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$f-$Lacunary Statistical Convergence and Strong $f-$Lacunary Summability of Order α

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Abstract. The main object of this article is to introduce the concepts of $f-$lacunary statistical convergence of order α and strong $f-$lacunary summability of order α of sequences of real numbers and give some inclusion relations between these spaces.

1. Introduction

In 1951, Steinhaus [33] and Fast [18] introduced the concept of statistical convergence and later in 1959, Schoenberg [32] reintroduced independently. Bhardwaj and Dhawan [3], Caserta et al. [4], Connor [5], Çakallı [10], Çınar et al. [11], Çolak [12], Et et al. ([14], [16]), Fridy [20], Işık [24], Salat [31], Di Maio and Kočinac [13] and many authors investigated some arguments related to this notion.

A modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that

i) $f(x)=0$ if and only if $x=0$,

ii) $f(x+y)\leq f(x)+f(y)$ for $x,y\geq 0$,

iii) f is increasing,

iv) f is continuous from the right at 0.

It follows that f must be continuous in everywhere on $[0,\infty)$. A modulus may be unbounded or bounded.

Aizpuru et al. [1] defined $f-$density of a subset $E\subset\mathbb{N}$ for any unbounded modulus f by

$$d^{f}(E)=\lim_{n\to\infty}\frac{f(|\{k\leq n: k\in E\}|)}{f(n)}, \text{if the limit exists}$$

and defined $f-$statistical convergence for any unbounded modulus f by

$$d^{f}(\{k\in\mathbb{N}: |x_k-\ell|\geq\varepsilon\})=0$$

i.e.

$$\lim_{n\to\infty}\frac{1}{f(n)}f(|\{k\leq n: |x_k-\ell|\geq\varepsilon\}|)=0,$$

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and we write it as $S^f - \lim x_k = \ell$ or $x_k \to \ell\left(S^f\right)$. Every $f-$statistically convergent sequence is statistically convergent, but a statistically convergent sequence does not need to be $f-$statistically convergent for every unbounded modulus f.

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ such that $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$.

In [21], Fridy and Orhan introduced the concept of lacunary statistically convergence in the sense that a sequence (x_k) of real numbers is called lacunary statistically convergent to a real number ℓ, if

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\{k \in I_r : |x_k - \ell| \geq \varepsilon\}\right| = 0$$

for every positive real number ε.

Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r. Lacunary sequence spaces were studied in ([6], [7], [8], [9], [17], [19], [21], [23], [25], [29], [35], [36]).

First of all, the notion of a modulus was given by Nakano [27]. Maddox [26] used a modulus function to construct some sequence spaces. Afterwards different sequence spaces defined by modulus have been studied by Altın and Et [2], Et et al. [15], Işık [24], Gaur and Mursaleen [22], Nuray and Savaş [28], Pehlivan and Fisher [30], Şengül [34] and everybody else.

2. Main Results

In this section we will introduce the concepts of $f-$lacunary statistically convergent sequences of order α and strongly $f-$lacunary summable sequences of order α of real numbers, where f is an unbounded modulus and give some inclusion relations between these concepts.

Definition 2.1. *Let f be an unbounded modulus, $\theta = (k_r)$ be a lacunary sequence and α be a real number such that $0 < \alpha \leq 1$. We say that the sequence $x = (x_k)$ is $f-$lacunary statistically convergent of order α, if there is a real number ℓ such that*

$$\lim_{r\to\infty} \frac{1}{f(h_r)^\alpha} f\left(\left|\{k \in I_r : |x_k - \ell| \geq \varepsilon\}\right|\right) = 0,$$

where $I_r = (k_{r-1}, k_r]$ and $f(h_r)^\alpha$ denotes the αth power of $f(h_r)$, that is $\left(f(h_r)^\alpha\right) = \left(f(h_1)^\alpha, f(h_2)^\alpha, ..., f(h_r)^\alpha, ...\right)$. This space will be denoted by $S_\theta^{f,\alpha}$. In this case, we write $S_\theta^{f,\alpha} - \lim x_k = \ell$ or $x_k \to \ell\left(S_\theta^{f,\alpha}\right)$.

Definition 2.2. *Let f be a modulus function, $p = (p_k)$ be a sequence of strictly positive real numbers and α be a real number such that $0 < \alpha \leq 1$. We say that the sequence $x = (x_k)$ is strongly $w^\alpha[\theta, f, p]-$summable to ℓ (a real number) such that*

$$w^\alpha[\theta, f, p] = \left\{x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} \left[f(|x_k - \ell|)\right]^{p_k} = 0, \text{ for some } \ell\right\}.$$

In the present case, we denote $w^\alpha[\theta, f, p] - \lim x_k = \ell$.

Definition 2.3. *Let f be an unbounded modulus, $p = (p_k)$ be a sequence of strictly positive real numbers and α be a real number such that $0 < \alpha \leq 1$. We say that the sequence $x = (x_k)$ is strongly $w_\theta^{f,\alpha}(p)-$summable to ℓ (a real number) such that*

$$w_\theta^{f,\alpha}(p) = \left\{x = (x_k) : \lim_{r\to\infty} \frac{1}{f(h_r)^\alpha} \sum_{k\in I_r} \left[f(|x_k - \ell|)\right]^{p_k} = 0, \text{ for some } \ell\right\}.$$

In the present case, we write $w_\theta^{f,\alpha}(p) - \lim x_k = \ell$. In case of $p_k = p$ for all $k \in \mathbb{N}$ we write $w_\theta^{f,\alpha}[p]$ instead of $w_\theta^{f,\alpha}(p)$.

Definition 2.4. *Let f be an unbounded modulus, $p = (p_k)$ be a sequence of strictly positive real numbers and α be a real number such that $0 < \alpha \leq 1$. We say that the sequence $x = (x_k)$ is strongly $w^{\alpha}_{\theta,f}(p)$ −summable to ℓ (a real number) such that*

$$w^{\alpha}_{\theta,f}(p) = \left\{ x = (x_k) : \lim_{r\to\infty} \frac{1}{f(h_r)^{\alpha}} \sum_{k\in I_r} |x_k - \ell|^{p_k} = 0, \text{ for some } \ell \right\}.$$

In the present case, we write $w^{\alpha}_{\theta,f}(p) - \lim x_k = \ell$. In case of $p_k = p$ for all $k \in \mathbb{N}$ we write $w^{\alpha}_{\theta,f}[p]$ instead of $w^{\alpha}_{\theta,f}(p)$.

The proof of each of the following results is fairly straightforward, so we choose to state these results without proof, where we shall assume that the sequence $p = (p_k)$ is bounded and $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$.

Theorem 2.5. *Let f be an unbounded modulus. The classes of sequences $w^{f,\alpha}_{\theta}(p)$ and $S^{f,\alpha}_{\theta}$ are linear spaces.*

Theorem 2.6. *The space $w^{f,\alpha}_{\theta}(p)$ is paranormed by*

$$g(x) = \sup_r \left\{ \frac{1}{f(h_r)^{\alpha}} \sum_{k\in I_r} [f(|x_k|)]^{p_k} \right\}^{\frac{1}{M}}$$

where $0 < \alpha \leq 1$ and $M = \max(1, H)$.

Proposition 2.7. *([30]) Let f be a modulus and $0 < \delta < 1$. Then for each $\|u\| \geq \delta$, we have $f(\|u\|) \leq 2f(1)\delta^{-1}\|u\|$.*

Theorem 2.8. *Let f be an unbounded modulus, α be a real number such that $0 < \alpha \leq 1$ and $p > 1$. If $\lim_{u\to\infty} \inf \frac{f(u)}{u} > 0$, then $w^{f,\alpha}_{\theta}[p] = w^{\alpha}_{\theta,f}[p]$.*

Proof. Let $p > 1$ be a positive real number and $x \in w^{f,\alpha}_{\theta}[p]$. If $\lim_{u\to\infty} \inf \frac{f(u)}{u} > 0$ then there exists a number $c > 0$ such that $f(u) > cu$ for $u > 0$. Clearly

$$\frac{1}{f(h_r)^{\alpha}} \sum_{k\in I_r} [f(|x_k - \ell|)]^{p} \geq \frac{1}{f(h_r)^{\alpha}} \sum_{k\in I_r} [c|x_k - \ell|]^{p} = \frac{c^p}{f(h_r)^{\alpha}} \sum_{k\in I_r} |x_k - \ell|^{p},$$

and therefore $w^{f,\alpha}_{\theta}[p] \subset w^{\alpha}_{\theta,f}[p]$.

Now let $x \in w^{\alpha}_{\theta,f}[p]$. Then we have

$$\frac{1}{f(h_r)^{\alpha}} \sum_{k\in I_r} |x_k - \ell|^{p} \to 0 \text{ as } r \to \infty.$$

Let $0 < \delta < 1$. We can write

$$\begin{aligned} \frac{1}{f(h_r)^{\alpha}} \sum_{k\in I_r} |x_k - \ell|^{p} &\geq \frac{1}{f(h_r)^{\alpha}} \sum_{\substack{k\in I_r \\ |x_k-\ell|\geq\delta}} |x_k - \ell|^{p} \\ &\geq \frac{1}{f(h_r)^{\alpha}} \sum_{\substack{k\in I_r \\ |x_k-\ell|\geq\delta}} \left[\frac{f(|x_k - \ell|)}{2f(1)\delta^{-1}} \right]^{p} \\ &\geq \frac{1}{f(h_r)^{\alpha}} \frac{\delta^p}{2^p f(1)^p} \sum_{k\in I_r} [f(|x_k - \ell|)]^{p} \end{aligned}$$

by Proposition 2.7. Therefore $x \in w_{\theta}^{f,\alpha}[p]$.

If $\lim_{u\to\infty}\inf \frac{f(u)}{u} = 0$, the equality $w_{\theta}^{f,\alpha}[p] = w_{\theta,f}^{\alpha}[p]$ can not be hold as shown the following example:

Let $f(x) = 2\sqrt{x}$ and define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} \sqrt{h_r}, & \text{if } k = k_r \\ 0, & \text{otherwise.} \end{cases} \quad r = 1, 2, \dots.$$

For $\ell = 0$, $\alpha = \frac{4}{5}$ and $p = \frac{6}{5}$, we have

$$\frac{1}{f(h_r)^{\alpha}} \sum_{k\in I_r} [f(|x_k|)]^p = \frac{\left(2h_r^{\frac{1}{4}}\right)^{\frac{6}{5}}}{\left(2\sqrt{h_r}\right)^{\frac{4}{5}}} \to 0 \text{ as } r \to \infty$$

hence $x \in w_{\theta}^{f,\alpha}[p]$, but

$$\frac{1}{f(h_r)^{\alpha}} \sum_{k\in I_r} |x_k|^p = \frac{\left(\sqrt{h_r}\right)^{\frac{6}{5}}}{\left(2\sqrt{h_r}\right)^{\frac{4}{5}}} \to \infty \text{ as } r \to \infty$$

and so $x \notin w_{\theta,f}^{\alpha}[p]$. □

Maddox [26] showed that the existence of an unbounded modulus f for which there is a positive constant c such that $f(xy) \geq cf(x)f(y)$, for all $x \geq 0$, $y \geq 0$.

Theorem 2.9. *Let f be an unbounded modulus, α be a real number such that $0 < \alpha \leq 1$ and $p_k = 1$ for all $k \in \mathbb{N}$. If $\lim_{u\to\infty} \frac{f(u)^{\alpha}}{u^{\alpha}} > 0$, then $w^{\alpha}[\theta, f, p] \subset S_{\theta}^{f,\alpha}$.*

Proof. Let $x \in w^{\alpha}[\theta, f, p]$ and $\lim_{u\to\infty} \frac{f(u)^{\alpha}}{u^{\alpha}} > 0$. For $\varepsilon > 0$, we have

$$\begin{aligned} \frac{1}{h_r^{\alpha}} \sum_{k\in I_r} f(|x_k - \ell|) &\geq \frac{1}{h_r^{\alpha}} f\left(\sum_{k\in I_r} |x_k - \ell|\right) \geq \frac{1}{h_r^{\alpha}} f\left(\sum_{\substack{k\in I_r \\ |x_k-\ell|\geq\varepsilon}} |x_k - \ell|\right) \\ &\geq \frac{1}{h_r^{\alpha}} f(|\{k \in I_r : |x_k - \ell| \geq \varepsilon\}| \varepsilon) \\ &\geq \frac{c}{h_r^{\alpha}} f(|\{k \in I_r : |x_k - \ell| \geq \varepsilon\}|) f(\varepsilon) \\ &= \frac{c}{h_r^{\alpha}} \frac{f(|\{k \in I_r : |x_k - \ell| \geq \varepsilon\}|)}{f(h_r)^{\alpha}} f(h_r)^{\alpha} f(\varepsilon). \end{aligned}$$

Therefore, $w^{\alpha}[\theta, f, p] - \lim x_k = \ell$ implies $S_{\theta}^{f,\alpha} - \lim x_k = \ell$. □

Theorem 2.10. *Let α_1, α_2 be two real numbers such that $0 < \alpha_1 \leq \alpha_2 \leq 1$, f be an unbounded modulus function and let $\theta = (k_r)$ be a lacunary sequence, then we have $w_{\theta}^{f,\alpha_1}(p) \subset S_{\theta}^{f,\alpha_2}$.*

Proof. Let $x \in w_{\theta}^{f,\alpha_1}(p)$ and $\varepsilon > 0$ be given and $\sum_1$, $\sum_2$ denote the sums over $k \in I_r$, $|x_k - \ell| \geq \varepsilon$ and $k \in I_r$, $|x_k - \ell| < \varepsilon$ respectively. Since $f(h_r)^{\alpha_1} \leq f(h_r)^{\alpha_2}$ for each r, we may write

$$\begin{aligned}
\frac{1}{f(h_r)^{\alpha_1}} \sum_{k \in I_r} [f(|x_k - \ell|)]^{p_k} &= \frac{1}{f(h_r)^{\alpha_1}} \left[\sum\nolimits_1 [f(|x_k - \ell|)]^{p_k} + \sum\nolimits_2 [f(|x_k - \ell|)]^{p_k}\right] \\
&\geq \frac{1}{f(h_r)^{\alpha_2}} \left[\sum\nolimits_1 [f(|x_k - \ell|)]^{p_k} + \sum\nolimits_2 [f(|x_k - \ell|)]^{p_k}\right] \\
&\geq \frac{1}{f(h_r)^{\alpha_2}} \left[\sum\nolimits_1 [f(\varepsilon)]^{p_k}\right] \\
&\geq \frac{1}{H.f(h_r)^{\alpha_2}} \left[f\left(\sum\nolimits_1 [\varepsilon]^{p_k}\right)\right] \\
&\geq \frac{1}{H.f(h_r)^{\alpha_2}} \left[f\left(\sum\nolimits_1 \min([\varepsilon]^h, [\varepsilon]^H)\right)\right] \\
&\geq \frac{1}{H.f(h_r)^{\alpha_2}} f\left(|\{k \in I_r : |x_k - \ell| \geq \varepsilon\}| \left[\min([\varepsilon]^h, [\varepsilon]^H)\right]\right) \\
&\geq \frac{c}{H.f(h_r)^{\alpha_2}} f(|\{k \in I_r : |x_k - \ell| \geq \varepsilon\}|)\ f\left(\left[\min([\varepsilon]^h, [\varepsilon]^H)\right]\right).
\end{aligned}$$

Hence $x \in S_{\theta}^{f,\alpha_2}$. □

Theorem 2.11. *Let $\theta = (k_r)$ be a lacunary sequence and α be a fixed real number such that $0 < \alpha \leq 1$. If $\liminf_r q_r > 1$ and $\lim_{u\to\infty} \frac{f(u)^\alpha}{u^\alpha} > 0$, then $S^{f,\alpha} \subset S_{\theta}^{f,\alpha}$.*

Proof. Suppose first that $\liminf_r q_r > 1$; then there exists a $\lambda > 0$ such that $q_r \geq 1 + \lambda$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r} \geq \frac{\lambda}{1+\lambda} \Longrightarrow \left(\frac{h_r}{k_r}\right)^\alpha \geq \left(\frac{\lambda}{1+\lambda}\right)^\alpha.$$

If $S^{f,\alpha} - \lim x_k = \ell$, then for every $\varepsilon > 0$ and for sufficiently large r, we have

$$\begin{aligned}
\frac{1}{f(k_r)^\alpha} f(|\{k \leq k_r : |x_k - \ell| \geq \varepsilon\}|) &\geq \frac{1}{f(k_r)^\alpha} f(|\{k \in I_r : |x_k - \ell| \geq \varepsilon\}|) \\
&= \frac{f(h_r)^\alpha}{f(k_r)^\alpha} \frac{1}{f(h_r)^\alpha} f(|\{k \in I_r : |x_k - \ell| \geq \varepsilon\}|) \\
&= \frac{f(h_r)^\alpha}{h_r^\alpha} \frac{k_r^\alpha}{f(k_r)^\alpha} \frac{h_r^\alpha}{k_r^\alpha} \frac{f(|\{k \in I_r : |x_k - \ell| \geq \varepsilon\}|)}{f(h_r)^\alpha} \\
&\geq \frac{f(h_r)^\alpha}{h_r^\alpha} \frac{k_r^\alpha}{f(k_r)^\alpha} \left(\frac{\lambda}{1+\lambda}\right)^\alpha \frac{f(|\{k \in I_r : |x_k - \ell| \geq \varepsilon\}|)}{f(h_r)^\alpha}.
\end{aligned}$$

This proves the sufficiency. □

Theorem 2.12. *Let f be an unbounded modulus and $0 < \alpha \leq 1$. If $(x_k) \in S^f \cap S_{\theta}^{f,\alpha}$, then $S^f - \lim x_k = S_{\theta}^{f,\alpha} - \lim x_k$ such that $\left|f(x) - f(y)\right| = f\left(\left|x - y\right|\right)$, for $x \geq 0$, $y \geq 0$.*

Proof. Suppose $S^f - \lim x_k = \ell_1$, $S_{\theta}^{f,\alpha} - \lim x_k = \ell_2$ and $\ell_1 \neq \ell_2$. Let $0 < \varepsilon < \frac{|\ell_1 - \ell_2|}{2}$. Then for $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \frac{f(|\{k \leq n : |x_k - \ell_1| \geq \varepsilon\}|)}{f(n)} = 0,$$

and

$$\lim_{r\to\infty} \frac{f\left(\left|\left\{k \in I_r : |x_k - \ell_2| \geq \varepsilon\right\}\right|\right)}{f\left(h_r\right)^{\alpha}} = 0.$$

On the other hand we can write

$$\frac{f\left(\left|\left\{k \leq n : |\ell_1 - \ell_2| \geq 2\varepsilon\right\}\right|\right)}{f\left(n\right)} \leq \frac{f\left(\left|\left\{k \leq n : |x_k - \ell_1| \geq \varepsilon\right\}\right|\right)}{f\left(n\right)} + \frac{f\left(\left|\left\{k \leq n : |x_k - \ell_2| \geq \varepsilon\right\}\right|\right)}{f\left(n\right)}.$$

Taking limit as $n \to \infty$, we get

$$1 \leq 0 + \lim_{n\to\infty} \frac{f\left(\left|\left\{k \leq n : |x_k - \ell_2| \geq \varepsilon\right\}\right|\right)}{f\left(n\right)} \leq 1,$$

and so

$$\lim_{n\to\infty} \frac{f\left(\left|\left\{k \leq n : |x_k - \ell_2| \geq \varepsilon\right\}\right|\right)}{f\left(n\right)} = 1.$$

We consider the subsequence

$$\frac{1}{f\left(k_m\right)} f\left(\left|\left\{k \leq k_m : |x_k - \ell_2| \geq \varepsilon\right\}\right|\right)$$

of sequence

$$\frac{1}{f\left(n\right)} f\left(\left|\left\{k \leq n : |x_k - \ell_2| \geq \varepsilon\right\}\right|\right).$$

Then

$$\begin{aligned} \tfrac{1}{f(k_m)} f\left(\left|\left\{k \leq k_m : |x_k - \ell_2| \geq \varepsilon\right\}\right|\right) &= \tfrac{1}{f(k_m)} f\left(\left|\left\{k \in \bigcup_{r=1}^{m} I_r : |x_k - \ell_2| \geq \varepsilon\right\}\right|\right) \\ &= \tfrac{1}{f(k_m)} f\left(\sum_{r=1}^{m} \left|\left\{k \in I_r : |x_k - \ell_2| \geq \varepsilon\right\}\right|\right) \\ &\leq \tfrac{1}{f(k_m)} \sum_{r=1}^{m} f\left(\left|\left\{k \in I_r : |x_k - \ell_2| \geq \varepsilon\right\}\right|\right) \\ &= \tfrac{1}{f(k_m)} \sum_{r=1}^{m} f\left(h_r\right)^{\alpha} \tfrac{1}{f(h_r)^{\alpha}} f\left(\left|\left\{k \in I_r : |x_k - \ell_2| \geq \varepsilon\right\}\right|\right) \end{aligned} \tag{1}$$

and

$$\begin{aligned} \sum_{r=1}^{m} f\left(h_r\right)^{\alpha} &= f\left(h_1\right)^{\alpha} + f\left(h_2\right)^{\alpha} + \ldots + f\left(h_m\right)^{\alpha} \\ &= f\left(k_1 - k_0\right)^{\alpha} + f\left(k_2 - k_1\right)^{\alpha} + \ldots + f\left(k_m - k_{m-1}\right)^{\alpha} \\ &= f\left(|k_1 - k_0|\right)^{\alpha} + f\left(|k_2 - k_1|\right)^{\alpha} + \ldots + f\left(|k_m - k_{m-1}|\right)^{\alpha} \\ &= \left|f\left(k_1\right) - f\left(k_0\right)\right|^{\alpha} + \left|f\left(k_2\right) - f\left(k_1\right)\right|^{\alpha} + \ldots + \left|f\left(k_m\right) - f\left(k_{m-1}\right)\right|^{\alpha} \\ &\leq \left|f\left(k_1\right) - f\left(k_0\right)\right| + \left|f\left(k_2\right) - f\left(k_1\right)\right| + \ldots + \left|f\left(k_m\right) - f\left(k_{m-1}\right)\right| \\ &= f\left(k_1\right) - f\left(k_0\right) + f\left(k_2\right) - f\left(k_1\right) + \ldots + f\left(k_m\right) - f\left(k_{m-1}\right) \\ &= f\left(k_m\right). \end{aligned} \tag{2}$$

Using (2) in (1), we have

$$\frac{1}{f\left(k_m\right)} f\left(\left|\left\{k \leq k_m : |x_k - \ell_2| \geq \varepsilon\right\}\right|\right) \leq \frac{\sum_{r=1}^{m} f\left(h_r\right)^{\alpha}}{\sum_{r=1}^{m} f\left(h_r\right)^{\alpha}} \frac{1}{f\left(h_r\right)^{\alpha}} f\left(\left|\left\{k \in I_r : |x_k - \ell_2| \geq \varepsilon\right\}\right|\right)$$

so

$$\frac{1}{f(k_m)} f(|\{k \le k_m : |x_k - \ell_2| \ge \varepsilon\}|) \to 0,$$

but this is a contradiction to

$$\lim_{n\to\infty} \frac{f(|\{k \le n : |x_k - \ell_2| \ge \varepsilon\}|)}{f(n)} = 1.$$

As a result, $\ell_1 = \ell_2$. □

Now as a result of Theorem 2.12 we have the following Corollary 2.13.

Corollary 2.13. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences and $0 < \alpha \le 1$. If $(x_k) \in S^f \cap \left(S_\theta^{f,\alpha} \cap S_{\theta'}^{f,\alpha}\right)$, then $S_\theta^{f,\alpha} - \lim x_k = S_{\theta'}^{f,\alpha} - \lim x_k$.*

Theorem 2.14. *Let f be an unbounded modulus. If $\lim p_k > 0$, then $w_\theta^{f,\alpha}(p) - \lim x_k = \ell$ uniquely.*

Proof. Let $\lim p_k = s > 0$. Assume that $w_\theta^{f,\alpha}(p) - \lim x_k = \ell_1$ and $w_\theta^{f,\alpha}(p) - \lim x_k = \ell_2$. Then

$$\lim_r \frac{1}{f(h_r)^\alpha} \sum_{k\in I_r} [f(|x_k - \ell_1|)]^{p_k} = 0,$$

and

$$\lim_r \frac{1}{f(h_r)^\alpha} \sum_{k\in I_r} [f(|x_k - \ell_2|)]^{p_k} = 0.$$

By definition of f, we have

$$\begin{aligned}\frac{1}{f(h_r)^\alpha} \sum_{k\in I_r} [f(|\ell_1 - \ell_2|)]^{p_k} &\le \frac{D}{f(h_r)^\alpha}\left(\sum_{k\in I_r} [f(|x_k - \ell_1|)]^{p_k} + \sum_{k\in I_r} [f(|x_k - \ell_2|)]^{p_k}\right) \\ &= \frac{D}{f(h_r)^\alpha} \sum_{k\in I_r} [f(|x_k - \ell_1|)]^{p_k} + \frac{D}{f(h_r)^\alpha} \sum_{k\in I_r} [f(|x_k - \ell_2|)]^{p_k}\end{aligned}$$

where $\sup_k p_k = H$ and $D = \max\left(1, 2^{H-1}\right)$. Hence

$$\lim_r \frac{1}{f(h_r)^\alpha} \sum_{k\in I_r} [f(|\ell_1 - \ell_2|)]^{p_k} = 0.$$

Since $\lim_{k\to\infty} p_k = s$ we have $\ell_1 - \ell_2 = 0$. Thus the limit is unique. □

Theorem 2.15. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and α_1, α_2 two real numbers such that $0 < \alpha_1 \le \alpha_2 \le 1$. If*

$$\liminf_{r\to\infty} \frac{f(h_r)^{\alpha_1}}{f(\ell_r)^{\alpha_2}} > 0 \tag{3}$$

where $I_r = (k_{r-1}, k_r]$, $h_r = k_r - k_{r-1}$ and $J_r = (s_{r-1}, s_r]$, $\ell_r = s_r - s_{r-1}$, then $w_{\theta'}^{f,\alpha_2}(p) \subset w_\theta^{f,\alpha_1}(p)$.

Proof. Let $x \in w_{\theta'}^{f,\alpha_2}(p)$. We can write

$$\begin{aligned}
\frac{1}{f(\ell_r)^{\alpha_2}}\sum_{k\in J_r}[f(|x_k-\ell|)]^{p_k} &= \frac{1}{f(\ell_r)^{\alpha_2}}\sum_{k\in J_r-I_r}[f(|x_k-\ell|)]^{p_k}+\frac{1}{f(\ell_r)^{\alpha_2}}\sum_{k\in I_r}[f(|x_k-\ell|)]^{p_k}\\
&\geq \frac{1}{f(\ell_r)^{\alpha_2}}\sum_{k\in I_r}[f(|x_k-\ell|)]^{p_k}\\
&\geq \frac{f(h_r)^{\alpha_1}}{f(\ell_r)^{\alpha_2}}\frac{1}{f(h_r)^{\alpha_1}}\sum_{k\in I_r}[f(|x_k-\ell|)]^{p_k}.
\end{aligned}$$

Thus if $x \in w_{\theta'}^{f,\alpha_2}(p)$, then $x \in w_{\theta}^{f,\alpha_1}(p)$. □

From Theorem 2.15 we have the following results.

Corollary 2.16. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and α_1, α_2 two real numbers such that $0 < \alpha_1 \leq \alpha_2 \leq 1$. If (3) holds then*

(i) $w_{\theta'}^{f,\alpha}(p) \subset w_{\theta}^{f,\alpha}(p)$, *if* $\alpha_1 = \alpha_2 = \alpha$,

(ii) $w_{\theta'}^{f}(p) \subset w_{\theta}^{f,\alpha_1}(p)$, *if* $\alpha_2 = 1$,

(iii) $w_{\theta'}^{f}(p) \subset w_{\theta}^{f}(p)$, *if* $\alpha_1 = \alpha_2 = 1$.

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