.5,\ 0.75,\ 1\right\}$;

(b) $\left\{x_k\right\}_{k=0}^4 = \left\{\dfrac{1}{10}(2k+1)\right\}_{k=0}^4 = \left\{0.1,\ 0.3,\ 0.5,\ 0.7,\ 0.9\right\}$;

(c) $\left\{x_k\right\}_{k=0}^4 = \left\{\dfrac{1}{2}\left(1 + \cos\dfrac{(9-2k)\pi}{10}\right)\right\}_{k=0}^4 = \left\{0.0244717,\ 0.206107,\ 0.5,\ 0.793893,\ 0.975528\right\}$;

(d) $\left\{x_k\right\}_{k=0}^4 = \left\{\dfrac{1}{2}\left(1 + x_k^L\right)\right\}_{k=0}^4 = \left\{0.0469101,\ 0.230765,\ 0.5,\ 0.769235,\ 0.95309\right\}$,

we construct the five-point interpolatory quadrature rules on $[0, 1]$ of the form

$$I(f) = \int_0^1 f(x)\,\mathrm{d}x = \sum_{k=0}^4 A_k^{[s]} f\left(x_k^{[s]}\right) + R_5^{[s]}(f) \qquad (\mathrm{s} = \mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}), \tag{31}$$

where each of these rules is constructed as an integral over $(0, 1)$ of the corresponding generalized interpolation function $G_4^*(x; f; \Lambda_4)$, obtained by the nodes (a), (b), (c), (d) as

$$Q_5^{[s]}(f) = \sum_{k=0}^4 A_k^{[s]} f\left(x_k^{[s]}\right) = \int_0^1 G_4^*(x; f; \Lambda_4)\,\mathrm{d}x = \sum_{k=0}^4 d_k^* \int_0^1 N_k(x)\,\mathrm{d}x,$$

i.e.,

$$Q_5^{[s]}(f) = \sum_{k=0}^4 \left(\int_0^1 N_k(x)\,\mathrm{d}x\right)[x_0, x_1, \ldots, x_k]_{\Lambda_k} f.$$

Here,

$$N_k(x) = \prod_{\nu=1}^{k}\Big(\lambda_\nu(x) - \lambda_\nu(x_{\nu-1})\Big), \quad k = 1,\dots,4, \quad \text{with} \quad N_0(x) = 1.$$

In the cases (c) and (d), the nodes x_k are zeros of the Chebyshev and the Legendre polynomial of degree five (transformed to $[0,1]$), respectively. Here, x_k^L are zeros of the Legendre polynomial

$$P_5(x) = \frac{1}{8}\Big(63x^5 - 70x^3 + 15x\Big).$$

For a simple function $f(x) = 1/(x+1)$, for which $I(f) = \log 2 \cong 0.693147180559945\dots$, we test the previous obtained quadrature rules for the nodes given by (a), (b), (c) and (d). In Table 2 we give the errors $|R_5^{[s]}(f)| = |I(f) - Q_5^{[s]}(f)|$ for s = a, b, c, d. Also, in the second row of this table we give the corresponding errors for the quadrature formulas $Q_5^N(f)$, obtained by using the standard Newton interpolation polynomial at given nodes. Numbers in parenthesis indicate the decimal exponents.

Table 2: Quadrature errors for $f(x) = 1/(x+1)$ in quadrature rules in Example 5.2

nodes	(a)	(b)	(c)	(d)
quadrature formula	Regular system of functions Λ_4 given by (30)			
$Q_5^{[s]}(f)$	2.11(−5)	8.79(−6)	1.24(−8)	2.58(−6)
$Q_5^N(f)$	2.74(−5)	1.92(−5)	2.27(−8)	3.78(−6)
	Irregular system of functions $\bar{\Lambda}_4$ given by (32)			
$Q_5^{[s]}(f)$	8.56(−6)	2.01(−8)	1.30(−8)	9.60(−7)

As we can see, for all distributions of nodes (a)–(d), the errors of the generalized quadrature formulas (31) are slightly smaller than ones in standard Newton's formulas $Q_5^N(f)$. Moreover, according to Remark 3.5 these results can be improved by using some of irregular systems of functions. For example, for

$$\bar{\Lambda}_4 = \{\lambda_\nu(x)\}_{\nu=1}^4 = \Big\{x^3 - 2.81x\,,\ x+1\,,\ x^3 + 2x\,,\ x^3 - 6x - 3\Big\}, \tag{32}$$

the corresponding quadrature errors are presented in the last row of Table 2. Note although the first function in (32) is not monotonic on $[0,1]$, it satisfies the initial condition $\lambda_1(x_i) \neq \lambda_1(x_j)$ for any $i \neq j$ and distinct nodes $\{x_k\}_{k=0}^4$ in all distributions (a)-(d) .

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