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On Fixed Point and Variational Inclusion Problems

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Abstract. In this paper, fixed point and variational inclusion problems are investigated based on a proximal-type iterative algorithm. Strong convergence theorems are established in the framework of Hilbert spaces.

1. Introduction

Variational inclusion problems are being used as mathematical programming models to study a large number of optimization problems arising in finance, economics, network, transportation, and engineering sciences; see [1-29] and the references therein. In the real word, many nonlinear problems arising in applied areas are mathematically modeled as a nonlinear operator equation and this operator is decomposed as the sum of two nonlinear operators. One of the most popular techniques for solving the inclusion problem goes back to the work of Browder [30]. One of the basic ideas in the case of a Hilbert space H is reducing the above inclusion problem to a fixed point problem of the operator R_A defined by $R_A = (I + A)^{-1}$, which is called the classical resolvent resolvent of A. If A has some monotonicity conditions, the classical resolvent of A is with full domain and firmly nonexpnsive. The property of the resolvent ensures that the Picard iterative algorithm $x_{n+1} = R_A x_n$ converge weakly to a fixed point of R_A, which is necessarily a zero point of A. Rockafellar introduced this iteration method and call it the proximal point algorithm; for more detail, see [31] and [32] and the references therein. Methods for finding zero points of monotone mappings in the framework of Hilbert spaces are based on the good properties of the resolvent R_A, but these properties are not available in the framework of Banach spaces. It is known that the proximal point algorithm only has weak convergence even for nonexpansive mappings. In many disciplines, including economics, image recovery, and control theory, problems arises in infinite dimension spaces. In such problems, strong convergence (norm convergence) is often much more desirable than weak convergence, for it translates the physically tangible property that the energy $\|x_n - x\|$ of the error between the iterate x_n and the solution x eventually becomes arbitrarily small.

In this paper, we study fixed point and variational inclusion problems based on a proximal-type iterative algorithm. Strong convergence theorems are established in the framework of Hilbert spaces.

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2. Preliminaries

In what follows, we always assume that H is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and C is a nonempty, closed and convex subset of H. Let $S : C \to C$ be a mapping. $F(S)$ is denoted by the fixed point set of S. S is said to be *contractive* iff there exists a constant $\alpha \in (0,1)$ such that

$$\|Sx - Sy\| \le \alpha\|x - y\|, \quad \forall x, y \in C.$$

S is said to be *nonexpansive* iff

$$\|Sx - Sy\| \le \|x - y\|, \quad \forall x, y \in C.$$

Let $A : C \to H$ be a mapping. Recall that the classical variational inequality problem is to find a point $x \in C$ such that

$$\langle y - x, Ax\rangle \ge 0, \quad \forall y \in C.$$

Such a point $x \in C$ is called a solution of the variational inequality. In this paper, we use $VI(C,A)$ to denote the solution set of the variational inequality. Recall that A is said to be *monotone* iff

$$\langle Ax - Ay, x - y\rangle \ge 0, \quad \forall x, y \in C.$$

Recall that A is said to be *inverse-strongly monotone* iff there exists a constant $\kappa > 0$ such that

$$\langle Ax - Ay, x - y\rangle \ge \kappa\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is not hard to see that every inverse-strongly monotone mapping is monotone and continuous.

Recall that a set-valued mapping $B : H \rightrightarrows H$ is said to be *monotone* iff, for all $x, y \in H$, $f \in Bx$ and $g \in By$ imply $\langle x - y, f - g\rangle \ge 0$. In this paper, we use $B^{-1}(0)$ to stand for the zero point of B. A monotone mapping $B : H \rightrightarrows H$ is *maximal* iff the graph $Graph(B)$ of B is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping B is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \ge 0$, for all $(y, g) \in Graph(B)$ implies $f \in Bx$. For a maximal monotone operator B on H, and $r > 0$, we may define the single-valued resolvent $J_r : H \to Dom(B)$, where $Dom(B)$ denote the domain of B. It is known that J_r is firmly nonexpansive, and $B^{-1}(0) = F(J_r)$.

In order to prove our main results, we also need the following tools.

Lemma 2.1 [33] *Let E be a Banach space and let A be an m-accretive operator. For $\lambda > 0$, $\mu > 0$, and $x \in E$, we have $J_\lambda x = J_\mu\Big(\frac{\mu}{\lambda}x + \Big(1 - \frac{\mu}{\lambda}\Big)J_\lambda x\Big)$, where $J_\lambda = (I + \lambda A)^{-1}$ and $J_\mu = (I + \mu A)^{-1}$.*

Lemma 2.2 [34] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E, and $\{\beta_n\}$ be a sequence in $(0,1)$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$, $\forall n \ge 1$ and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.3 [35] *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the condition $a_{n+1} \le (1 - t_n)a_n + t_n b_n$, $\forall n \ge 0$, where $\{t_n\}$ is a number sequence in $(0,1)$ such that $\lim_{n\to\infty} t_n = 0$ and $\sum_{n=0}^{\infty} t_n = \infty$, $\{b_n\}$ is a number sequence such that $\limsup_{n\to\infty} b_n \le 0$. Then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.4. [13] *Let $A : C \to H$ be a mapping, and $B : H \rightrightarrows H$ a maximal monotone operator. Then $F(J_r(I - rB)) = (A + B)^{-1}(0)$.*

3. Main Results

Now, we are in a position to give our main results.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let B be a maximal monotone operator on H. Let $S : C \to C$ be a nonexpansive* *mapping with fixed points. Assume that* $Dom(B) \subset C$ *and* $F(S) \cap (A+B)^{-1}(0)$ *is not empty. Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be real number sequences in* $(0,1)$ *and* $\{r_n\}$ *be a positive real number sequence in* $(0,2\alpha)$*. Assume that the above sequences satisfy the following restrictions:*

(1) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(2) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;

(3) $0 < a \le r_n \le b < 2\alpha$ *and* $\sum_{n=1}^{\infty}|r_n - r_{n-1}| < \infty$,

where a *and* b *are two real numbers. Let* $\{x_n\}$ *be a sequence generated in the following process:* $x_1 \in C$ *and*

$$\begin{cases} y_n = \alpha_n u + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)SJ_{r_n}(y_n - r_nAy_n), \quad \forall n \ge 1, \end{cases}$$

where u *is fixed element in* C *and* $J_{r_n} = (I + r_nB)^{-1}$*. Then* $\{x_n\}$ *converges strongly to a point* $q \in F(S)\cap(A+B)^{-1}(0)$*, which is an unique solution to the following variational inequality*

$$\langle u - q, p - q\rangle \le 0, \quad \forall p \in F(S)\cap(A+B)^{-1}(0).$$

Proof. We first show that the sequence $\{x_n\}$ is bouned. Notice that $I - r_nA$ is nonexpansive. Indeed, we have

$$\begin{aligned} &\|(I - r_nA)x - (I - r_nA)y\|^2 \\ &= \|x-y\|^2 - 2r_n\langle x-y, Ax - Ay\rangle + {r_n}^2\|Ax - Ay\|^2 \\ &\le \|x-y\|^2 - r_n(2\alpha - r_n)\|Ax - Ay\|^2. \end{aligned}$$

It follows from the restriction (3) that $I - r_nA$ is nonexpansive. Let $p \in (A+B)^{-1}(0)\cap F(S)$. It follows that

$$\begin{aligned} \|x_{n+1} - p\| &\le \beta_n\|x_n - p\| + (1-\beta_n)\|SJ_{r_n}(y_n - r_nAy_n) - p\| \\ &\le \beta_n\|x_n - p\| + (1-\beta_n)\|J_{r_n}(y_n - r_nAy_n) - p\| \\ &\le \beta_n\|x_n - p\| + (1-\beta_n)\|(y_n - r_nAy_n) - p\| \\ &\le \beta_n\|x_n - p\| + (1-\beta_n)\|y_n - p\| \\ &\le \Big(1 - (1-\beta_n)\alpha_n\Big)\|x_n - p\| + (1-\beta_n)\alpha_n\|u - p\| \\ &\dots \\ &\le \max\{\|x_1 - p\|, \|u - p\|\}. \end{aligned}$$

This proves that the sequence $\{x_n\}$ is bounded. Notice that

$$\|y_n - y_{n-1}\| \le (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|u - x_{n-1}\|. \tag{3.1}$$

Set $z_n = y_n - r_nAy_n$. It follows from Lemma 2.1 that

$$\begin{aligned} \|J_{r_n}z_n - J_{r_{n-1}}z_{n-1}\| &= \|J_{r_{n-1}}\Big(\frac{r_{n-1}}{r_n}z_n + (1 - \frac{r_{n-1}}{r_n})J_{r_n}z_n\Big) - J_{r_{n-1}}z_{n-1}\| \\ &\le \|\frac{r_{n-1}}{r_n}(z_n - z_{n-1}) + (1 - \frac{r_{n-1}}{r_n})(J_{r_n}z_n - z_{n-1})\| \\ &\le \|z_n - z_{n-1}\| + \frac{|r_n - r_{n-1}|}{a}\|J_{r_n}z_n - z_n\| \\ &\le \|y_n - y_{n-1}\| + |r_{n-1} - r_n|(\|Ay_{n-1}\| + \frac{\|J_{r_n}z_n - z_n\|}{a}). \end{aligned} \tag{3.2}$$

Substituting (3.1) into (3.2), we find that

$$\|J_{r_n}z_n - J_{r_{n-1}}z_{n-1}\| \le \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|u - x_{n-1}\| + |r_{n-1} - r_n|(\|Ay_{n-1}\| + \frac{\|J_{r_n}z_n - z_n\|}{a}).$$

It follows that

$$\begin{aligned}&\|SJ_{r_n}z_n - SJ_{r_{n-1}}z_{n-1}\|\\ &\le \|J_{r_n}z_n - J_{r_{n-1}}z_{n-1}\|\\ &\le \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|u - x_{n-1}\| + |r_{n-1} - r_n|M,\end{aligned}$$

where M is an appropriate constant. In view of the restrictions (1) and (3), we find that

$$\limsup_{n\to\infty}\Big(\|SJ_{r_n}z_n - SJ_{r_{n-1}}z_{n-1}\| - \|x_n - x_{n-1}\|\Big) \le 0.$$

It follows from Lemma 2.2 that $\lim_{n\to\infty}\|SJ_{r_n}z_n - x_n\| = 0$. Notice that

$$x_{n+1} - x_n = (1 - \beta_n)(SJ_{r_n}z_n - x_n).$$

This yields that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.3}$$

On the other hand, we also have

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.4}$$

Since $\|\cdot\|^2$ is convex, we find that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|SJ_{r_n}z_n - p\|^2\\ &\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|J_{r_n}(I - r_nA)y_n - p\|^2\\ &\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|(I - r_nA)y_n - (I - r_nA)p\|^2\\ &\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|y_n - p\|^2 - r_n(1 - \beta_n)(2\alpha - r_n)\|Ay_n - Ap\|^2\\ &\le \beta_n\|x_n - p\|^2 + \alpha_n(1 - \beta_n)\|u - p\|^2 + (1 - \beta_n)(1 - \alpha_n)\|x_n - p\|^2\\ &\quad - r_n(1 - \beta_n)(2\alpha - r_n)\|Ay_n - Ap\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}&r_n(1 - \beta_n)(2\alpha - r_n)\|Ay_n - Ap\|^2\\ &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n(1 - \beta_n)\|u - p\|^2\\ &\le (\|x_n - p\| + \|x_{n+1} - p\|)\|x_{n+1} - x_n\| + \alpha_n\|u - p\|^2.\end{aligned}$$

In view of the restrictions (1), (2), and (3), we find from (3.3) that

$$\lim_{n\to\infty}\|Ay_n - Ap\| = 0. \tag{3.5}$$

Since J_{r_n} is firmly nonexpansive, thus we have

$$\begin{aligned}\|J_{r_n}z_n - p\|^2 &\le \langle J_{r_n}z_n - p, (y_n - r_nAy_n) - (p - r_nAp)\rangle\\ &= \frac{1}{2}\Big(\|J_{r_n}z_n - p\|^2 + \|(y_n - r_nAy_n) - (p - r_nAp)\|^2\\ &\quad - \|(J_{r_n}z_n - p) - \big((y_n - r_nAy_n) - (p - r_nAp)\big)\|^2\Big)\\ &\le \frac{1}{2}\Big(\|J_{r_n}z_n - p\|^2 + \|y_n - p\|^2 - \|J_{r_n}z_n - y_n\|^2\\ &\quad - \|r_nAy_n - r_nAp\|^2 + 2r_n\|Ay_n - Ap\|\|J_{r_n}z_n - y_n\|\Big).\end{aligned}$$

This implies that

$$\begin{aligned}\|J_{r_n}z_n - p\|^2 &\le \|y_n - p\|^2 - \|J_{r_n}z_n - y_n\|^2\\ &\quad - \|r_nAy_n - r_nAp\|^2 + 2r_n\|Ay_n - Ap\|\|J_{r_n}z_n - y_n\|\\ &\le \alpha_n\|u - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \|J_{r_n}z_n - y_n\|^2\\ &\quad + 2r_n\|Ay_n - Ap\|\|J_{r_n}z_n - y_n\|.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|SJ_{r_n}z_n - p\|^2\\ &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|J_{r_n}z_n - p\|^2\\ &\le \|x_n - p\|^2 + \alpha_n\|u - p\|^2 - (1-\beta_n)\|J_{r_n}z_n - y_n\|^2\\ &\quad + 2r_n\|Ay_n - Ap\|\|J_{r_n}z_n - y_n\|.\end{aligned}$$

This implies that

$$\begin{aligned}(1-\beta_n)\|J_{r_n}z_n - y_n\|^2 &\le (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + \alpha_n\|u - p\|^2\\ &\quad + 2r_n\|Ay_n - Ap\|\|J_{r_n}z_n - y_n\|.\end{aligned}$$

In view of the restrictions (1) and (2), we find from (3.3) and (3.5) that

$$\lim_{n\to\infty}\|J_{r_n}z_n - y_n\| = 0. \tag{3.6}$$

Next, we show that $\limsup_{n\to\infty}\langle u - \bar{x}, y_n - \bar{x}\rangle \le 0$. To show it, we can choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle u - \bar{x}, y_n - \bar{x}\rangle = \lim_{i\to\infty}\langle u - \bar{x}, y_{n_i} - \bar{x}\rangle.$$

Since $\{y_{n_i}\}$ is bounded, we can choose a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly some point x. We may assume, without loss of generality, that y_{n_i} converges weakly to x.

Now, we are in a position to show that $x \in (A+B)^{-1}(0)$. Set $m_n = J_{r_n}(y_n - r_nAy_n)$. It follows that

$$y_n - r_nAy_n \in (I + r_nB)m_n$$

That is, $\frac{y_n - m_n}{r_n} - Ay_n \in Bm_n$. Since B is monotone, we get, for any $(\mu, \nu) \in B$, that

$$\langle m_n - \mu, \frac{y_n - m_n}{r_n} - Ay_n - \nu\rangle \ge 0.$$

Replacing n by n_i and letting $i \to \infty$, we obtain from (3.6) that

$$\langle x - \mu, -Ax - \nu\rangle \ge 0.$$

This gives that $-Ax \in Bx$, that is, $0 \in (A+B)(x)$. This proves that $x \in (A+B)^{-1}(0)$. Next, we prove that $x \in F(S)$. Notice that

$$\|Sm_n - y_n\| \le \frac{1}{1-\beta_n}\|x_{n+1} - y_n\| + \frac{\beta_n}{1-\beta_n}\|y_n - x_n\|.$$

This implies that $\|Sm_n - y_n\| \to \infty 0$. On the other hand, we have

$$\|Sm_n - m_n\| \le \|Sm_n - y_n\| + \|y_n - m_n\|.$$

It follows from (3.6) that $\|Sm_n - m_n\| \to 0$. In view of demiclosed of the mapping, we find that $x \in F(S)$. This complete the proof that $x \in F(S) \cap (A+B)^{-1}(0)$. It follows that

$$\limsup_{n\to\infty}\langle u - \bar{x}, y_n - \bar{x}\rangle \le 0.$$

Finally, we show that $x_n \to \bar{x}$. Notice that

$$\begin{aligned}\|y_n - \bar{x}\|^2 &\le \alpha_n\langle u - \bar{x}, y_n - \bar{x}\rangle + (1-\alpha_n)\|x_n - \bar{x}\|\|y_n - \bar{x}\| \\ &\le \alpha_n\langle u - \bar{x}, y_n - \bar{x}\rangle + \frac{1-\alpha_n}{2}(\|x_n - \bar{x}\|^2 + \|y_n - \bar{x}\|^2).\end{aligned}$$

This implies that

$$\|y_n - \bar{x}\|^2 \le \alpha_n\langle u - \bar{x}, y_n - \bar{x}\rangle + (1-\alpha_n)\|x_n - \bar{x}\|^2.$$

It follows that

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 &\le \beta_n\|x_n - \bar{x}\|^2 + (1-\beta_n)\|SJ_{r_n}(I - r_nA)y_n - \bar{x}\|^2 \\ &\le \beta_n\|x_n - \bar{x}\|^2 + (1-\beta_n)\|y_n - \bar{x}\|^2 \\ &\le \Big(1 - \alpha_n(1-\beta_n)\Big)\|x_n - \bar{x}\|^2 + \alpha_n(1-\beta_n)\langle u - \bar{x}, y_n - \bar{x}\rangle\end{aligned}$$

In view of the restrictions (1) and (2), we find from Lemma 2.3 that $x_n \to \bar{x}$. This completes the proof.

4. Applications

Recall the classical variational inequality is to find $u \in C$ such that

$$\langle Au, v - u\rangle \ge 0, \quad \forall v \in C.$$

The solution set of the inequality is denoted by $VI(C, A)$ in this section. Let $f : H \to (-\infty, +\infty]$ a proper convex lower semicontinuous function. Then the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{y \in H : f(z) \ge f(x) + \langle z - x, y\rangle, \quad z \in H\}, \quad \forall x \in H.$$

From Rockafellar [36], we know that ∂f is maximal monotone. It is easy to verify that $0 \in \partial f(x)$ if and only if $f(x) = \min_{y\in H} f(y)$. Let I_C be the indicator function of C, i.e.,

$$I_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases} \tag{4.1}$$

Since I_C is a proper lower semicontinuous convex function on H, we see that the subdifferential ∂I_C of I_C is a maximal monotone operator.

Lemma 4.1 [5] *Let C be a nonempty closed convex subset of a real Hilbert space H, $Proj_C$ the metric projection from H onto C, ∂I_C the subdifferential of I_C, where I_C is as defined in (4.1) and $J_\lambda = (I + \lambda\partial I_C)^{-1}$. Then $y = J_\lambda x \Longleftrightarrow y = Proj_C x$, $\forall x \in H, y \in C$.*

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $S : C \to C$ be a nonexpansive mapping with fixed points. Assume that $F(S) \cap VI(C, A)$ is not empty. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real number sequences in $(0, 1)$ and $\{r_n\}$ be a positive real number sequence in $(0, 2\alpha)$. Assume that the above sequences satisfy the following restrictions:*

(1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(2) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(3) $0 < a \le r_n \le b < 2\alpha$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$,

where a and b are two real numbers. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$ and

$$\begin{cases} y_n = \alpha_n u + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)SProj_C(y_n - r_nAy_n), \quad \forall n \ge 1, \end{cases}$$

where u is fixed element in C and $J_{r_n} = (I + r_nB)^{-1}$*. Then* $\{x_n\}$ *converges strongly to a point* $q \in F(S) \cap VI(C, A)$*, which is an unique solution to the following variational inequality*

$$\langle u - q, p - q\rangle \le 0, \quad \forall p \in F(S) \cap VI(C, A).$$

Proof Putting $Bx = \partial I_C$, we find from Lemma 4.1 the desired conclusion immediately.

First we consider the following inclusion problem.

Theorem 4.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let* $A : C \to H$ *be an* α*-inverse-strongly monotone mapping and let B be a maximal monotone operator on H. Assume that* $Dom(B) \subset C$ *and* $(A + B)^{-1}(0)$ *is not empty. Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be real number sequences in* $(0, 1)$ *and* $\{r_n\}$ *be a positive real number sequence in* $(0, 2\alpha)$*. Assume that the above sequences satisfy the following restrictions:*

(1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(2) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(3) $0 < a \le r_n \le b < 2\alpha$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$,

where a and b are two real numbers. Let $\{x_n\}$ *be a sequence generated in the following process:* $x_1 \in C$ *and*

$$\begin{cases} y_n = \alpha_n u + (1 - \alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)J_{r_n}(y_n - r_nAy_n), \quad \forall n \ge 1, \end{cases}$$

where u is fixed element in C and $J_{r_n} = (I + r_nB)^{-1}$*. Then* $\{x_n\}$ *converges strongly to a point* $q \in (A + B)^{-1}(0)$*, which is an unique solution to the following variational inequality*

$$\langle u - q, p - q\rangle \le 0, \quad \forall p \in (A + B)^{-1}(0).$$

Proof. Putting $S = I$, the identity mapping, the desired conclusion can be immediately concluded.

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem.

$$\text{Find } x \in C \text{ such that } F(x, y) \ge 0, \quad \forall y \in C. \tag{4.2}$$

In this paper, we use $EP(F)$ to denote the solution set of the equilibrium problem (4.2).

To study the equilibrium problems (4.2), we may assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \le 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\limsup_{t\downarrow 0} F(tz + (1 - t)x, y) \le F(x, y);$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and weakly lower semi-continuous.

Putting $F(x, y) = \langle Ax, y - x\rangle$ for every $x, y \in C$, we see that the equilibrium problem (4.2) is reduced to a variational inequality.

Lemma 4.4. [5] *Let C be a nonempty closed convex subset of a real Hilbert space H, F a bifunction from* $C \times C$ *to* $\mathbb{R}$ *which satisfies* $(A1)$*-*$(A4)$ *and* A_F *a multivalued mapping of H into itself defined by*

$$A_F x = \begin{cases} \{z \in H : F(x, y) \ge \langle y - x, z\rangle, \quad \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases} \tag{4.3}$$

Then A_F is a maximal monotone operator with the domain $D(A_F) \subset C$, $EP(F) = A_F^{-1}(0)$ and

$$T_r x = (I + rA_F)^{-1}x, \quad \forall x \in H, r > 0,$$

where T_r is defined as

$$T_r x = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\}$$

Theorem 4.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and Let F_B be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies $(A1)$-$(A4)$. Let $S : C \to C$ be a nonexpansive mapping with fixed points. Assume that $F(S) \cap EP(F)$ is not empty. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real number sequences in $(0, 1)$ and $\{r_n\}$ be a positive real number sequence in $(0, 2\alpha)$. Assume that the above sequences satisfy the following restrictions:*

(1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(3) $0 < a \leq r_n \leq b < 2\alpha$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$,

where a and b are two real numbers. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$ and

$$\begin{cases} y_n = \alpha_n u + (1 - \alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)ST_{r_n}(y_n - r_n A y_n), \quad \forall n \geq 1, \end{cases}$$

where u is fixed element in C and $J_{r_n} = (I + r_n B)^{-1}$. Then $\{x_n\}$ converges strongly to a point $q \in F(S) \cap EP(F)$, which is an unique solution to the following variational inequality

$$\langle u - q, p - q\rangle \leq 0, \quad \forall p \in F(S) \cap EP(F).$$

If $S = I$, the identity mapping, we have the following result.

Corollary 4.6. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and Let F_B be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies $(A1)$-$(A4)$. Assume that $EP(F)$ is not empty. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real number sequences in $(0, 1)$ and $\{r_n\}$ be a positive real number sequence in $(0, 2\alpha)$. Assume that the above sequences satisfy the following restrictions:*

(1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(3) $0 < a \leq r_n \leq b < 2\alpha$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$,

where a and b are two real numbers. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$ and

$$\begin{cases} y_n = \alpha_n u + (1 - \alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)T_{r_n}(y_n - r_n A y_n), \quad \forall n \geq 1, \end{cases}$$

where u is fixed element in C and $J_{r_n} = (I + r_n B)^{-1}$. Then $\{x_n\}$ converges strongly to a point $q \in EP(F)$, which is an unique solution to the following variational inequality $\langle u - q, p - q\rangle \leq 0, \forall p \in EP(F)$.

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