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On Totally Real Statistical Submanifolds

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Abstract. In the present paper, first we prove some results by using fundamental properties of totally real statistical submanifolds immersed into holomorphic statistical manifolds. Further, we obtain the generalized Wintgen inequality for Lagrangian statistical submanifolds of holomorphic statistical manifolds with constant holomorphic sectional curvature c. The paper finishes with some geometric consequences of obtained results.

1. Introduction

In 1985, the notion of statistical manifolds has been studied by Amari [1]. The abstract generalizations of statistical models are considered as the statistical manifolds. The geometry of statistical manifolds lies at a junction of several branches of geometry (information geometry, affine differential geometry [10] and Hessian geometry [11]). A statistical structure can be considered as a generalization of a Riemannian structure (a pair of a Riemannian metric and its Levi-Civita connection). It includes the notion of dual connection, also called conjugate connection. The theory of statistical manifold and its statistical submanifold plays a role of central importance in many research fields of differential geometry.

Recently, H. Furuhata has constructed complex structure and contact structure on the statistical manifolds, and defined a holomorphic statistical manifold (see [5, 6]) and Sasakian statistical manifold (see [7]). The theory of statistical manifold and its statistical submanifold is a very recent geometry. Therefore, it attracts the geometers and several interesting results have been obtained by many of them ([2, 3, 6, 7, 9]).

On another hand, in 1999, Wintgen inequality has been conjectured by De Smet, et al. [12] for all submanifolds N^m of all real space forms $\mathcal{M}^{m+s}(c)$ with constant sectional curvature c, for all dimensions $m \geq 2$ and for all codimensions $s \geq 1$. This is also known as the DDVV conjecture and it is proved by Ge and Tang (2008) and by Lu (2011), independently.

In the present paper, first we study holomorphic statistical manifolds with some examples (see Examples 2.7,2.8 and 2.9). We give some basic results on totally real and Lagrangian statistical submanifolds immersed into holomorphic statistical manifolds (see Section 3). Then we establish Wintgen inequality for Lagrangian statistical submanifolds of a holomorphic statistical manifold with constant holomorphic sectional curvature c (see Theorem 4.1). Also, we obtain a general inequality for totally real statistical submanifold in the same ambient (see Theorem 4.2). At the end, some immediate consequences are also obtained (see Section 5).

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2. Statistical Manifolds and Statistical Submanifolds

Let $\overline{\nabla}$ be an affine connection on a Riemannian manifold $\left(\overline{\mathcal{M}}, \overline{g}\right)$. The affine connection $\overline{\nabla}^*$ on $\overline{\mathcal{M}}$ satisfying [6]

$$\mathcal{Z}\overline{g}(\mathcal{X},\mathcal{Y}) = \overline{g}(\overline{\nabla}_{\mathcal{Z}}\mathcal{X},\mathcal{Y}) + \overline{g}(\mathcal{X},\overline{\nabla}^*_{\mathcal{Z}}\mathcal{Y})$$

for any $\mathcal{X},\mathcal{Y},\mathcal{Z} \in \Gamma(T\overline{\mathcal{M}})$ is called the dual connection of $\overline{\nabla}$ with respect to $\overline{g}$.

Definition 2.1. *[6] The statistical manifold $\left(\overline{\mathcal{M}},\overline{\nabla},\overline{g}\right)$ is a Riemannian manifold equipped with torsion-free affine connection $\overline{\nabla}$ satisfying $(\overline{\nabla}_{\mathcal{X}}\overline{g})(\mathcal{Y},\mathcal{Z}) = (\overline{\nabla}_{\mathcal{Y}}\overline{g})(\mathcal{X},\mathcal{Z})$, for any $\mathcal{X},\mathcal{Y},\mathcal{Z} \in \Gamma(T\overline{\mathcal{M}})$.*

Remark 2.2. *A pair of affine connections $\overline{\nabla}$ and $\overline{\nabla}^*$ on $\overline{\mathcal{M}}$ satisfies [6]*

$$(\overline{\nabla}^*)^* = \overline{\nabla}. \tag{1}$$

We remark that if $(\overline{\mathcal{M}},\overline{\nabla},\overline{g})$ is a statistical structure, so is $(\overline{\mathcal{M}},\overline{\nabla}^*,\overline{g})$.

Definition 2.3. *[6] Let $\left(\overline{\mathcal{M}},\overline{\nabla},\overline{g}\right)$ be a statistical manifold and $\mathcal{M}$ be a submanifold of $\overline{\mathcal{M}}$. Then $\left(\mathcal{M},\nabla,g\right)$ is also a statistical manifold with the induced statistical structure (∇,g) on $\mathcal{M}$ from $(\overline{\nabla},\overline{g})$ and we call $\left(\mathcal{M},\nabla,g\right)$ as a statistical submanifold in $\left(\overline{\mathcal{M}},\overline{\nabla},\overline{g}\right)$.*

Let $(\mathcal{M},\nabla,g)$ be a submanifold with any codimension of a statistical manifold $(\overline{\mathcal{M}},\overline{\nabla},\overline{g})$. The fundamental equations in the geometry of Riemannian submanifolds are the Gauss and Weingarten formulae and the equations of Gauss, Codazzi and Ricci (see [15]). In the statistical setting, Gauss and Weingarten formulae are respectively defined by [6]

$$\left.\begin{aligned} &\overline{\nabla}_{\mathcal{X}}\mathcal{Y} = \nabla_{\mathcal{X}}\mathcal{Y} + \zeta(\mathcal{X},\mathcal{Y}), \qquad \overline{\nabla}^*_{\mathcal{X}}\mathcal{Y} = \nabla^*_{\mathcal{X}}\mathcal{Y} + \zeta^*(\mathcal{X},\mathcal{Y}),\\ &\overline{\nabla}_{\mathcal{X}}\mathcal{U} = -\Lambda_{\mathcal{U}}(\mathcal{X}) + D_{\mathcal{X}}\mathcal{U}, \qquad \overline{\nabla}^*_{\mathcal{X}}\mathcal{U} = -\Lambda^*_{\mathcal{U}}(\mathcal{X}) + D^*_{\mathcal{X}}\mathcal{U}, \end{aligned}\right\} \tag{2}$$

for any $\mathcal{X},\mathcal{Y} \in \Gamma(T\mathcal{M})$ and $\mathcal{U} \in \Gamma(T^{\perp}\mathcal{M})$, where $\overline{\nabla}$ and $\overline{\nabla}^*$ (resp., ∇ and ∇^*) are the dual connections on $\overline{\mathcal{M}}$ (resp., on $\mathcal{M}$). Also, D and D^* are the dual connections of a vector bundle $T^{\perp}\mathcal{M}$. The symmetric and bilinear imbedding curvature tensor of $\mathcal{M}$ in $\overline{\mathcal{M}}$ for $\overline{\nabla}$ and $\overline{\nabla}^*$ are denoted by ζ and ζ^*, respectively.

The relation between ζ (resp., ζ^*) and Λ (resp., Λ^*) is defined by [6]

$$\left.\begin{aligned} &\overline{g}(\zeta(\mathcal{X},\mathcal{Y}),\mathcal{U}) = g(\Lambda^*_{\mathcal{U}}(\mathcal{X}),\mathcal{Y}),\\ &\overline{g}(\zeta^*(\mathcal{X},\mathcal{Y}),\mathcal{U}) = g(\Lambda_{\mathcal{U}}(\mathcal{X}),\mathcal{Y}), \end{aligned}\right\} \tag{3}$$

for any $\mathcal{X},\mathcal{Y} \in \Gamma(T\mathcal{M})$ and $\mathcal{U} \in \Gamma(T^{\perp}\mathcal{M})$.

Definition 2.4. *[9] Let $(\mathcal{M},\nabla,g)$ be a submanifold with any codimension of a statistical manifold $(\overline{\mathcal{M}},\overline{\nabla},\overline{g})$. Then $\mathcal{M}$ is said to be*

(i) totally geodesic with respect to $\overline{\nabla}$ if $\zeta = 0$.

(i) totally geodesic with respect to $\overline{\nabla}^*$ if $\zeta^* = 0$.*

(ii) totally tangentially umbilical with respect to $\overline{\nabla}$ if $\zeta(\mathcal{X},\mathcal{Y}) = g(\mathcal{X},\mathcal{Y})\mathcal{H}$ for any $\mathcal{X},\mathcal{Y} \in \Gamma(T\mathcal{M})$. Here $\mathcal{H}$ is the mean curvature vector of $\mathcal{M}$ in $\overline{\mathcal{M}}$ for $\overline{\nabla}$.

(ii) totally tangentially umbilical with respect to $\overline{\nabla}^*$ if $\zeta^*(\mathcal{X},\mathcal{Y}) = g(\mathcal{X},\mathcal{Y})\mathcal{H}^*$ for any $\mathcal{X},\mathcal{Y} \in \Gamma(T\mathcal{M})$. Here $\mathcal{H}^*$ is the mean curvature vector of $\mathcal{M}$ in $\overline{\mathcal{M}}$ for $\overline{\nabla}^*$.*

(iii) *totally normally umbilical with respect to* $\overline{\nabla}$ *if* $\Lambda_{\mathcal{U}}\mathcal{X} = g(\mathcal{H},\mathcal{U})\mathcal{X}$ *for any* $\mathcal{X} \in \Gamma(T\mathcal{M})$ *and* $\mathcal{U} \in \Gamma(T^{\perp}\mathcal{M})$.

(iii)* *totally normally umbilical with respect to* $\overline{\nabla}^*$ *if* $\Lambda^*_{\mathcal{U}}\mathcal{X} = g(\mathcal{H}^*,\mathcal{U})\mathcal{X}$ *for any* $\mathcal{X} \in \Gamma(T\mathcal{M})$ *and* $\mathcal{U} \in \Gamma(T^{\perp}\mathcal{M})$.

The curvature tensors with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$ are denoted by $\overline{\mathcal{R}}$ and $\overline{\mathcal{R}}^*$, respectively. Also, $\mathcal{R}$ and $\mathcal{R}^*$ are the curvature tensors with respect to ∇ and ∇^*, respectively. Then the corresponding Gauss, Codazzi and Ricci equations, respectively, are given by [6]

$$\left.\begin{aligned}
\overline{\mathcal{R}}(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{W}) =\ & \mathcal{R}(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{W}) + \overline{g}(\zeta(\mathcal{X},\mathcal{Z}),\zeta^*(\mathcal{Y},\mathcal{W}))\\
& - \overline{g}(\zeta^*(\mathcal{X},\mathcal{W}),\zeta(\mathcal{Y},\mathcal{Z})),\\
\overline{\mathcal{R}}(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{U}) =\ & \overline{g}((\overline{\nabla}_{\mathcal{X}}\zeta)(\mathcal{Y},\mathcal{Z}),\mathcal{U}) - \overline{g}((\overline{\nabla}_{\mathcal{Y}}\zeta)(\mathcal{X},\mathcal{Z}),\mathcal{U}),\\
\overline{\mathcal{R}}(\mathcal{X},\mathcal{Y},\mathcal{U},\mathcal{Z}) =\ & g((\overline{\nabla}_{\mathcal{Y}}\Lambda)_{\mathcal{U}}\mathcal{X},\mathcal{Z}) - g((\overline{\nabla}_{\mathcal{X}}\Lambda)_{\mathcal{U}}\mathcal{Y},\mathcal{Z}),\\
\overline{\mathcal{R}}(\mathcal{X},\mathcal{Y},\mathcal{U},\mathcal{V}) =\ & \mathcal{R}^{\perp}(\mathcal{X},\mathcal{Y},\mathcal{U},\mathcal{V}) + \overline{g}(\zeta(\mathcal{Y},\Lambda_{\mathcal{U}}\mathcal{X},\mathcal{V}) - \overline{g}(\zeta(\mathcal{X},\Lambda_{\mathcal{U}}\mathcal{Y},\mathcal{V}),
\end{aligned}\right\} \tag{4}$$

for any $\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{W} \in \Gamma(T\mathcal{M})$ and $\mathcal{U},\mathcal{V} \in \Gamma(T^{\perp}\mathcal{M})$.

The curvature tensor fields of $\overline{\mathcal{M}}$ and $\mathcal{M}$ are respectively defined as [6]

$$\overline{\mathcal{S}} = \frac{1}{2}(\overline{\mathcal{R}} + \overline{\mathcal{R}}^*),$$

and

$$\mathcal{S} = \frac{1}{2}(\mathcal{R} + \mathcal{R}^*).$$

Thus, the sectional curvature $\mathbb{K}$ on $\mathcal{M}$ of $\overline{\mathcal{M}}$ is given by [3]

$$\mathbb{K}(\mathcal{X}\wedge\mathcal{Y}) = g(\mathcal{S}(\mathcal{X},\mathcal{Y})\mathcal{Y},\mathcal{X}) = \frac{1}{2}(g(\mathcal{R}(\mathcal{X},\mathcal{Y})\mathcal{Y},\mathcal{X}) + g(\mathcal{R}^*(\mathcal{X},\mathcal{Y})\mathcal{Y},\mathcal{X})),$$

for any orthonormal vectors $\mathcal{X},\mathcal{Y} \in T_{\wp}\mathcal{M}$, $\wp \in \mathcal{M}$.

Definition 2.5. *[6] Let* $\left(\overline{\mathcal{M}},\mathcal{J},\overline{g}\right)$ *be a Kaehler manifold and* $\overline{\nabla}$ *be an affine connection on* $\overline{\mathcal{M}}$. *Then* $\left(\overline{\mathcal{M}},\overline{\nabla},\overline{g},\mathcal{J}\right)$ *is said to be a holomorphic statistical manifold if*

(i) $\left(\overline{\mathcal{M}},\overline{\nabla},\overline{g}\right)$ *is a statistical manifold, and*

(ii) *a* $2-$*form* ϖ *on* $\overline{\mathcal{M}}$, *given by*

$$\varpi(\mathcal{X},\mathcal{Y}) = \overline{g}(\mathcal{X},\mathcal{J}\mathcal{Y}),$$

for any $\mathcal{X},\mathcal{Y} \in \Gamma(T\overline{\mathcal{M}})$, *is a* $\overline{\nabla}-$*parallel, that is,* $\overline{\nabla}\varpi = 0$.

For a holomorphic statistical manifold $\left(\overline{\mathcal{M}},\mathcal{J},\overline{g}\right)$, we have the following relation [6]:

$$\overline{\nabla}_{\mathcal{X}}(\mathcal{J}\mathcal{Y}) = \mathcal{J}\overline{\nabla}^*_{\mathcal{X}}\mathcal{Y}, \tag{5}$$

for any $\mathcal{X},\mathcal{Y} \in \Gamma(T\overline{\mathcal{M}})$.

Lemma 2.6. *[5] Let* $(\overline{\mathcal{M}},\overline{g},\mathcal{J})$ *be a Kaehler manifold and a connection* $\overline{\nabla}$ *is defined as* $\overline{\nabla} := \nabla^{\overline{g}} + K$, *where* K *is a* $(1,2)-$*tensor field satisfying the following conditions:*

$$K(\mathcal{X},\mathcal{Y}) = K(\mathcal{Y},\mathcal{X}), \tag{6}$$

$$\overline{g}(K(\mathcal{X},\mathcal{Y}),\mathcal{Z}) = \overline{g}(\mathcal{Y},K(\mathcal{X},\mathcal{Z})), \tag{7}$$

and

$$K(\mathcal{X},\mathcal{J}\mathcal{Y}) + \mathcal{J}K(\mathcal{X},\mathcal{Y}) = 0, \tag{8}$$

for any $\mathcal{X},\mathcal{Y},\mathcal{Z} \in \Gamma(T\overline{\mathcal{M}})$. *Then* $(\overline{\mathcal{M}},\overline{\nabla},\overline{g},\mathcal{J})$ *is a holomorphic statistical manifold.*

By using the above Lemma 2.6, we construct the following examples:

Example 2.7. *Let us consider a Kaehler manifold* $(\overline{\mathcal{M}} = \{(x^1, x^2)' \in \mathbb{R}^2 | x^1 > 0, x^2 > 0\}, \overline{g}, \mathcal{J})$*, where a Riemanian metric* $\overline{g}$ *and the standard complex structure* $\mathcal{J}$ *on* $\overline{\mathcal{M}}$ *are defined by*

$$\overline{g} = x^1\{(dx^1)^2 + (dx^2)^2\},$$

and

$$\mathcal{J}\partial_1 = \partial_2, \quad \mathcal{J}\partial_2 = -\partial_1,$$

where $\partial_i = \frac{\partial}{\partial x^i}$ *for* $i = 1, 2$*. Now, for any* $\lambda \in \mathbb{R}$*, we define a* $(1, 2)-$*tensor field K on* $\mathbb{R}^2$ *as follows:*

$$K = \sum_{i,j,l=1}^{2} k_{ij}^l \partial_l \otimes dx^i \otimes dx^j,$$

where $-k_{11}^1 = k_{12}^2 = k_{21}^2 = k_{22}^1 = \lambda$ *and* $k_{11}^2 = k_{12}^1 = k_{21}^1 = k_{22}^2 = 0$*. Then K satisfies all three conditions of Lemma 2.6, and hence we get a holomorphic statistical manifold* $(\overline{\mathcal{M}}, \overline{\nabla} := \nabla^{\overline{g}} + K, \overline{g}, \mathcal{J})$*, where an affine connection* $\overline{\nabla}$ *on* $\overline{\mathcal{M}}$ *is given by*

$$\begin{aligned}
\overline{\nabla}_{\partial_1}\partial_1 &= \Big(\frac{1}{2}(x^1)^{-1} - \lambda\Big)\partial_1, \\
\overline{\nabla}_{\partial_1}\partial_2 &= \overline{\nabla}_{\partial_2}\partial_1 = \Big(\frac{1}{2}(x^1)^{-1} + \lambda\Big)\partial_2, \\
\overline{\nabla}_{\partial_2}\partial_2 &= -\Big(\frac{1}{2}(x^1)^{-1} - \lambda\Big)\partial_1.
\end{aligned}$$

Example 2.8. *Let* $(\overline{g}, \mathcal{J})$ *be a Kaehler structure on* $\overline{\mathcal{M}}$*. We take a vector field* $\Omega \in \Gamma(T\overline{\mathcal{M}})$ *and set a tensor field* $K_1 \in \Gamma(T\overline{\mathcal{M}}^{(1,2)})$ *as follows:*

$$\begin{aligned}
K_1(\mathcal{X}, \mathcal{Y}) &= \Big[\overline{g}(\mathcal{J}\Omega, \mathcal{X})\overline{g}(\mathcal{J}\Omega, \mathcal{Y}) - \overline{g}(\Omega, \mathcal{X})\overline{g}(\Omega, \mathcal{Y})\Big]\Omega \\
&\quad + \Big[\overline{g}(\mathcal{J}\Omega, \mathcal{X})\overline{g}(\Omega, \mathcal{Y}) + \overline{g}(\Omega, \mathcal{X})\overline{g}(\mathcal{J}\Omega, \mathcal{Y})\Big]\mathcal{J}\Omega,
\end{aligned} \tag{9}$$

for any $\mathcal{X}, \mathcal{Y} \in \Gamma(T\overline{\mathcal{M}})$*. Then, by simple computation, we see that* K_1 *satisfies three conditions of Lemma 2.6, and hence a holomorphic statistical manifold* $(\overline{\mathcal{M}}, \overline{\nabla} := \nabla^{\overline{g}} + K_1, \overline{g}, \mathcal{J})$ *is obtained.*

Example 2.9. *For a Kaehler manifold* $(\overline{\mathcal{M}}, \overline{g}, \mathcal{J})$*, we take a vector field* $\Omega \in \Gamma(T\overline{\mathcal{M}})$ *and set* K_2 *as follows:*

$$\begin{aligned}
K_2(\mathcal{X}, \mathcal{Y}) &= \Big[\overline{g}(\Omega, \mathcal{J}\mathcal{X})\overline{g}(\Omega, \mathcal{J}\mathcal{Y}) - \overline{g}(\Omega, \mathcal{X})\overline{g}(\Omega, \mathcal{Y}) \\
&\quad - \overline{g}(\Omega, \mathcal{J}\mathcal{X})\overline{g}(\Omega, \mathcal{Y}) - \overline{g}(\Omega, \mathcal{X})\overline{g}(\Omega, \mathcal{J}\mathcal{Y})\Big]\Omega \\
&\quad + \Big[\overline{g}(\Omega, \mathcal{X})\overline{g}(\Omega, \mathcal{Y}) - \overline{g}(\Omega, \mathcal{J}\mathcal{X})\overline{g}(\Omega, \mathcal{J}\mathcal{Y}) \\
&\quad - \overline{g}(\Omega, \mathcal{J}\mathcal{X})\overline{g}(\Omega, \mathcal{Y}) - \overline{g}(\Omega, \mathcal{X})\overline{g}(\Omega, \mathcal{J}\mathcal{Y})\Big]\mathcal{J}\Omega,
\end{aligned} \tag{10}$$

for any $\mathcal{X}, \mathcal{Y} \in \Gamma(T\overline{\mathcal{M}})$*. Then* $K_2 \in \Gamma(T\overline{\mathcal{M}}^{(1,2)})$ *satisfies three conditions of Lemma 2.6 as in Example 2.8, and hence* $(\overline{\mathcal{M}}, \overline{\nabla} := \nabla^{\overline{g}} + K_2, \overline{g}, \mathcal{J})$ *becomes a holomorphic statistical manifold.*

For any $\mathcal{X} \in \Gamma(T\mathcal{M})$ and $\mathcal{V} \in \Gamma(T^{\perp}\mathcal{M})$, respectively, we put [15]

$$\mathcal{J}\mathcal{X} = G\mathcal{X} + L\mathcal{X}, \tag{11}$$

and

$$\mathcal{J}\mathcal{V} = C\mathcal{V} + P\mathcal{V}, \tag{12}$$

where $G\mathcal{X}$ and $C\mathcal{V}$ are tangential parts $\mathcal{J}\mathcal{X}$ and $\mathcal{J}\mathcal{V}$, respectively. And $L\mathcal{X}$ and $P\mathcal{V}$ are normal parts of $\mathcal{J}\mathcal{X}$ and $\mathcal{J}\mathcal{V}$, respectively.

Definition 2.10. *[6] A holomorphic statistical manifold $\overline{\mathcal{M}}$ is said to be of constant holomorphic curvature $c \in \mathcal{R}$ if the following curvature equation holds*

$$\begin{aligned}\overline{\mathcal{R}}(\mathcal{X},\mathcal{Y})\mathcal{Z} &= \frac{c}{4}\Big\{g(\mathcal{Y},\mathcal{Z})\mathcal{X} - g(\mathcal{X},\mathcal{Z})\mathcal{Y} + g(\mathcal{J}\mathcal{Y},\mathcal{Z})\mathcal{J}\mathcal{X} \\ &\quad - g(\mathcal{J}\mathcal{X},\mathcal{Z})\mathcal{J}\mathcal{Y} + 2g(\mathcal{X},\mathcal{J}\mathcal{Y})\mathcal{J}\mathcal{Z}\Big\},\end{aligned} \tag{13}$$

for any $\mathcal{X},\mathcal{Y},\mathcal{Z} \in \Gamma(T\overline{\mathcal{M}})$.

Definition 2.11. *[6] A statistical submanifold $\mathcal{M}$ of a holomorphic statistical manifold $\overline{\mathcal{M}}$ is called holomorphic ($L = 0$ and $C = 0$) if the almost complex structure $\mathcal{J}$ of $\overline{\mathcal{M}}$ carries each tangent space of $\mathcal{M}$ into itself whereas it is said to be totally real ($G = 0$) if the almost complex structure $\mathcal{J}$ of $\overline{\mathcal{M}}$ carries each tangent space of $\mathcal{M}$ into its corresponding normal space.*

Definition 2.12. *A totally real statistical submanifold of maximal dimension is called Lagrangian statistical submanifold.*

3. Totally Real Statistical Submanifolds

In the following, we assume that $(\overline{\mathcal{M}},\overline{\nabla},g,\mathcal{J})$ is a holomorphic statistical manifold and $\overline{\mathcal{M}}(c)$ is a holomorphic statistical manifold with constant holomorphic sectional curvature c.

We prove the following:

Theorem 3.1. *Let $\mathcal{M}$ be a totally real statistical submanifold immersed into $\overline{\mathcal{M}}(c)$. If $\mathcal{M}$ is totally umbilical submanifold with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$, then sectional curvature $\mathbb{K} = \frac{c}{4}$ if and only if any one of the following holds:*

- *(i) Both $\mathcal{H}$ and $\mathcal{H}^*$ are perpendicular to each other;*
- *(ii) $\mathcal{H} = 0$;*
- *(iii) $\mathcal{H}^* = 0$.*

Proof. For any orthonormal vectors $\mathcal{X},\mathcal{Y} \in \Gamma(T\mathcal{M})$, the sectional curvature $\mathbb{K}$ of $\mathcal{M}$ is defined as follows

$$\mathbb{K}(\mathcal{X}\wedge\mathcal{Y}) = \frac{1}{2}\Big[g(\mathcal{R}(\mathcal{X},\mathcal{Y})\mathcal{X},\mathcal{Y}) + g(\mathcal{R}^*(\mathcal{X},\mathcal{Y})\mathcal{X},\mathcal{Y})\Big]. \tag{14}$$

Putting curvature tensor fields $\mathcal{R}$ and its dual $\mathcal{R}^*$ into (14), we have

$$\begin{aligned}\mathbb{K}(\mathcal{X}\wedge\mathcal{Y}) &= \frac{1}{2}\Big[\frac{c}{2} + g(\Lambda_{\zeta(\mathcal{Y},\mathcal{X})}\mathcal{X},\mathcal{Y}) - g(\Lambda_{\zeta(\mathcal{X},\mathcal{X})}\mathcal{Y},\mathcal{Y}) \\ &\quad + g(\Lambda^*_{\zeta(\mathcal{Y},\mathcal{X})}\mathcal{X},\mathcal{Y}) - g(\Lambda^*_{\zeta(\mathcal{X},\mathcal{X})}\mathcal{Y},\mathcal{Y})\Big] \\ &= \frac{1}{2}\Big[\frac{c}{2} + \overline{g}(\zeta^*(\mathcal{X},\mathcal{Y}),\zeta(\mathcal{Y},\mathcal{X})) - \overline{g}(\zeta^*(\mathcal{Y},\mathcal{Y}),\zeta(\mathcal{X},\mathcal{X})) \\ &\quad + \overline{g}(\zeta(\mathcal{X},\mathcal{Y}),\zeta^*(\mathcal{Y},\mathcal{X})) - \overline{g}(\zeta^*(\mathcal{Y},\mathcal{Y}),\zeta(\mathcal{X},\mathcal{X}))\Big] \\ &= \frac{c}{4} + \overline{g}(\zeta^*(\mathcal{X},\mathcal{Y}),\zeta(\mathcal{Y},\mathcal{X})) - \overline{g}(\zeta^*(\mathcal{Y},\mathcal{Y}),\zeta(\mathcal{X},\mathcal{X})).\end{aligned} \tag{15}$$

Since, $\mathcal{M}$ is totally umbilical submanifold with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$, then (15) becomes

$$\mathbb{K}(\mathcal{X} \wedge \mathcal{Y}) = \frac{c}{4} - g(\mathcal{H}, \mathcal{H}^*).$$

If we take $\mathbb{K} = \frac{c}{4}$, then any one of the following holds:

(i) $\mathcal{H} \perp \mathcal{H}^*$, or

(ii) $\mathcal{H} = 0$, or

(iii) $\mathcal{H}^* = 0$.

Thus, our assertions follow. Converse part is trivial if any one of the above holds. □

Theorem 3.2. *Let $\mathcal{M}$ be a totally real statistical submanifold immersed into $\overline{\mathcal{M}}$. If $\mathcal{M}$ is totally umbilical submanifold with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$, then $\|\mathcal{H}^0\|^2 = g(\mathcal{H}, \mathcal{H}^*)$.*

Proof. Since, we have [6]

$$\begin{aligned} 0 &= D_{\mathcal{X}}\mathcal{H} + D^*_{\mathcal{X}}\mathcal{H}^* \\ &= \overline{\nabla}_{\mathcal{X}}\mathcal{H} + \|\mathcal{H}\|^2\mathcal{X} + \overline{\nabla}^*_{\mathcal{X}}\mathcal{H}^* + \|\mathcal{H}^*\|^2\mathcal{X}. \end{aligned}$$

By taking inner product with unit vector field $\mathcal{X}$ on $\mathcal{M}$, we get

$$\begin{aligned} 0 &= g(\overline{\nabla}_{\mathcal{X}}\mathcal{H}, \mathcal{X}) + g(\overline{\nabla}^*_{\mathcal{X}}\mathcal{H}^*, \mathcal{X}) + \|\mathcal{H}^*\|^2 + \|\mathcal{H}\|^2 \\ &= \mathcal{X}g(\mathcal{H}, \mathcal{X}) - g(\mathcal{H}, \overline{\nabla}^*_{\mathcal{X}}\mathcal{X}) + \mathcal{X}g(\mathcal{H}^*, \mathcal{X}) - g(\mathcal{H}^*, \overline{\nabla}_{\mathcal{X}}\mathcal{X}) + \|\mathcal{H}^*\|^2 + \|\mathcal{H}\|^2 \\ &= -g(\mathcal{H}, \nabla^*_{\mathcal{X}}\mathcal{X} + \mathcal{H}^*) - g(\mathcal{H}^*, \nabla_{\mathcal{X}}\mathcal{X} + \mathcal{H}) + \|\mathcal{H}^*\|^2 + \|\mathcal{H}\|^2 \\ &= \|\mathcal{H}^*\|^2 + \|\mathcal{H}\|^2 - 2g(\mathcal{H}, \mathcal{H}^*). \end{aligned}$$

Hence, we get the desired result. □

Theorem 3.3. *Let $\mathcal{M}$ be a Lagrangian statistical submanifold immersed into $\overline{\mathcal{M}}$. If $\mathcal{M}$ is totally umbilical submanifold with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$, then $\mathcal{H} = \mathcal{H}^* = 0$.*

Proof. For any $\mathcal{X}, \mathcal{Y} \in \Gamma(T\mathcal{M})$, then

$$\overline{\nabla}_{\mathcal{X}}\mathcal{J}\mathcal{Y} = -\Lambda_{\mathcal{J}\mathcal{X}}\mathcal{Y} + D_{\mathcal{X}}\mathcal{J}\mathcal{Y}. \tag{16}$$

On the other hand,

$$\begin{aligned} \overline{\nabla}_{\mathcal{X}}\mathcal{J}\mathcal{Y} &= \mathcal{J}\overline{\nabla}^*_{\mathcal{X}}\mathcal{Y} \\ &= \mathcal{J}\left(\nabla^*_{\mathcal{X}}\mathcal{Y} + g(\mathcal{X}, \mathcal{Y})\mathcal{H}^*\right) \\ &= \mathcal{J}\nabla^*_{\mathcal{X}}\mathcal{Y} + g(\mathcal{X}, \mathcal{Y})\mathcal{J}\mathcal{H}^*. \end{aligned} \tag{17}$$

On combining (16) and (17), we find that

$$\begin{aligned} 0 &= g(\mathcal{X}, \mathcal{Y})\mathcal{J}\mathcal{H}^* + \Lambda_{\mathcal{J}\mathcal{X}}\mathcal{Y} \\ &= g(\mathcal{X}, \mathcal{Y})\mathcal{J}\mathcal{H}^* - g(\mathcal{X}, \mathcal{J}\mathcal{H})\mathcal{Y} \end{aligned}$$

for any $\mathcal{X}, \mathcal{Y} \in \Gamma(T\mathcal{M})$. Putting $\mathcal{X} = \mathcal{J}\mathcal{H}$ and $0 \neq \mathcal{Y} \perp \mathcal{J}\mathcal{H}$ in the last equation, we have our assertions. □

Theorem 3.4. *In an $m-$dimensional Lagrangian statistical submanifold $\mathcal{M}$ of a holomorphic statistical manifold $\overline{\mathcal{M}}$, the following holds:*

$$\mathcal{S}^{\perp}(\mathcal{X}, \mathcal{Y})\mathcal{J}\mathcal{Z} = (\mathcal{S}(\mathcal{X}, \mathcal{Y})\mathcal{Z})^T$$

for any $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \Gamma(T\mathcal{M})$.

Proof. For any $\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{W}\in\Gamma(T\mathcal{M})$, we have

$$\begin{aligned}2\overline{g}(\overline{\mathcal{S}}(\mathcal{X},\mathcal{Y})\mathcal{J}\mathcal{Z},\mathcal{J}\mathcal{W}) &= 2g(\mathcal{S}^{\perp}(\mathcal{X},\mathcal{Y})\mathcal{J}\mathcal{Z},\mathcal{J}\mathcal{W})-\overline{g}(\zeta(\mathcal{X},\Lambda_{\mathcal{J}\mathcal{Z}}\mathcal{Y}),\mathcal{J}\mathcal{W})\\ &\quad+\overline{g}(\zeta(\mathcal{Y},\Lambda_{\mathcal{J}\mathcal{Z}}\mathcal{X}),\mathcal{J}\mathcal{W})-\overline{g}(\zeta^{*}(\mathcal{X},\Lambda^{*}_{\mathcal{J}\mathcal{Z}}\mathcal{Y}),\mathcal{J}\mathcal{W})\\ &\quad+\overline{g}(\zeta^{*}(\mathcal{Y},\Lambda^{*}_{\mathcal{J}\mathcal{Z}}\mathcal{X}),\mathcal{J}\mathcal{W})\\ &= 2g(\mathcal{S}^{\perp}(\mathcal{X},\mathcal{Y})\mathcal{J}\mathcal{Z},\mathcal{J}\mathcal{W})-g(\Lambda^{*}_{\zeta^{*}(\mathcal{Y},\mathcal{Z})}\mathcal{X},\mathcal{W})\\ &\quad+g(\Lambda^{*}_{\zeta^{*}(\mathcal{X},\mathcal{Z})}\mathcal{Y},\mathcal{W})-g(\Lambda_{\zeta(\mathcal{Y},\mathcal{Z})}\mathcal{X},\mathcal{W})\\ &\quad+g(\Lambda_{\zeta(\mathcal{X},\mathcal{Z})}\mathcal{Y},\mathcal{W}).\end{aligned}$$

Thus, we get

$$g(\mathcal{S}^{\perp}(\mathcal{X},\mathcal{Y})\mathcal{J}\mathcal{Z},\mathcal{J}\mathcal{W})=g(\mathcal{S}(\mathcal{X},\mathcal{Y})\mathcal{Z},\mathcal{W}).$$

This is the required assertion. □

Theorem 3.5. *Let $\mathcal{M}$ be a statistical submanifold immersed into $\overline{\mathcal{M}}(c)$. Then $\overline{\mathcal{S}}(\mathcal{X},\mathcal{Y})T\mathcal{M}\subset T\mathcal{M}$ for all vector fields $\mathcal{X}$ and $\mathcal{Y}$ on $\mathcal{M}$ if and only if $\mathcal{M}$ is either a holomorphic or a totally real statistical submanifold of $\overline{\mathcal{M}}(c)$.*

Proof. For any vector fields $\mathcal{X}$ and $\mathcal{Y}$ on $\mathcal{M}$ and $\mathcal{Z}\in\Gamma(T\mathcal{M})$, we have

$$\begin{aligned}\overline{\mathcal{S}}(\mathcal{X},\mathcal{Y})\mathcal{Z} &= \frac{1}{2}\Big[\overline{\mathcal{R}}(\mathcal{X},\mathcal{Y})\mathcal{Z}+\overline{\mathcal{R}}^{*}(\mathcal{X},\mathcal{Y})\mathcal{Z}\Big]\\ &= \frac{c}{4}\Big[g(\mathcal{Y},\mathcal{Z})\mathcal{X}-g(\mathcal{X},\mathcal{Z})\mathcal{Y}+\overline{g}(\mathcal{J}\mathcal{Y},\mathcal{Z})\mathcal{J}\mathcal{X}\\ &\quad-\overline{g}(\mathcal{J}\mathcal{X},\mathcal{Z})\mathcal{J}\mathcal{Y}+2\overline{g}(\mathcal{X},\mathcal{J}\mathcal{Y})\mathcal{J}\mathcal{Z}\Big].\end{aligned}\tag{18}$$

Now, if we say $\mathcal{M}$ is holomorphic statistical submanifold, then we have $\overline{\mathcal{S}}(\mathcal{X},\mathcal{Y})\mathcal{Z}\in\Gamma(T\mathcal{M})$. Further, if we consider $\mathcal{M}$ is totally real statistical submanifold, then also $\overline{\mathcal{S}}(\mathcal{X},\mathcal{Y})\mathcal{Z}\in\Gamma(T\mathcal{M})$ because $\overline{g}(\mathcal{J}\mathcal{Y},\mathcal{Z})=\overline{g}(\mathcal{J}\mathcal{X},\mathcal{Z})=\overline{g}(\mathcal{X},\mathcal{J}\mathcal{Y})=0$. Hence, $\overline{\mathcal{S}}(\mathcal{X},\mathcal{Y})T\mathcal{M}\subset T\mathcal{M}$.

Conversely, we put $\mathcal{Z}=\mathcal{Y}$ into (18), then we have the following:

$$\begin{aligned}\overline{\mathcal{S}}(\mathcal{X},\mathcal{Y})\mathcal{Y} &= \frac{c}{4}\Big[g(\mathcal{Y},\mathcal{Y})\mathcal{X}-g(\mathcal{X},\mathcal{Y})\mathcal{Y}+\overline{g}(\mathcal{J}\mathcal{Y},\mathcal{Y})\mathcal{J}\mathcal{X}\\ &\quad-\overline{g}(\mathcal{J}\mathcal{X},\mathcal{Y})\mathcal{J}\mathcal{Y}+2\overline{g}(\mathcal{X},\mathcal{J}\mathcal{Y})\mathcal{J}\mathcal{Y}\Big]\\ &= \frac{c}{4}\Big[g(\mathcal{Y},\mathcal{Y})\mathcal{X}-g(\mathcal{X},\mathcal{Y})\mathcal{Y}+3\overline{g}(\mathcal{J}\mathcal{Y},\mathcal{X})\mathcal{J}\mathcal{Y}\Big].\end{aligned}\tag{19}$$

But from the assumption $\overline{\mathcal{S}}(\mathcal{X},\mathcal{Y})\mathcal{Y}\in\Gamma(T\mathcal{M})$, the last term of above equation (19), that is, $\overline{g}(\mathcal{J}\mathcal{Y},\mathcal{X})\mathcal{J}\mathcal{Y}$ must be in $T\mathcal{M}$, which means that either $\overline{g}(\mathcal{J}\mathcal{Y},\mathcal{X})=0$ or $\mathcal{J}\mathcal{Y}\in\Gamma(T\mathcal{M})$. Hence, we conclude that $\mathcal{M}$ is either a holomorphic or a totally real statistical submanifold. □

Theorem 3.6. *Let $\mathcal{M}$ be a statistical submanifold immersed into $\overline{\mathcal{M}}(c)$. If $\mathcal{M}$ is either a holomorphic or a totally real statistical submanifold of $\overline{\mathcal{M}}(c)$, then $\overline{\mathcal{S}}(\mathcal{X},\mathcal{Y})T^{\perp}\mathcal{M}\subset T^{\perp}\mathcal{M}$ for all vector fields $\mathcal{X}$ and $\mathcal{Y}$ on $\mathcal{M}$.*

Proof. Since, $\overline{g}(\overline{\mathcal{S}}(\mathcal{X},\mathcal{Y})\mathcal{Z},\mathcal{V})=-\overline{g}(\overline{\mathcal{S}}(\mathcal{X},\mathcal{Y})\mathcal{V},\mathcal{Z})$ for any $\mathcal{X},\mathcal{Y},\mathcal{Z}$ tangent to $\mathcal{M}$ and any $\mathcal{V}$ normal to $\mathcal{M}$. Therefore, the proof follows directly from above Theorem 3.5. □

4. Wintgen Inequality

In [8], Mihai has proved the generalized Wintgen inequality for Lagrnagian submanifolds in complex space forms. In the same paper, he also has obtained another Wintgen inequality for totally real submanifolds in same ambient. Motivated by his result, we obtain another Wintgen inequality for an $m-$dimensional Lagrangian statistical submanifold $\mathcal{M}$ of a $2m-$dimensional holomorphic statistical manifold $(\overline{\mathcal{M}}, \overline{\nabla}, g, \mathcal{J})$ with constant holomorphic sectional curvature c, $\overline{\mathcal{M}}(c)$. For this, we consider $\{\mathcal{E}_1, \ldots, \mathcal{E}_m\}$ is an orthonormal frame on $\mathcal{M}^m$, then $\{\xi_1 = \mathcal{J}\mathcal{E}_1, \ldots, \xi_m = \mathcal{J}\mathcal{E}_m\}$ is an orthonormal frame in the normal bundle $T^{\perp}\mathcal{M}$. We need the scalar normal curvature $\mathcal{K}_N$ [14] and the normalized scalar normal curvature ϱ_N [8] of $\mathcal{M}^m$. Both terms are defined below:

$$\mathcal{K}_N = \frac{1}{4}\Big[\sum_{1\le a<b\le m}\sum_{1\le i<j\le m}\Big(g([\Lambda_a^*, \Lambda_b]\mathcal{E}_i, \mathcal{E}_j) + g([\Lambda_a, \Lambda_b^*]\mathcal{E}_i, \mathcal{E}_j)\Big)^2\Big] \tag{20}$$

and

$$\varrho_N = \frac{2}{m(m-1)}\sqrt{\mathcal{K}_N} \tag{21}$$

Now, we prove the following:

Theorem 4.1. *In an $m-$dimensional Lagrangian statistical submanifold $\mathcal{M}$ of $\overline{\mathcal{M}}^{2m}(c)$, the following inequality holds:*

$$m(m-1)\varrho_N^2 + \frac{c}{m(m-1)}\Big[\|\mathcal{H}\|^2 + \|\mathcal{H}^*\|^2\Big] \ge \Big(\frac{2}{m(m-1)}\varrho - \frac{c}{2}\Big)^2 + \frac{4c}{m(m-1)}\|\mathcal{H}^0\|^2.$$

Proof. From (20) and (21), we arrive at

$$\varrho_N^2 = \frac{1}{m^2(m-1)^2}\Big[\sum_{1\le a<b\le m}\sum_{1\le i<j\le m}\Big(g([\Lambda_a, \Lambda_b^*]\mathcal{E}_i - [\Lambda_b, \Lambda_a^*]\mathcal{E}_i, \mathcal{E}_j)\Big)^2\Big]. \tag{22}$$

For calculating RHS of above equation (22), we use equation (7.101) of [6], that is,

$$\begin{aligned}\mathcal{S}(\mathcal{X}, \mathcal{Y})\mathcal{Z} &= \frac{c}{4}\{g(\mathcal{Y}, \mathcal{Z})\mathcal{X} - g(\mathcal{X}, \mathcal{Z})\mathcal{Y}\} \\ &\quad + \frac{1}{2}\Big[[\Lambda_{\mathcal{J}\mathcal{X}}, \Lambda^*_{\mathcal{J}\mathcal{Y}}]\mathcal{Z} - [\Lambda_{\mathcal{J}\mathcal{Y}}, \Lambda^*_{\mathcal{J}\mathcal{X}}]\mathcal{Z}\Big]\end{aligned} \tag{23}$$

for any $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \Gamma(T\mathcal{M})$. Now we consider $\{\mathcal{E}_1, \ldots, \mathcal{E}_m\}$ is an orthonormal frame on $\mathcal{M}^m$, then $\{\xi_1 = \mathcal{J}\mathcal{E}_1, \ldots, \xi_m = \mathcal{J}\mathcal{E}_m\}$ is an orthonormal frame in the normal bundle $T^{\perp}\mathcal{M}$. Equation (23) gives

$$\varrho = \frac{cn(n-1)}{4} + g(\mathcal{H}, \mathcal{H}^*) - \sum_{i=1}^{n} Tr(\Lambda_i\Lambda_i^*), \tag{24}$$

where $\Lambda_i = \Lambda_{\mathcal{J}\mathcal{E}_i}$, $\Lambda_i^* = \Lambda^*_{\mathcal{J}\mathcal{E}_i}$. Further, we have [6]

$$\|\mathcal{S}\|^2 = \frac{c^2m(m-1)}{8} - c\sum_{i=1}^{m} Tr(\Lambda_i\Lambda_i^*) + \frac{1}{4}\sum_{a,b=1}^{m}\sum_{i=1}^{m}\|([\Lambda_a, \Lambda_b^*] - [\Lambda_b, \Lambda_a^*])\mathcal{E}_i\|^2.$$

Obviously,

$$\begin{aligned} \|\mathcal{S}\|^2 &= \frac{c^2 m(m-1)}{8} - c\sum_{i=1}^{m} Tr(\Lambda_i\Lambda_i^*) \\ &+\frac{1}{2}\sum_{1\le a<b\le m}\sum_{i=1}^{m}\|([\Lambda_a,\Lambda_b^*]-[\Lambda_b,\Lambda_a^*])\mathcal{E}_i\|^2. \end{aligned} \tag{25}$$

From (22) and (25), we get

$$\varrho_N^2 = \frac{1}{m^2(m-1)^2}\Big[2\|\mathcal{S}\|^2 - \frac{c^2 m(m-1)}{4} + 2c\sum_{i=1}^{m} Tr(\Lambda_i\Lambda_i^*)\Big].$$

Using (24) and the above equation becomes

$$\begin{aligned} \varrho_N^2 &= \frac{1}{m^2(m-1)^2}\Big[2\|\mathcal{S}\|^2 - \frac{c^2 m(m-1)}{4} \\ &+\frac{c^2 m(m-1)}{2} + 2cg(\mathcal{H},\mathcal{H}^*) - 2c\varrho\Big]. \end{aligned} \tag{26}$$

Since, for any $m-$dimensional statistical manifold $(\mathcal{M},\nabla,g)$ the following inequality holds [6]:

$$\|\mathcal{S}\|^2 \ge \frac{2}{m(m-1)}\varrho^2. \tag{27}$$

On combining (26) and (27), we find that

$$\begin{aligned} \varrho_N^2 &\ge \frac{1}{m(m-1)}\Big[\Big(\frac{2}{m(m-1)}\varrho\Big)^2 + \frac{c^2}{4} + \frac{2c}{m(m-1)}g(\mathcal{H},\mathcal{H}^*) - \frac{2c}{m(m-1)}\varrho\Big] \\ &= \frac{1}{m(m-1)}\Big[\Big(\frac{2}{m(m-1)}\varrho - \frac{c}{2}\Big)^2 + \frac{2c}{m(m-1)}g(\mathcal{H},\mathcal{H}^*)\Big]. \end{aligned} \tag{28}$$

From (1), we get $2\mathcal{H}^0 = \mathcal{H} + \mathcal{H}^*$ and thus

$$2g(\mathcal{H},\mathcal{H}^*) = 4\|\mathcal{H}^0\|^2 - \|\mathcal{H}\|^2 - \|\mathcal{H}^*\|^2. \tag{29}$$

Therefore, from (28) and (29), we derive our desired inequality. □

A general inequality for totally real statistical submanifolds is as follows:

Theorem 4.2. *Let $\mathcal{M}$ be an $m-$dimensional totally real statistical submanifold immersed into $\overline{\mathcal{M}}^{2m}(c)$. Then*

$$\varrho \ge \frac{c}{4} + \frac{m}{m-1}g(\mathcal{H},\mathcal{H}^*) - \frac{1}{m(m-1)}\|\zeta\|\|\zeta^*\|. \tag{30}$$

Proof. Let $\{\mathcal{E}_1,\dots,\mathcal{E}_m\}$ be an orthonormal frame of $\mathcal{M}^m$ and $\{\mathcal{E}_{m+1},\dots,\mathcal{E}_{2m}\}$ be an orthonormal frame in the normal bundle $T^\perp\mathcal{M}$. From Proposition 3 of [6], we get

$$2\sigma = \frac{c}{4}(m(m-1)) + m^2 g(\mathcal{H},\mathcal{H}^*) - \sum_{a=m+1}^{2m}\sum_{i,j=1}^{m}\hbar_{ij}^a\hbar_{ij}^{*a},$$

where

$$
\begin{aligned}
\mathcal{H} &= \frac{1}{m}\sum_{a=m+1}^{2m}\Big(\sum_{i=1}^{m}\hbar_{ii}^{a}\Big)\mathcal{E}_a,\\
\mathcal{H}^* &= \frac{1}{m}\sum_{a=m+1}^{2m}\Big(\sum_{i=1}^{m}\hbar_{ii}^{*a}\Big)\mathcal{E}_a,\\
\hbar_{ij}^{a} &= \overline{g}(\zeta(\mathcal{E}_i,\mathcal{E}_j),\mathcal{E}_a),\\
\hbar_{ij}^{*a} &= \overline{g}(\zeta^*(\mathcal{E}_i,\mathcal{E}_j),\mathcal{E}_a).
\end{aligned}
$$

Further, we apply Cauchy-Buniakowski-Schwarz, we have

$$2\sigma \geq \frac{c}{4}(m(m-1)) + m^2 g(\mathcal{H},\mathcal{H}^*) - \|\zeta\|\|\zeta^*\|.$$

From last inequality, we can easily obtain the following:

$$\varrho \geq \frac{c}{4} + \frac{m}{m-1}g(\mathcal{H},\mathcal{H}^*) - \frac{1}{m(m-1)}\|\zeta\|\|\zeta^*\|.$$

This is the desired inequality. □

5. Some Geometric Applications

In this section, we provide some immediate statistical significance of obtained results in the previous section.

If a statistical submanifold $\mathcal{M}$ of statistical manifold $\overline{\mathcal{M}}$ is minimal, that is, $\mathcal{H}^0 = 0$, then $\mathcal{H} + \mathcal{H}^* = 0$. Thus, we have the following corollary follows from Theorem 3.2:

Corollary 5.1. *Let $\mathcal{M}$ be a minimal totally real statistical submanifold immersed into $\overline{\mathcal{M}}$. If $\mathcal{M}$ is totally umbilical submanifold with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$, then any one of the following holds:*

(i) Both $\mathcal{H}$ and $\mathcal{H}^*$ are perpendicular to each other;

(ii) $\mathcal{H} = 0$;

(iii) $\mathcal{H}^* = 0$.

By using Theorem 3.4, we can easily obtain the following corollary:

Corollary 5.2. *Let $\mathcal{M}$ be a Lagrangian statistical submanifold immersed into $\overline{\mathcal{M}}$. Then $\mathcal{S} = 0$ if and only if $\mathcal{S}^\perp = 0$.*

Wintgen inequality for minimal statistical submanifold is directly from Theorem 4.1:

Corollary 5.3. *In an $m-$dimensional minimal Lagrangian statistical submanifold $\mathcal{M}$ of $\overline{\mathcal{M}}^{2m}(c)$, the following inequality holds:*

$$m(m-1)\varrho_N^2 + \frac{c}{m(m-1)}\Big[\|\mathcal{H}\|^2 + \|\mathcal{H}^*\|^2\Big] \geq \Big(\frac{2}{m(m-1)}\varrho - \frac{c}{2}\Big)^2.$$

Corollary 5.4. *Let $\mathcal{M}$ be an $m-$dimensional Lagrangian statistical submanifold immersed into $\overline{\mathcal{M}}^{2m}(c)$. Suppose that*

(i) $\varrho_N = 0$ and

(ii) $\varrho = \frac{cm(m-1)}{4}$.

Then $c \leq 0$.

Remark 5.5. *The proof of above Corollary 5.4 follows directly from equation (28).*

Now, we observe the following consequences of Theorem 4.2:

Corollary 5.6. *Let* $\mathcal{M}$ *be an* $m-$*dimensional totally real statistical submanifold immersed into* $\overline{\mathcal{M}}^{2m}(c)$. *If* $\mathcal{M}$ *is totally umbilical and totally geodesic with respect to* $\overline{\nabla}$ *and* $\overline{\nabla}^*$, *then* $\varrho \geq \frac{c}{4}$.

Remark 5.7. *In the above Corollary 5.6, we have* $\mathcal{M}$ *is totally umbilical and totally geodesic with respect to* $\overline{\nabla}$ *and* $\overline{\nabla}^*$, *that is, for any* $\mathcal{X}, \mathcal{Y} \in \Gamma(T\mathcal{M})$, $0 = \zeta(\mathcal{X}, \mathcal{Y}) = g(\mathcal{X}, \mathcal{Y})\mathcal{H}$, *which gives* $\mathcal{H} = 0$. *Similarly,* $0 = \zeta^*(\mathcal{X}, \mathcal{Y}) = g(\mathcal{X}, \mathcal{Y})\mathcal{H}^*$ *implies* $\mathcal{H}^* = 0$. *Hence, an inequality (30) reduces to* $\varrho \geq \frac{c}{4}$.

Corollary 5.8. *Let* $\mathcal{M}$ *be an* $m-$*dimensional totally real statistical submanifold immersed into* $\overline{\mathcal{M}}^{2m}(c)$. *If* $\varrho = \frac{c}{4}$, *then* $\mathcal{M}$ *is not totally geodesic with respect to* $\overline{\nabla}$ *and* $\overline{\nabla}^*$.

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