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Global Stability of Solutions in a Reaction-diffusion System of Predator-prey Model

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Abstract. In this article, we investigate the global asymptotic stability of a reaction-diffusion system of predator-prey model. By applying the comparison principle and iteration method, we prove the global asymptotic stability of the unique positive equilibrium solution of (1.1).

1. Introduction

The main goal of this paper is to study the following reaction-diffusion system of predator-prey model

$$\begin{cases} u_t - d_1\Delta u = au - bu^2 - \dfrac{\delta uv}{\beta u + v + \alpha}, & x \in \Omega, t \in (0,\infty), \\ v_t - d_2\Delta v = rv - \gamma \dfrac{v^2}{u}, & x \in \Omega, t \in (0,\infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t \in (0,\infty), \\ u(x,0) = u_0(x) > 0, \quad v(x,0) = v_0(x) \geq 0, & x \in \overline{\Omega}, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N (N = 1,2,3)$ with smooth boundary $\partial\Omega$, $0 < T \leq +\infty$, initial condition $u_0(x)$ and $v_0(x)$ are continuous functions on $\overline{\Omega}$ and compatible on $\partial\Omega$, constants $d_1, d_2, a, b, r, \alpha, \beta, \gamma, \delta > 0$, and ν is the outward directional derivative normal to $\partial\Omega$.

There are the Beddington-DeAngelis and Tanner type functional response contained in the first and second equation of model (1.1), respectively, where $u(x,t)$ and $v(x,t)$ represent the population density of the predator and the prey at time t with diffusion rates d_1 and d_2, respectively. We suppose that the two diffusion rate are positive and equal, but not necessary constants. a denote the death rate of the predator u. The constant r is called the intrinsic growth rate of the prey v. The constants δ is the conversion rate of the predator. The term βu measures the mutual interference between predators.

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The Beddington-DeAngelis type functional response term $\frac{\delta uv}{\beta u+v+\alpha}$ was proposed by Beddington [1] and DeAngelis [2]. They introduced the following predator-prey model with this functional response term

$$\begin{cases} \dot{x} = rx - \theta x^2 - \dfrac{\gamma xy}{a+bx+cy}, \\ \dot{y} = -dy + \dfrac{\delta xy}{a+bx+cy}. \end{cases} \tag{1.2}$$

Huang et al. [3, 4] introduced a class of virus dynamics model with intracellular delay and nonlinear Beddington-DeAngelis infection rate. Liu and Kong [5] considered the dynamics of a predator-prey system with Beddington-DeAngelis functional response and delays.

Besides the Beddington-DeAngelis and Tanner type functional responses term mentioned above, there exist several other famous functional responses, such as well-known Holling type (I, II, III, IV), Hassel-Verley type and Monod-Haldane type functional responses and so on. Some researchers investigated and raised some well-known open questions for structured predator-prey models with different types of functional responses. Particularly, Peng and Wang [6] studied the steady states of a diffusive Holling-Tanner type predator-prey system

$$\begin{cases} u_t - d_1\Delta u = au - u^2 - \dfrac{uv}{u+m}, & x \in \Omega, t \in (0,\infty), \\ v_t - d_2\Delta v = rv - \dfrac{v^2}{\gamma u}, & x \in \Omega, t \in (0,\infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t \in (0,\infty), \\ u(x,0) = u_0(x) > 0, \quad v(x,0) = v_0(x) \geq 0, & x \in \overline{\Omega}, \end{cases} \tag{1.3}$$

They proved the existence and non-existence of positive non-constant steady solutions for (1.3), and argued that (1.3) possesses no positive non-constant steady solution under a certain condition. In the another paper [7], the authors studied the stability of diffusive predator-prey model of Holling-Tanner type (1.3) by the construction of a standard linearization procedure and a Lyapunov function. Chen and Shi [8] focused attention on the steady states of (1.3). They applied the defined iteration and comparison principle sequences to prove the global asymptotic stability. Their scientific research achievement improves the earlier one proposed by Wang and Peng [7] which used Lyapunov method. We also note here that the (non-spatial) kinetic equation of system (1.3) was first introduced by May [10] and Tanner [9], see also [11, 12] and references therein.

Recently, Qi and Zhu [13] studied the global stability of a reaction-diffusion system of predator-prey model (1.3). Indeed, they established improved global asymptotic stability of the unique positive equilibrium solution in [13]. Besides the papers mentioned above, one can see [14–20] for more detailed information and biological significances of the studied system.

In the present paper by incorporating the ratio-dependent Beddington-DeAngelis functional response and diffusion term into system (1.3), motivated by the previous works [13], we will study the global stability of the positive equilibrium solution. Therefore, we argue that it is interesting, beneficial and significant to study the global asymptotic stability of (1.1) since it possesses biological implications and extends the former researches.

It is known via a direct and simple computation that (1.1) possesses a unique positive equilibrium $(\overline{u}, \overline{v})$, where

$$\overline{u} = \frac{a\left(\beta + \frac{r}{\gamma}\right) - \alpha b - \delta\frac{r}{\gamma} + \sqrt{\left(a\left(\beta + \frac{r}{\gamma}\right) - \alpha b - \delta\frac{r}{\gamma}\right)^2 + 4ab\alpha\left(\beta + \frac{r}{\gamma}\right)}}{2b\left(\beta + \frac{r}{\gamma}\right)}, \quad \overline{v} = \frac{r}{\gamma}\overline{u}.$$

2. Main result

We define $w = \frac{v}{u}$, then we obtain

$$
\begin{aligned}
w_t &= \frac{v_t u - v u_t}{u^2} = \frac{v_t}{u} - \frac{u_t v}{u^2},\\
\nabla w &= \frac{\nabla v \cdot u - \nabla u \cdot v}{u^2} = \frac{\nabla v}{u} - \frac{\nabla u \cdot v}{u^2},\\
\Delta w &= \frac{\Delta v u^3 + u^2 \nabla u \cdot \nabla v - u^2 v \Delta u - u^2 \nabla u \cdot \nabla v - (\nabla v \cdot u - \nabla u \cdot v)\, 2u \nabla u}{u^4}\\
&= \frac{\Delta v}{u} - \frac{v \Delta u}{u^2} - \frac{2\nabla u \cdot \nabla v}{u^2} + \frac{2v|\nabla u|^2}{u^3}.
\end{aligned}
$$

Therefore the equation satisfied by $w(x,t)$ is

$$
\begin{aligned}
w_t - d\Delta w =& \left(\frac{v_t}{u} - \frac{u_t v}{u^2}\right) - d\left(\frac{\Delta v}{u} - \frac{v\Delta u}{u^2} - \frac{2\nabla u \cdot \nabla v}{u^2} + \frac{2v|\nabla u|^2}{u^3}\right)\\
=& \frac{v_t - d\Delta v}{u} - \frac{v(u_t - d\Delta u)}{u^2} + 2d\frac{\nabla u}{u}\left(\frac{\nabla v}{u} - \frac{v \nabla u}{u^2}\right)\\
=& \frac{v(r - \gamma\frac{v}{u})}{u} - \frac{v\left(au - bu^2 - \frac{\delta u v}{\beta u + v + \alpha}\right)}{u^2} + 2d\frac{\nabla u}{u}\cdot \nabla w\\
=& w\left(r - a + bu + w\left(\frac{\delta u}{\beta u + v + \alpha} - \gamma\right)\right) + 2d\frac{\nabla u}{u} \cdot \nabla w.
\end{aligned}
\tag{2.1}
$$

Theorem 2.1. *Suppose $d = d(x,t)$ is strictly positive, bounded and continuous in $\Omega \times [0, +\infty)$, $a, b, r, \alpha, \beta, \gamma$, and δ are positive constants, $r < a$, then the positive equilibrium solution $(\overline{u}, \overline{v})$ is globally asymptotically stable in the sense that every solution $u(x,t)$ of (1.1) satisfies*

$$\lim_{t\to\infty} (u(x,t), v(x,t)) = (\overline{u}, \overline{v})$$

uniformly in $x \in \Omega$.

Proposition 2.2. *Suppose $r < a$ and $\varepsilon_1 > 0$ small. There exists a sufficiently large constant $T > 0$ such that the solution u of (1.1) satisfies*

$$u \le \overline{u}_2(\varepsilon_1) \equiv \frac{a\beta - (\alpha + \frac{r}{\gamma}\underline{u}_1)b + \sqrt{\left[a\beta - (\alpha + \frac{r}{\gamma}\underline{u}_1)b\right]^2 + 4ab\beta(\alpha + \frac{r}{\gamma}\underline{u}_1)}}{2b\beta} + O(\varepsilon_1),$$

for $x \in \Omega$ and $t \ge T$, where

$$
\begin{aligned}
\underline{u}_1 =& \frac{a(\beta + \underline{w}_1(\varepsilon_1)) - \alpha b - \delta \underline{w}_1(\varepsilon_1) + \sqrt{\left[a(\beta + \underline{w}_1(\varepsilon_1)) - \alpha b - \delta \underline{w}_1(\varepsilon_1)\right]^2 + 4ab\alpha(\beta + \underline{w}_1(\varepsilon_1))}}{2b(\beta + \overline{w}_1(\varepsilon_1))},\\
\overline{w}_1 =& \frac{(r+\delta)\overline{u}_1 + b\overline{u}_1^2 - (a + \beta\gamma)\overline{u}_1 - \alpha\gamma}{2\gamma\overline{u}_1}\\
&+ \frac{\sqrt{\left[(a+\beta\gamma)\overline{u}_1 - (r+\delta)\overline{u}_1 - b\overline{u}_1^2 + \alpha\gamma\right]^2 + 4\gamma\overline{u}_1\left[\alpha(r-a) + (r\beta - a\beta + b\alpha + b\beta\overline{u}_1)\overline{u}_1\right]}}{2\gamma\overline{u}_1}.
\end{aligned}
$$

and $\overline{u}_1 \equiv \frac{a}{b}$.

Proof of Proposition 2.2

Proof. Since $u > 0, v \geq 0$, a simple computation gives

$$u_t - d\Delta u \leq u(a - bu), \quad \text{in } \Omega \times (0, \infty).$$

By a simple comparison and the well established fact that any positive solution of

$$\begin{cases} u_t - d_1\Delta u = u(a - bu), & x \in \Omega, t \in (0, \infty), \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, t \in (0, \infty), \end{cases}$$

converges to the asymptotic stable equilibrium $\frac{a}{b}$ as $t \to \infty$, we obtain that for all $\varepsilon_1 > 0$, there exists a constant $t_1 > 0$, such that

$$u(x, t) < \overline{u}_1(\varepsilon_1) \equiv \frac{a}{b} + \frac{\varepsilon_1}{5} \tag{2.2}$$

for $x \in \Omega$ and $t \geq t_1$. Hence

$$w_t - d\Delta w \leq w\left(r - a + b\overline{u}_1(\varepsilon_1) + w\left(\frac{\delta\overline{u}_1(\varepsilon_1)}{\beta\overline{u}_1(\varepsilon_1) + \overline{u}_1(\varepsilon_1)w + \alpha} - \gamma\right)\right) + 2d\frac{\nabla u}{u} \cdot \nabla w.$$

for $x \in \Omega$ and $t \geq t_1$.

It is obvious that the following equation about $\mathcal{W}(t)$

$$\mathcal{W}_t = \mathcal{W}\left(r - a + b\overline{u}_1(\varepsilon_1) + \mathcal{W}\left(\frac{\delta\overline{u}_1(\varepsilon_1)}{\beta\overline{u}_1(\varepsilon_1) + \overline{u}_1(\varepsilon_1)\mathcal{W} + \alpha} - \gamma\right)\right) \tag{2.3}$$

possesses three solutions:

$$\begin{aligned} \mathcal{W}_0 =& 0, \\ \mathcal{W}_{1,2} =& \frac{(r + \delta)\overline{u}_1(\varepsilon_1) + b\overline{u}_1^2(\varepsilon_1) - (a + \beta\gamma)\overline{u}_1(\varepsilon_1) - \alpha\gamma}{2\gamma\overline{u}_1(\varepsilon_1)} \\ &\pm \frac{\sqrt{\left[(a + \beta\gamma)\overline{u}_1(\varepsilon_1) - (r + \delta)\overline{u}_1(\varepsilon_1) - b\overline{u}_1^2(\varepsilon_1) + \alpha\gamma\right]^2 + 4\gamma\overline{u}_1(\varepsilon_1)(r - a + b\overline{u}_1)(\alpha + \beta\overline{u}_1)}}{2\gamma\overline{u}_1(\varepsilon_1)}. \end{aligned} \tag{2.4}$$

It is obvious that $\mathcal{W}_1(t)$ is the unique asymptotically stable positive equilibrium point of (2.3), and $\mathcal{W}_0(t) = 0$ is unstable. Since the trajectories of (2.3) cannot cross the x-axis, then all positive solutions $\mathcal{W}(t)$ of (2.3) will converge to the unique positive asymptotically stable equilibrium point $\mathcal{W}_1(t)$. By a simple comparison argument, we obtain that there possesses a positive constant $t_2 \geq t_1$ satisfies

$$0 < w(x, t) = \frac{v(x, t)}{u(x, t)} \leq \overline{w}_1(\varepsilon_1) \equiv \mathcal{W}_1 + \frac{\varepsilon_1}{5} \tag{2.5}$$

for all $x \in \Omega$ and $t \geq t_2$. Therefore, $v \leq \overline{w}_1(\varepsilon_1)u$, and

$$u_t - d\Delta u \geq u(a - bu) - \frac{\delta\overline{w}_1(\varepsilon_1)u}{\frac{\alpha}{u} + \beta + \overline{w}_1(\varepsilon_1)} = \frac{u\left[(a - bu)\left(\frac{\alpha}{u} + \beta + \overline{w}_1(\varepsilon_1)\right) - \delta\overline{w}_1(\varepsilon_1)\right]}{\frac{\alpha}{u} + \beta + \overline{w}_1(\varepsilon_1)}$$

for all $x \in \Omega$ and $t \geq t_2$. Let

$$(a - bu)\left(\frac{\alpha}{u} + \beta + \overline{w}_1(\varepsilon_1)\right) - \delta\overline{w}_1(\varepsilon_1) = 0,$$

then we obtain only one positive solution

$$u_+ = \frac{a(\beta+\overline{w}_1(\varepsilon_1)) - \alpha b - \delta\overline{w}_1(\varepsilon_1) + \sqrt{[a(\beta+\overline{w}_1(\varepsilon_1)) - \alpha b - \delta\overline{w}_1(\varepsilon_1)]^2 + 4ab\alpha(\beta+\overline{w}_1(\varepsilon_1))}}{2b(\beta+\overline{w}_1(\varepsilon_1))},$$

which is a stable equilibrium point of the corresponding ordinary differential equation

$$u_t = \frac{u\left[(a-bu)\left(\frac{\alpha}{u}+\beta+\overline{w}_1(\varepsilon_1)\right) - \delta\overline{w}_1(\varepsilon_1)\right]}{\frac{\alpha}{u}+\beta+\overline{w}_1(\varepsilon_1)}. \tag{2.6}$$

Hence, all positive solution of (2.6) will converge to u_+, which means that there exists $t_3 > t_2$ such that

$$\begin{aligned} &u \geq \underline{u}_1(\varepsilon_1) \equiv u_+ - \frac{\varepsilon_1}{5} \\ =&\frac{a(\beta+\overline{w}_1(\varepsilon_1)) - \alpha b - \delta\overline{w}_1(\varepsilon_1) + \sqrt{[a(\beta+\overline{w}_1(\varepsilon_1)) - \alpha b - \delta\overline{w}_1(\varepsilon_1)]^2 + 4ab\alpha(\beta+\overline{w}_1(\varepsilon_1))}}{2b(\beta+\overline{w}_1(\varepsilon_1))} - \frac{\varepsilon_1}{5}, \end{aligned} \tag{2.7}$$

for all $x \in \Omega$ and $t \geq t_3$. On the other hand, by applying the second equation of (1.1), we have

$$v_t - d\Delta v \geq rv - \gamma\frac{v^2}{\underline{u}_1(\varepsilon_1)}$$

for all $x \in \Omega$ and $t \geq t_3$. Therefore, there exists a constant $t_4 > t_3$ such that

$$v \geq \underline{v}_1(\varepsilon_1) = \frac{r\underline{u}_1(\varepsilon_1)}{\gamma} - \frac{\varepsilon_1}{5} \tag{2.8}$$

for all $x \in \Omega$ and $t \geq t_4$. Setting the estimate $v \geq \underline{v}_1(\varepsilon_1)$ into the first equation of (1.1), we have

$$u_t - d\Delta u \leq au - bu^2 - \frac{\delta u\underline{v}_1(\varepsilon_1)}{\beta u + \underline{v}_1(\varepsilon_1) + \alpha} = \frac{u\left[(a-bu)(\beta u + \underline{v}_1(\varepsilon_1) + \alpha) - \delta\underline{v}_1(\varepsilon_1)\right]}{\beta u + \underline{v}_1(\varepsilon_1) + \alpha}$$

The quadratic equation of one variable

$$(a-bu)(\beta u + \underline{v}_1(\varepsilon_1) + \alpha) - \delta\underline{v}_1(\varepsilon_1) = 0$$

possesses only one positive solution

$$u^+ = \frac{a\beta - (\alpha+\underline{v}_1(\varepsilon_1))b + \sqrt{\left[a\beta - (\alpha+\underline{v}_1(\varepsilon_1))b\right]^2 + 4\left[a(\alpha+\underline{v}_1(\varepsilon_1)) - \delta\underline{v}_1(\varepsilon_1)\right]b\beta}}{2b\beta} \tag{2.9}$$

By comparison principle, we can draw a conclusion that there exists $t_5 > t_4$ such that if $t \geq t_5$,

$$\begin{aligned} &u \leq \overline{u}_2(\varepsilon_1) \equiv u^+ + \frac{\varepsilon_1}{5} \\ =&\frac{a\beta - (\alpha+\underline{v}_1(\varepsilon_1))b + \sqrt{\left[a\beta - (\alpha+\underline{v}_1(\varepsilon_1))b\right]^2 + 4\left[a(\alpha+\underline{v}_1(\varepsilon_1)) - \delta\underline{v}_1(\varepsilon_1)\right]b\beta}}{2b\beta} + \frac{\varepsilon_1}{5}. \end{aligned} \tag{2.10}$$

The expression of $\overline{u}_2(\varepsilon_1)$ and that of $\underline{u}_1(\varepsilon_1)$ and $\overline{w}_1(\varepsilon_1)$ are valid by a simple computation using (2.2), (2.4), (2.5) and (2.7)-(2.10). The proof is complete. □

By repeating the above step, there exists a sufficiently large T such that when $t \geq T$,

$$\begin{aligned} u \leq\, & \overline{u}_{n+1}(\varepsilon_1) \\ \equiv & \frac{a\beta - (\alpha + \underline{v}_n(\varepsilon_1))b + \sqrt{\left[a\beta - (\alpha + \underline{v}_n(\varepsilon_1))b\right]^2 + 4\left[a(\alpha + \underline{v}_n(\varepsilon_1)) - \delta \underline{v}_n(\varepsilon_1)\right] b\beta}}{2b\beta} + \frac{\varepsilon_1}{5}, \\ u \geq\, & \underline{u}_n(\varepsilon_1) \\ \equiv & \frac{a(\beta + \overline{w}_n(\varepsilon_1)) - \alpha b - \delta \overline{w}_n(\varepsilon_1) + \sqrt{[a(\beta + \overline{w}_n(\varepsilon_1)) - \alpha b - \delta \overline{w}_n(\varepsilon_1)]^2 + 4ab\alpha(\beta + \overline{w}_n(\varepsilon_1))}}{2b(\beta + \overline{w}_n(\varepsilon_1))} - \frac{\varepsilon_1}{5} \end{aligned}$$

uniformly in Ω for any positive integer n, where

$$\begin{aligned} \underline{v}_n(\varepsilon_1) = & \frac{r\underline{u}_n(\varepsilon_1)}{\gamma} - \frac{\varepsilon_1}{5}, \\ \overline{w}_n = & \frac{(r+\delta)\overline{u}_n + b\overline{u}_n^2 - (a + \beta\gamma)\overline{u}_n - \alpha\gamma}{2\gamma \overline{u}_n} \\ & + \frac{\sqrt{\left[(a+\beta\gamma)\overline{u}_n - (r+\delta)\overline{u}_n - b\overline{u}_n^2 + \alpha\gamma\right]^2 + 4\gamma\overline{u}_n\left[\alpha(r-a) + (r\beta - a\beta + b\alpha + b\beta\overline{u}_n)\overline{u}_n\right]}}{2\gamma\overline{u}_n}. \end{aligned}$$

When setting $\varepsilon_1 = 0$, we obtain

$$\begin{aligned} \overline{u}_{n+1} = & \frac{a\beta - (\alpha + \frac{r}{\gamma}\underline{u}_n)b + \sqrt{\left[a\beta - (\alpha + \frac{r}{\gamma}\underline{u}_n)b\right]^2 + 4\left[a(\alpha + \frac{r}{\gamma}\underline{u}_n) - \delta\frac{r}{\gamma}\underline{u}_n\right] b\beta}}{2b\beta}, \\ \underline{u}_n = & \frac{a(\beta + \overline{w}_n) - \alpha b - \delta \overline{w}_n + \sqrt{[a(\beta + \overline{w}_n) - \alpha b - \delta \overline{w}_n]^2 + 4ab\alpha(\beta + \overline{w}_n)}}{2b(\beta + \overline{w}_n)}, \\ \underline{v}_n = & \frac{r}{\gamma}\underline{u}_n \end{aligned}$$

and $\overline{u}_1 = \frac{a}{b}$, $\overline{u}_1 > \overline{u}$, $\underline{u}_1 < \overline{u}$. It is known by a direct calculation with the first equality proposed above that

$$\begin{aligned} & \left[a\beta - \left(\alpha + \frac{r}{\gamma}\underline{u}_1\right)b\right]^2 + 4\left[a\left(\alpha + \frac{r}{\gamma}\underline{u}_1\right) - \delta\frac{r}{\gamma}\underline{u}_1\right]b\beta \\ = & (a\beta + \alpha b)^2 + \frac{r^2 b^2 \underline{u}_1^2}{\gamma^2} + 2(a\beta + \alpha b)\frac{rb}{\gamma}\underline{u}_1 - \frac{4\delta b\beta r \underline{u}_1}{\gamma} \\ < & \left[a\beta + \left(\alpha + \frac{r}{\gamma}\underline{u}_1\right)b\right]^2. \end{aligned}$$

Therefore,

$$\overline{u}_2 = \frac{a\beta - (\alpha + \frac{r}{\gamma}\underline{u}_1)b + \sqrt{\left[a\beta - (\alpha + \frac{r}{\gamma}\underline{u}_1)b\right]^2 + 4\left[a(\alpha + \frac{r}{\gamma}\underline{u}_1) - \delta\frac{r}{\gamma}\underline{u}_1\right] b\beta}}{2b\beta} < \frac{2a\beta}{2b\beta} = \frac{a}{b} = \overline{u}_1.$$

Then, by induction, we can obtain that the sequence $\{\overline{u}_n\}$ is decreasing as $n \to \infty$. Similarly, since

$$\begin{aligned} \overline{w}_n = & \frac{r+\delta}{2\gamma} + \frac{b\overline{u}_n}{2\gamma} - \frac{a+\beta\gamma}{2\gamma} - \frac{\alpha}{2\overline{u}_n} \\ & + \sqrt{\left(\frac{r+\delta}{2\gamma} + \frac{b\overline{u}_n}{2\gamma} - \frac{a+\beta\gamma}{2\gamma} - \frac{\alpha}{2\overline{u}_n}\right)^2 + \frac{1}{\gamma}\left(\frac{\alpha(r-a)}{\overline{u}_n} + r\beta - a\beta + b\alpha + b\beta\overline{u}_n\right)}, \end{aligned}$$

and

$$\underline{u}_n = \frac{1}{2}\left[\left(\frac{a}{b} - \frac{\alpha b + \delta\overline{w}_n}{b(\beta + \overline{w}_n)}\right) + \sqrt{\left(\frac{a}{b} + \frac{\alpha b + \delta\overline{w}_n}{b(\beta + \overline{w}_n)}\right)^2 - \frac{4a\delta\overline{w}_n}{b^2(\beta + \overline{w}_n)}}\right],$$

where $r < a$, we obtain that the sequence $\{\overline{w}_n\}$ is decreasing and the sequence $\{\underline{u}_n\}$ is increasing. Hence, we obtain

$$\lim_{n\to\infty} \overline{u}_n = \lim_{n\to\infty} \underline{u}_n = \overline{u}$$

under the assumption of Theorem 2.1. Thus, we obtain

$$\lim_{n\to\infty} \overline{v}_n = \lim_{n\to\infty} \underline{v}_n = \overline{v}.$$

Now, we prove $\lim_{t\to\infty} (u(x,t), v(x,t)) = (\overline{u}, \overline{v})$ uniformly in $x \in \Omega$.

Proof of Theorem 2.1

Proof. For any $\varepsilon > 0$, there exists $N_1 \in \mathbb{Z}^+$ such that when $n > N_1$,

$$\left|\overline{u}_n - \overline{u}\right| + \left|\underline{u}_n - \overline{u}\right| < \frac{\varepsilon}{4}. \tag{2.11}$$

We can choose a sufficiently small positive number $\varepsilon_1 > 0$ such that

$$\left|\overline{u}_{N_1}(\varepsilon_1) - \overline{u}_{N_1}\right| + \left|\underline{u}_{N_1}(\varepsilon_1) - \underline{u}_{N_1}\right| < \frac{\varepsilon}{4}. \tag{2.12}$$

For any $\varepsilon > 0$, there exists $N_2 \in \mathbb{Z}^+$ such that when $n > N_2$,

$$\left|\overline{v}_n - \overline{v}\right| + \left|\underline{v}_n - \overline{v}\right| < \frac{\varepsilon}{4}. \tag{2.13}$$

We can choose a sufficiently small positive number $\varepsilon_2 > 0$ such that

$$\left|\overline{v}_{N_2}(\varepsilon_1) - \overline{v}_{N_2}\right| + \left|\underline{v}_{N_2}(\varepsilon_1) - \underline{v}_{N_2}\right| < \frac{\varepsilon}{4}. \tag{2.14}$$

Furthermore, there exists $t_{M_1}, t_{M_2} \gg a$ such that when $t \geq t_{M_1}$ and $t \geq t_{M_2}$, we have

$$\underline{u}_{N_1}(\varepsilon_1) \leq u(x,t) \leq \overline{u}_{N_1}(\varepsilon_1) \quad \text{in } \Omega$$
$$\underline{v}_{N_2}(\varepsilon_2) \leq u(x,t) \leq \overline{v}_{N_2}(\varepsilon_2) \quad \text{in } \Omega$$

respectively.

Let $N = \max\{N_1, N_2\}$, $\epsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and $t_M = \max\{t_{M_1}, t_{M_2}\}$. Hence, by (2.11)- (2.14), when $t \geq t_M$, we obtain

$$\left|u(x,t) - \overline{u}\right| < \varepsilon \quad \text{and} \quad \left|v(x,t) - \overline{v}\right| < \varepsilon \quad \text{in } \Omega$$

This proves $\lim_{t\to\infty} u(x,t) = \overline{u}$ and $\lim_{t\to\infty} v(x,t) = \overline{v}$ uniformly in $x \in \Omega$. This completes the proof of Theorem 2.1. □

3. Generalization and future works

The method we introduce in this paper is motivated by [13, 21, 22] and can be used to several well-known reaction-diffusion system where the stability of a unique positive equilibrium solution is a significant question to be investigated. For instance, the following generalized Gierer-Meinhardt system,

$$\begin{cases} u_t = \varepsilon^2 \Delta u - u + \dfrac{u^p}{v^q}, & x \in \Omega, t \in (0, \infty), \\ \tau v_t = \Delta v - v + \dfrac{u^m}{v^s}, & x \in \Omega, t \in (0, \infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t \in (0, \infty), \\ u(x, 0) = u_0(x) > 0, \quad v(x, 0)) = v_0(x) \geq 0, & x \in \overline{\Omega}, \end{cases} \tag{3.1}$$

is an interesting model worth of research.

It will be interesting to see how can we corporate other interesting features such as time delay into our scheme.

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