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Convex Sequences of Higher Order

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Abstract. In this paper we study the class of convex sequences of higher order defined using the difference operators and investigate their properties. The notion of the convex sequence of order $r \in \mathbb{N}$ will be extended for r a real number. Some necessary and sufficient conditions such that a real sequence belongs to the class of convex sequences of higher order $r \in \mathbb{R}$ are introduced. Using different types of means we will investigate the convexity of higher order for real sequences.

1. Introduction

Let K be the set of real sequences. A sequence $(a_n)_{n\in\mathbb{N}} \in K$ is called convex if $a_{n+2} - 2a_{n+1} + a_n \geq 0$ for all $n \in \mathbb{N}$. Many generalizations of convexity of sequences were studied in the last years. A sequence $(a_n)_{n\in\mathbb{N}}$ is r-convex for a positive real number r if $L_r a_n \geq 0$ for all $n \in \mathbb{N}$, where the operator L_r is given by

$$L_r a_n = a_{n+2} - (1+r)a_{n+1} + ra_n.$$

This concept was introduced by Lacković and Jovanović [5]. Another generalization was introduced by Kocić [4] using the operator:

$$L_{p,q} a_n = a_{n+2} + (p+q)a_{n+1} + pqa_n,\ \ p, q \in \mathbb{R}.$$

The higher order of convexity was introduced using the difference operator of order r, $r \in \mathbb{N}$ defined as follows:

$$\Delta^r a_n = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} a_{n+k},\ n \in \mathbb{N}.$$

A sequence $(a_n)_{n\in\mathbb{N}}$ is called convex of order r if $\Delta^r a_n \geq 0$ (see [6]). Let K_r be the set of all convex real sequences of order r. The class of convex sequences of higher order has attracted the attention of many mathematicians. The reader should consult the papers of Lupaş [1]-[2], Nicová [7], Ţincu [11]-[12].

Denote $(x)_l = x(x+1)...(x+l-1)$ the Pochhammer symbols and $\lfloor x \rfloor$ the integer part of x. A natural question was raised whether the assertion $r \in \mathbb{N}$ for the convex sequences of order r could be extended to

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$\mathbb{R}$. An answer of this question was given in [10] considering the operator $\Delta^r : K \to \mathbb{R}$, defined by

$$\Delta^r a_n = (-1)^{\lfloor r \rfloor} \sum_{k=0}^{\infty} \frac{(-r)_k}{k!} a_{n+k} \ , \tag{1}$$

with the convention $\Delta^0 a_n = a_n$, for every $n \in \mathbb{N}$, where $\sum_{k=0}^{\infty} \frac{(-r)_k}{k!} a_{n+k}$ is a convergent series.

Let

$$M_r := \{(a_n)_{n\in\mathbb{N}} \in K | \Delta^r a_n \geq 0, \text{ for every } n \in \mathbb{N}\}$$

be the class of convex sequences of order r, $r \in \mathbb{R}$. Note that for $r \in \mathbb{N}$, it follows $M_r = K_r$.

A few properties of the operator Δ^r, $r \in \mathbb{R}$ are summarized in the following lemmas (see [10]).

Lemma 1.1. *[10] The operator Δ^r defined in the relation (1) verifies:*

i) Δ^r is a linear operator;

ii) $\Delta^{r+1} a_n = \Delta^r a_{n+1} - \Delta^r a_n$, for any $r \in \mathbb{R}_+$.

Proof. From the relation (1) it follows imediately that Δ^r is a linear operator. In order to prove the identity ii) we calculate

$$\begin{aligned}\Delta^r a_{n+1} - \Delta^r a_n &= (-1)^{\lfloor r \rfloor} \sum_{k=0}^{\infty} \frac{(-r)_k}{k!} a_{n+k+1} - (-1)^{\lfloor r \rfloor} \sum_{k=0}^{\infty} \frac{(-r)_k}{k!} a_{n+k} \\ &= (-1)^{\lfloor r \rfloor} \left\{ \sum_{k=1}^{\infty} \left[\frac{(-r)_{k-1}}{(k-1)!} - \frac{(-r)_k}{k!} \right] a_{n+k} - a_n \right\} = (-1)^{1+\lfloor r \rfloor} \sum_{k=0}^{\infty} \frac{(-r-1)_k}{k!} a_{n+k}.\end{aligned}$$

Therefore, $\Delta^{r+1} a_n = \Delta^r a_{n+1} - \Delta^r a_n$. □

In the condition that the series $\Delta^r a_n$, $r \in \mathbb{R}$ is convergent, the following result is obtained:

Lemma 1.2. *[10] Let $\mathbb{Z}$ be the set of integer numbers. For $r_1 \in \mathbb{Z}$, $r_2 \in \mathbb{R}_+$ and $n \in \mathbb{N}$, the identities*

$$\Delta^{r_1+r_2} a_n = \Delta^{r_1} \left(\Delta^{r_2} a_n \right) = \Delta^{r_2} \left(\Delta^{r_1} a_n \right), \tag{2}$$

hold.

Proof. We have

$$\begin{aligned}\Delta^{r_1} \left(\Delta^{r_2} a_n \right) &= (-1)^{\lfloor r_1 \rfloor} \sum_{k=0}^{\infty} \frac{(-r_1)_k}{k!} \Delta^{r_2} a_{n+k} = (-1)^{\lfloor r_1 \rfloor + \lfloor r_2 \rfloor} \sum_{k=0}^{\infty} \frac{(-r_1)_k}{k!} \left[\sum_{i=0}^{\infty} \frac{(-r_2)_i}{i!} a_{n+k+i} \right] \\ &= (-1)^{\lfloor r_1 \rfloor + \lfloor r_2 \rfloor} \sum_{k=0}^{\infty} c(-r_1, k) \left[\sum_{i=0}^{\infty} c(-r_2, i) a_{n+k+i} \right], \text{ where } c(r,k) = \frac{(r)_k}{k!}.\end{aligned} \tag{3}$$

Therefore, we can write

$$\begin{aligned}\Delta^{r_1} \left(\Delta^{r_2} a_n \right) &= (-1)^{\lfloor r_1 \rfloor + \lfloor r_2 \rfloor} \sum_{k=0}^{\infty} a_{n+k} \left[\sum_{i=0}^{k} c(-r_1, i) c(-r_2, k-i) \right] \\ &= (-1)^{\lfloor r_1 \rfloor + \lfloor r_2 \rfloor} \sum_{k=0}^{\infty} \frac{a_{n+k}}{k!} \sum_{i=0}^{k} \binom{k}{i} (-r_1)_i (-r_2)_{k-i}.\end{aligned}$$

Using the Chu-Vandermonde formula (see [3])

$$(x+y)_k = \sum_{i=0}^{k} \binom{k}{i} (x)_i (y)_{k-i},\ k \in \mathbb{N},$$

we get

$$\Delta^{r_1}(\Delta^{r_2} a_n) = (-1)^{\lfloor r_1 \rfloor + \lfloor r_2 \rfloor} \sum_{k=0}^{\infty} \frac{(-r_1 - r_2)_k}{k!} a_{n+k} = \Delta^{r_1+r_2} a_n, \text{ for any } \ n \in \mathbb{N}.$$

□

The aim of this paper is to extend the notion of convex sequence and to investigate the properties of this new class of convex sequences of higher order. The necessary and suficient conditions such that a real sequence belongs to the class of convex sequences of higher order $r \in \mathbb{R}$ are given. Considerable attention has been given by various authors to the study of mean-convex sequences. In the last section we are focused on the u-mean-convex sequences introduced by G. Toader [8]. We will prove that the set of convex sequences of higher order is a subset of u-mean-convex sequences of higher order.

2. Properties of convex sequences of higher-order

In this section we determine some necessary and sufficient conditions such that a real sequence to belong to the class M_r. In order to give these conditions we will calculate the value of the operator Δ^r, considering the real sequence $(a_n)_{n\in\mathbb{N}}$ with the general term defined as follows

$$a_n = \sum_{k=0}^{n} \frac{(r)_{n-k}}{(n-k)!} b_k \ , \quad n = 0, 1, ..., \text{ where the sequence } (b_n)_{n\in\mathbb{N}} \in K. \tag{4}$$

We have $\Delta^r a_n = (-1)^{\lfloor r \rfloor} \sum_{k=0}^{\infty} \frac{(-r)_k}{k!} a_{n+k} = (-1)^{\lfloor r_1 \rfloor} \sum_{k=0}^{\infty} c(-r,k) \sum_{j=0}^{n+k} c(r, n+k-j) b_j$, where $c(r,k)$ was defined in (3). Therefore, we get

$$\begin{aligned}
\Delta^r a_n &= (-1)^{\lfloor r \rfloor} \left[\sum_{k=0}^{\infty} c(-r,k) \sum_{j=0}^{n-1} c(r, n+k-j) b_j + \sum_{k=0}^{\infty} c(-r,k) \sum_{j=n}^{n+k} c(r, n+k-j) b_j \right] \\
&= (-1)^{\lfloor r \rfloor} \left[\sum_{j=0}^{n-1} b_j \sum_{k=0}^{\infty} c(-r,k) c(r, n+k-j) + \sum_{k=0}^{\infty} c(-r,k) \sum_{j=0}^{k} b_{j+n} c(r, k-j) \right] \\
&= (-1)^{\lfloor r \rfloor} \left[\sum_{j=0}^{n-1} b_j S(0, n-j) + \sum_{\nu=0}^{\infty} b_{n+\nu} S(\nu, 0) \right],
\end{aligned}$$

where $S(\nu,\mu) = \sum_{k=0}^{\infty} c(-r, k+\nu) c(r, k+\mu)$.

We will consider three cases:

Case 1. Let $r \in (0,1)$.

We have

$$
\begin{aligned}
S(0,n-j) &= \sum_{k=0}^{\infty} c(-r,k)c(r,n+k-j) = \frac{1}{\Gamma(r)\Gamma(1-r)} \sum_{k=0}^{\infty} \frac{(-r)_k}{k!} B(r+n+k-j,1-r) \\
&= \frac{1}{\Gamma(r)\Gamma(1-r)} \int_0^1 t^{-r}(1-t)^{r+n-j-1} \left[\sum_{k=0}^{\infty} \frac{(-r)_k}{k!}(1-t)^k \right] dt \\
&= \frac{1}{\Gamma(r)\Gamma(1-r)} \int_0^1 (1-t)^{r+n-j-1} dt = \frac{1}{\Gamma(r)\Gamma(1-r)} \cdot \frac{1}{r+n-j} = \frac{\sin r\pi}{\pi} \cdot \frac{1}{r+n-j}.
\end{aligned}
$$

For $\nu \geq 1$, we get

$$
\begin{aligned}
S(\nu,0) &= \sum_{k=0}^{\infty} c(-r,k+\nu)c(r,k) = \sum_{k=0}^{\infty} \frac{(-r)_{k+\nu}}{(k+\nu)!} \cdot \frac{(r)_k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(r)_k}{(k)!} \frac{(-r)\Gamma(k+\nu-r)}{\Gamma(1-r)} \frac{1}{\Gamma(k+\nu+1)} \frac{\Gamma(r+1)}{\Gamma(r+1)} = \frac{-1}{\Gamma(1-r)\Gamma(r)} \sum_{k=0}^{\infty} \frac{(r)_k}{k!} B(k+\nu-r,r+1) \\
&= -\frac{\sin \pi r}{\pi} \int_0^1 t^{r}(1-t)^{\nu-r-1} \left[\sum_{k=0}^{\infty} \frac{(r)_k}{k!}(1-t)^k \right] dt = -\frac{\sin \pi r}{\pi} \int_0^1 (1-t)^{\nu-r-1} dt = \frac{-\sin \pi r}{\pi(\nu-r)}.
\end{aligned}
$$

For $\nu = 0$, we have

$$
\begin{aligned}
S(0,0) &= \sum_{k=0}^{\infty} \frac{(-r)_k}{k!} \frac{(r)_k}{k!} = \sum_{k=0}^{\infty} \frac{(-r)_k}{k!} \cdot \frac{\Gamma(r+k)}{\Gamma(r)} \cdot \frac{1}{\Gamma(k+1)} = \frac{1}{\Gamma(r)\Gamma(1-r)} \sum_{k=0}^{\infty} \frac{(-r)_k}{k!} B(1-r,r+k) \\
&= \frac{1}{\Gamma(r)\Gamma(1-r)} \int_0^1 t^{-r}(1-t)^{r-1} \left[\sum_{k=0}^{\infty} \frac{(-r)_k}{k!}(1-t)^k \right] dt = \frac{\sin \pi r}{\pi} \int_0^1 (1-t)^{r-1} dt = \frac{\sin \pi r}{\pi r}.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\Delta^r a_n &= \sum_{j=0}^{n-1} b_j S(0,n-j) + b_n S(0,0) + \sum_{\nu=1}^{\infty} b_{n+\nu} S(\nu,0) = \frac{\sin \pi r}{\pi} \left[\sum_{j=0}^{n-1} \frac{b_j}{r+n-j} + \sum_{\nu=0}^{\infty} \frac{b_{n+\nu}}{r-\nu} \right] \\
&= \frac{\sin \pi r}{\pi} \left[\sum_{j=0}^{n-1} \frac{b_j}{r+n-j} + \sum_{\nu=n}^{\infty} \frac{b_\nu}{r+n-\nu} \right] = \frac{\sin \pi r}{\pi} \sum_{j=0}^{\infty} \frac{b_j}{n+r-j}. \qquad (5)
\end{aligned}
$$

Case 2. Let $r > 1$, $r \notin \mathbb{N}$.

Denote $m = \lfloor r \rfloor$, $m \in \mathbb{N}$ and $l = \{r\}$, $l \in (0,1)$, where $\lfloor \cdot \rfloor$ and $\{\cdot\}$ are the integer part and the fractional part, respectively. From (1), (2) and (5), we have

$$
\begin{aligned}
\Delta^r a_n &= \Delta^{m+l} a_n = \Delta^m \left(\Delta^l a_n \right) = \Delta^m \left(\frac{\sin \pi l}{\pi} \sum_{j=0}^{\infty} \frac{b_j}{n+l-j} \right) \\
&= (-1)^m \sum_{k=0}^{m} \frac{(-m)_k}{k!} \left[\frac{\sin \pi l}{\pi} \sum_{j=0}^{\infty} \frac{b_j}{n+k+l-j} \right] = \frac{\sin \pi l}{\pi} \sum_{j=0}^{\infty} b_j T_m(l,j),
\end{aligned}
$$

where $T_m(l,j) = (-1)^m \sum_{k=0}^{m} \frac{(-m)_k}{k!} \frac{1}{n+k+l-j}$.

According to the relation $\dfrac{1}{n+k+l-j}=\dfrac{1}{n+l-j}\cdot\dfrac{(n+l-j)_k}{(n+l-j+1)_k}$, we can write

$$T_m(l,j)=\frac{(-1)^m}{n+l-j}\sum_{k=0}^{m}\frac{(-m)_k(n+l-j)_k}{(n+l-j+1)_k}\frac{1^k}{k!}=\frac{(-1)^m}{n+l-j}{}_2F_1(-m,n+l-j;n+l-j+1;1).$$

In the following we will apply the identity ${}_2F_1(a,b;c;1)=\dfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$, for $c\neq 0,-1,-2,...$ and $Re(c-a-b)>0$. Considering $a=-m$, $b=n+l-j$, $c=n+l-j+1$, it follows $T_m(l,j)=\dfrac{(-1)^m m!}{(n+l-j)_{m+1}}$. Therefore, $\Delta^r a_n=\dfrac{\sin\pi l}{\pi}(-1)^m m!\displaystyle\sum_{j=0}^{\infty}\frac{b_j}{(n+l-j)_{m+1}}$.

Case 3. Let $r=p$, $p\in\mathbb{N}^*$.

Using (4) we can write $a_n=\displaystyle\sum_{k=0}^{n}b_k\frac{(p)_{n-k}}{(n-k)!}=\sum_{k=0}^{n}b_k\frac{(n-k+1)_{p-1}}{(p-1)!}$. Therefore,

$$\Delta^p a_n=(-1)^p\sum_{k=0}^{p}\frac{(-p)_k}{k!}a_{n+k}=(-1)^p\sum_{k=0}^{p}c(-p,k)\sum_{i=0}^{n+k}b_i c(n+k-i+1,p-1),$$

where $c(r,k)$ was defined in (3).

We have

$$\begin{aligned}\Delta^p a_n&=(-1)^p\left[\sum_{i=0}^{n}b_i\sum_{k=0}^{p}c(-p,k)c(n+k-i+1,p-1)+\sum_{i=1}^{p}b_{n+i}\sum_{k=i}^{p}c(-p,k)c(k-i+1,p-1)\right]\\&=\sum_{i=0}^{n}b_iS_1+\sum_{i=1}^{p}b_{n+i}S_2,\ \text{where}\\ S_1&:=(-1)^p\sum_{k=0}^{p}c(-p,k)c(n+k-i+1,p-1)\text{ and }S_2:=(-1)^p\sum_{k=i}^{p}c(-p,k)c(k-i+1,p-1).\end{aligned}$$

For the above sums we will obtain the following results:

$$\begin{aligned}S_1&=(-1)^p\sum_{k=0}^{p}(-1)^k\binom{p}{k}\frac{(n+k-i+1)_{p-1}}{(p-1)!}=\frac{1}{(p-1)!}\sum_{k=0}^{p}(-1)^{p-k}\binom{p}{k}\frac{\Gamma(n+k-i+p)}{\Gamma(n+k-i+1)}\frac{\Gamma(n-i+p)}{\Gamma(n-i+p)}\\&=\frac{1}{(p-1)!}\sum_{k=0}^{p}\binom{p}{k}\frac{(n-i+p)_k(-n+i-p)_{p-k}}{n-i+p}=\frac{1}{(p-1)!}\frac{(0)_p}{n-i+p}=0,\\ S_2&=(-1)^p\sum_{k=i}^{p}(-1)^k\binom{p}{k}\frac{(k-i+1)_{p-1}}{(p-1)!}=\frac{(-1)^p}{(p-1)!}\sum_{k=0}^{p-i}(-1)^{k+i}\binom{p}{k+i}(k+1)_{p-1}\\&=\frac{(-1)^p}{(p-1)!}\sum_{k=0}^{p-i}(-1)^{k+i}\binom{p-i}{k}\frac{\binom{p}{k+i}}{\binom{p-i}{k}}(k+1)_{p-1}=\frac{p}{(p-i)!}\sum_{k=0}^{p-i}(-1)^{k+p+i}\binom{p-i}{k}\frac{\Gamma(k+p)}{\Gamma(k+i+1)}\\&=\frac{1}{(p-i)!}\sum_{k=0}^{p-i}\binom{p-i}{k}(p)_k(-p)_{p-k-i}=0,\ \text{for } i\leq p-1.\end{aligned}$$

For $i=p$, we have $S_2=\dfrac{(1)_{p-1}}{(p-1)!}=1$. It follows that $\Delta^p a_n=b_{n+p}$.

These cases lead to the following result:

Lemma 2.1. *Let $(b_n)_{n\in\mathbb{N}}$ be a real sequence. If $a_n = \sum_{k=0}^{n} \frac{(r)_{n-k}}{(n-k)!} b_k$, $n = 0, 1, \dots,$ then*

$$\Delta^r a_n = \begin{cases} \dfrac{\sin \pi r}{\pi} \displaystyle\sum_{j=0}^{\infty} \frac{b_j}{n+r-j}, \ r \in (0,1), \\ (-1)^m \dfrac{\sin \pi l}{\pi} m! \displaystyle\sum_{j=0}^{\infty} \frac{b_j}{(n+l-j)_{m+1}}, \ r > 1, r \notin \mathbb{N}, \\ b_{r+n}, \ r \in \mathbb{N}, \end{cases}$$

where m and l are the integer part and the fractional part of r.

In the following we will raise the problem:

Problem 2.2. *Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence and $r \in \mathbb{R}_+$. What are the conditions such that*

$$\Delta^r a_n \geq 0, \ \text{for all } n \in \mathbb{N}. \tag{6}$$

As a response of this problem the next two theorems are proven.

Theorem 2.3. *A real sequence $(a_n)_{n\in\mathbb{N}}$ verify the relation (6) for all $r \in \mathbb{N}^*$ if and only if there exists a real sequence $(b_n)_{n\in\mathbb{N}}$ such that*

$$a_n = \sum_{k=0}^{n} \frac{(r)_{n-k}}{(n-k)!} b_k, \ \text{and } b_n \geq 0 \text{ for all } n \geq r. \tag{7}$$

Proof. From Lemma 2.1 it follows that the conditions (7) are sufficient such that $\Delta^r a_n \geq 0$. Now, we consider $\Delta^r a_n \geq 0$ for all $n \in \mathbb{N}$ and we construct recursively a real sequence $(b_n)_{n\in\mathbb{N}}$ as follows:

$$b_0 = a_0, \ b_n = a_n - \sum_{k=0}^{n-1} \frac{(r)_{n-k}}{(n-k)!} b_k.$$

The above relations lead to $b_0 = \Delta^0 a_0 \geq 0$, $b_n = \Delta^n a_{n-1} \geq 0$ and $a_n = \sum_{k=0}^{n} \frac{(r)_{n-k}}{(n-k)!} b_k$. □

In a similar way the following results can be obtained:

Theorem 2.4. *A real sequence $(a_n)_{n\in\mathbb{N}}$ verify the relation (6) for all $r \in \mathbb{R}_+ \setminus \mathbb{N}^*$ if and only if there exists a real sequence $(b_n)_{n\in\mathbb{N}}$ such that*

$$a_n = \sum_{k=0}^{n} \frac{(r)_{n-k}}{(n-k)!} b_k, \ \text{and} \ (-1)^m \frac{\sin \pi l}{\pi} m! \sum_{j=0}^{\infty} \frac{b_j}{(n+l-j)_{m+1}} \geq 0, \ \text{where } m = \lfloor r \rfloor, l = \{r\}.$$

3. The weighted arithmetic means of real sequence

Let $a = (a_n)_{n\in\mathbb{N}} \in K$. The sequence $(C_n)_{n\in\mathbb{N}}$ of Cesaro means is defined as follows

$$C_n = \frac{a_0 + \cdots + a_n}{n+1}.$$

Definition 3.1. *[9] A sequence $(a_n)_{n\in\mathbb{N}}$ is called mean-convex if the sequence $(C_n)_{n\in\mathbb{N}}$ of Cesaro means is convex.*

In [13] is considered the weighted mean:

$$A_n = \frac{p_0 a_0 + \cdots + p_n a_n}{p_0 + \cdots + p_n}, \text{ with } p_n > 0 \text{ for } n \geq 0.$$

Toader [8] proved that the sequence $(A_n)_{n\in\mathbb{N}}$ is convex for any convex sequence $(a_n)_{n\in\mathbb{N}}$ if and only there is $u > 0$ such that

$$p_n = p_0 \binom{u+n-1}{n}, \text{ for } n \geq 1,$$

where

$$\binom{u}{0} = 1, \binom{u}{n} = \frac{1}{n!}\prod_{k=0}^{n-1}(u-k), \text{ for } n \geq 1,\ u \in \mathbb{R}.$$

In this case A_n become

$$A_n = A_n^u = \frac{1}{\binom{n+u}{n}} \sum_{i=0}^{n} \binom{u+i-1}{i} a_i.$$

Definition 3.2. *[8] The sequence $(a_n)_{n\in\mathbb{N}}$ is called u-mean-convex if $(A_n^u)_{n\in\mathbb{N}}$ is convex.*

In the following we will study the convexity of higher order for the sequence $(A_n^u)_{n\in\mathbb{N}}$.

Theorem 3.3. *The sequence $(A_n^u)_{n\in\mathbb{N}}$ verify*

$$\Delta^r(A_n^u) = \sum_{j=0}^{n} c_j(n,r,u)\Delta^r(a_j),\ r \in \mathbb{R}_+,$$

where

$$c_j(n,r,u) = \begin{cases} \dfrac{n!}{(u+r+1)_n},\ j=0, \\ \dfrac{un!}{(u+r+1)_n} \cdot \dfrac{(u+r+1)_{j-1}}{j!},\ j=1,2,\ldots,n. \end{cases} \tag{8}$$

Proof. Using the following relation

$$a_j = \frac{u+j}{u}A_j^u - \frac{j}{u}A_{j-1}^u,$$

it follows

$$\begin{aligned}
\Delta^r a_j &= (-1)^{\lfloor r \rfloor} \sum_{i=0}^{\infty} \frac{(-r)_i}{i!} a_{j+i} = (-1)^{\lfloor r \rfloor} \sum_{i=0}^{\infty} \frac{(-r)_i}{i!} \left[\frac{u+j+i}{u} A_{j+i}^u - \frac{j+i}{u} A_{j+i-1}^u \right] \\
&= \frac{(-1)^{\lfloor r \rfloor}}{u} \left\{ \sum_{i=0}^{\infty} \left[\frac{(-r)_i}{i!}(u+r)A_{j+i}^u + j\frac{(-r)_i}{i!}\frac{r+1}{i+1}A_{j+i}^u \right] - jA_{j-1}^u \right\} \\
&= \frac{u+r}{u}\Delta^r A_j^u - j\frac{(-1)^{\lfloor r \rfloor}}{u} \sum_{i=0}^{\infty} \frac{(-r-1)_{i+1}}{(i+1)!} A_{j+i}^u - j\frac{(-1)^{\lfloor r \rfloor}}{u} A_{j-1}^u \\
&= \frac{u+r}{u}\Delta^r A_j^u - \frac{j}{u}(-1)^{\lfloor r \rfloor} \sum_{i=1}^{\infty} \frac{(-r-1)_i}{i!} A_{j+i-1}^u - j\frac{(-1)^{\lfloor r \rfloor}}{u} A_{j-1}^u \\
&= \frac{u+r}{u}\Delta^r A_j^u - \frac{j}{u}(-1)^{\lfloor r \rfloor} \sum_{i=0}^{\infty} \frac{(-r-1)_i}{i!} A_{j+i-1}^u = \frac{u+r}{u}\Delta^r A_j^u + \frac{j}{u}\Delta^{r+1}A_{j-1}.
\end{aligned}$$

But

$$\Delta^{r+1}A_{j-1}^u = \Delta^r A_j^u - \Delta^r A_{j-1}^u.$$

Therefore, we have

$$\Delta^r a_j = \frac{u+r}{u}\Delta^r A_j^u + \frac{j}{u}\Delta^r A_j^u - \frac{j}{u}\Delta^r A_{j-1}^u = \frac{u+r+j}{u}\Delta^r A_j^u - \frac{j}{u}\Delta^r A_{j-1}^u.$$

From the above relation we get

$$\frac{(r+u+1)_{j-1}}{j!}\Delta^r a_j = \frac{(r+u+1)_j}{uj!}\Delta^r A_j^u - \frac{(r+u+1)_{j-1}}{(j-1)!u}\Delta^r A_{j-1}^u.$$

If we sum for $j=\overline{1,n}$, it follows

$$\sum_{j=1}^{n}\frac{(r+u+1)_{j-1}}{j!}\Delta^r a_j = \frac{(r+u+1)_n}{un!}\Delta^r A_n^u - \frac{1}{u}\Delta^r A_0^u.$$

Since $\Delta^r A_0^u = \Delta^r a_0$, we have

$$\Delta^r A_n^u = \frac{n!}{(r+u+1)_n}\Delta^r a_0 + \sum_{j=1}^{n}\frac{un!}{(u+r+1)_n}\frac{(r+u+1)_{j-1}}{j!}\Delta^r a_j.$$

□

Corollary 3.4. *The coefficients (8) verify*

i) $c_j(n,r,u) > 0$ *for* $r\in\mathbb{R}_+$, $j=\overline{0,n}$*;*

ii) $\displaystyle\sum_{j=0}^{n} c_j(n,r,u) = \frac{rn!}{(u+r)(u+r+1)_n} + \frac{u}{u+r}.$

Proof. In order to prove the condition ii) we calculate:

$$\sum_{j=0}^{n} c_j(n,r,u) = \frac{n!}{(u+r+1)_n} + \frac{un!}{(u+r+1)_n}\sum_{j=1}^{n}\frac{(u+r+1)_{j-1}}{j!}$$

$$= \frac{n!}{(u+r+1)_n} + \frac{un!}{(u+r+1)_n}\frac{1}{u+r}\left\{\sum_{j=0}^{n}\frac{(u+r)_j}{j!} - 1\right\}$$

$$= \frac{n!}{(u+r+1)_n} + \frac{un!}{(u+r+1)_n}\frac{1}{u+r}\left\{\frac{1}{n!}(u+r+1)_n - 1\right\} = \frac{rn!}{(u+r)(u+r+1)_n} + \frac{u}{u+r}.$$

□

Theorem 3.5. *Let* $(a_n)_{n\in\mathbb{N}}\in K$*. Then*

i) $A_n^u(M_r)\subseteq M_r$*, for* $r\geq 0$*;*

ii) If there exists $A\in\mathbb{R}_+$*, such that*

$$|\Delta^r a_n| < A, \text{ for } n = 0,1,\dots,$$

then

$$|\Delta^r A_n^u| \leq A\left(\frac{rn!}{(u+r)(u+r+1)_n} + \frac{u}{u+r}\right).$$

Thus, all convex sequences of higher order are u-mean-convex sequence of higher order. Therefore, the set of convex sequences of higher order is a proper subset of u-mean-convex sequences of higher order.

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References

[1] A. Lupaş, *On convexity preserving matrix transformations*, Univ. Beograd, Publ. Elektrotehn. Fak. Ser. Mat.Fiz. No. 634-No. 677(1979), 189-191.

[2] A. Lupaş, *On the means of convex sequences,* Gazeta Matematică, Seria (A), Anul IV, Nr.1-2,(1983), 90-93.

[3] A. Lupaş , C. Manole, *Capitole de Analiză Numerică*, Editura Universităţii Lucian Blaga din Sibiu , Colectia Facultăţii de Stiinţe-Seria Matematica 3 , 1994.

[4] L. M. Kocić, *On generalized convexity preserving matrix transformation*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. 762 (1982), 3-10.

[5] I. B. Lacković, M. R. Jovanović, *On a class of real sequences which satisfy a difference inequality*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. 678 (1980), 99-104.

[6] I. B. Lackovic, M.K. Ljubisa, *Invariant transformation of some sequence classes*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 678-715 (1980), 117-126.

[7] K. Nicová, *A note on higher-order convexity of sequences*, International Journal of Pure and Applied Mathematics, Volume 83, No. 1 (2013), 81-90.

[8] G. Toader, *Starshaped sequences*, Revue d'Analyse Numérique et de Théorie de l'Approximation, 14(2), (1985), 147-151.

[9] G. Toader, *A hierarchy of convexity for sequences*, L'Analyse Num. et Th. de l'Approx., 12(1983), 2, 187-192.

[10] I. Ţincu, *Linear transformations and summability methods of real sequences*, Ph.D. Thesis, Babeş Bolyai University of Cluj-Napoca, 2005.

[11] I. Ţincu, *On some p-convex sequences*, Acta Universitatis Apulensis, No. 11 (2006), 249-257.

[12] I. Ţincu, *On the means sequences,* General Mathematics,14(3), (2006), 23-26.

[13] P.M. Vasić, P.M., J.D. Kećkić, I.B. Lacković, Z.M. Mitrović, *Some properties of arithmetic means of real sequences*, Mat. Vesnik, 9(24), (1972), 205-212.