Filomat 29:6 (2015), 1361–1368
DOI 10.2298/FIL1506361A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Graphs with Large Geodetic Number

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Abstract. For two vertices u and v of a graph G, the set $I[u,v]$ consists of all vertices lying on some $u-v$ geodesic in G. If S is a set of vertices of G, then $I[S]$ is the union of all sets $I[u,v]$ for $u,v \in S$. A subset S of vertices of G is a *geodetic set* if $I[S] = V$. The *geodetic number* $g(G)$ is the minimum cardinality of a geodetic set of G. It was shown that a connected graph G of order $n \geq 3$ has geodetic number $n-1$ if and only if G is the join of K_1 and pairwise disjoint complete graphs $K_{n_1}, K_{n_2}, \ldots, K_{n_r}$, that is, $G = (K_{n_1} \cup K_{n_2} \cup \ldots K_{n_r}) + K_1$, where $r \geq 2$, $n_1, n_2, \ldots, n_r$ are positive integers with $n_1 + n_2 + \ldots + n_r = n-1$. In this paper we characterize all connected graphs G of order $n \geq 3$ with $g(G) = n-2$.

1. Introduction

Throughout this paper, G is a simple connected graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). We refer the reader to the book [17] for graph theory notation and terminology not defined here. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. A vertex v is called a simplicial vertex in a graph G if the subgraph induced by its neighbors is complete. For vertices x and y in a connected graph G, the *distance* $d_G(x,y)$ is the length of a shortest $x-y$ path in G. For a vertex x of G, the *eccentricity* $e_G(x)$ is the distance between x and a vertex farthest from x. The maximum eccentricity among the vertices of G is the *diameter*, $\mathrm{diam}(G)$. An $x-y$ path of length $d_G(x,y)$ is called an $x-y$ *geodesic*. The *geodetic interval* $I[x,y]$ consists of x, y and all vertices lying in some $x-y$ geodesic of G, and for a nonempty subset S of $V(G)$, we define $I[S] = \cup_{x,y \in S} I[x,y]$.

A subset S of vertices of G is a *geodetic set* if $I[S] = V$. The *geodetic number* $g(G)$ is the minimum cardinality of a geodetic set of G. A $g(G)$-*set* is a geodetic set of G of size $g(G)$. The geodetic sets of a connected graph were introduced by Harary, Loukakis and Tsouros [8], as a tool for studying metric properties of connected graphs. It was shown in [1] that the determination of $g(G)$ is an NP-hard problem and its decision problem is NP-complete. The geodetic number and its variants have been studied by several authors (see for example [1, 5–7, 9–16, 18]). Clearly, a connected graph G of order $n \geq 2$ has geodetic number n if and only if $G = K_n$. It was shown in [3] that a connected graph G of order $n \geq 3$ has geodetic number $n-1$ if and only if G is

2010 *Mathematics Subject Classification*. 05C69
Keywords. geodetic set, geodetic number
Received: 08 December 2013; Accepted: 10 September 2014
Communicated by Francesco Belardo
This research was partially supported by the Research Office of Azarbaijan Shahid Madani University
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the join of K_1 and pairwise disjoint complete graphs $K_{n_1}, K_{n_2}, \ldots, K_{n_r}$, that is, $G = (K_{n_1} \cup K_{n_2} \cup \ldots K_{n_r}) + K_1$, where $r \geq 2$, $n_1, n_2, \ldots, n_r$ are positive integers with $n_1 + n_2 + \ldots + n_r = n - 1$.

The purpose of this paper is to characterize all connected graphs G of order $n \geq 3$ with $g(G) = n - 2$.

We make use of the following results in this paper.

Observation 1.1. ([4]) Every geodetic set of a graph contains its simplicial vertices.

Observation 1.2. Every connected graph G of order n different from K_n, has a geodetic set S of size $n - 1$ such that the vertex not in S, belongs to a geodesic path of length two.

Proof. Let G be a connected graph of order n different from K_n. Since $G \neq K_n$, G has three vertices u, v and w such that $uv, uw \in E(G)$ and $vw \notin E(G)$ (see Exercise 1.6.14 in [2]). It follows that $d_G(v, w) = 2$ and $u \in I[v, w]$ and hence $S = V(G) - \{u\}$ is a geodetic set of G with desired property. □

Observation 1.3. Let G be a connected graph of order n with $g(G) \leq n-2$ and let u be an arbitrary vertex of G. Then G has a geodetic set S of size $n - 1$ containing u such that the vertex not in S, belongs to a geodesic path of length two.

Proof. If there is a vertex v at distance 2 from u and $w \in N(u) \cap N(v)$, then $V(G) - \{w\}$ is a geodetic set of G with desired property. Thus, we assume u is adjacent to all vertices of G. Since $g(G) \leq n - 2$, $G - u$ has a component H that is not complete. By Observation 1.2, H has a geodetic set S of size $|V(H)| - 1$ such that the vertex not in S, say x, belongs to a geodesic path of length two. Then obviously $V(G) - \{x\}$ is a geodetic set of G with desired property. □

Proposition 1.4. Let G be a connected graph and H be a connected induced subgraph of G that is not complete. If

1. $G - V(H)$ has a cycle $(v_1v_2v_3v_4)$ in which $v_1v_3 \notin E(G)$, or
2. $G - V(H)$ has a path $v_1v_2v_3v_4$ in which $d_G(v_1, v_4) = 3$ and there is no edge between the sets $\{v_1, v_4\}$ and $V(H)$,

then $g(G) \leq n - 3$.

Proof. By Observation 1.2, H has a geodetic set S of size $|V(H)| - 1$ such that the vertex not in S, say x, belongs to a (y, z)-geodesic path where $d_H(y, z) = 2$. Clearly $d_G(y, z) = 2$. If (1) holds, then clearly $d_G(v_1, v_3) = 2$ and $\{v_2, v_4\} \subseteq I[v_1, v_3]$. It follows that $V(G) - \{x, v_2, v_4\}$ is a geodetic set of G that implies $g(G) \leq n - 3$. If (2) holds, then $x \in I[y, z]$ and $v_2, v_3 \in I[v_1, v_4]$ and so $V(G) - \{x, v_2, v_3\}$ is a geodetic set of G implying that $g(G) \leq n - 3$. □

2. Graphs with Large Geodetic Number

Chartrand, Harary and Zhang [4] established the following upper bound on geodetic number of a graph in terms of its order and diameter.

Theorem A. If G is a nontrivial connected graph of order n and diameter d, then $g(G) \leq n - d + 1$.

Corollary 2.1. If G is a connected graph of order n with $g(G) = n - i$, then $\mathrm{diam}(G) \leq i + 1$.

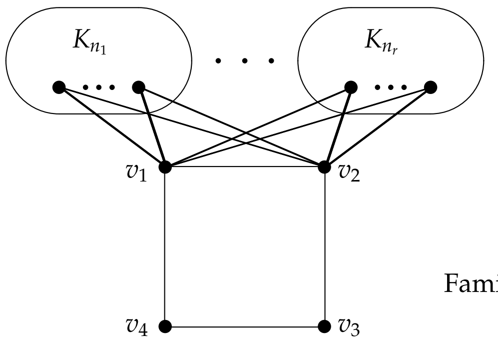
In what follows, we characterize all graphs G of order $n \geq 3$ with $g(G) = n - 2$. By Corollary 2.1 we need to consider connected graphs G for which $\mathrm{diam}(G) = 2$ or 3. First we introduce four families of graphs.

Let $\mathcal{F}_1$ be the collection of all graphs obtained from a cycle $C_4 = (v_1v_2v_3v_4)$ and complete graphs $K_{n_1}, \ldots, K_{n_r}$ (possibly no complete graphs) by joining v_1 and v_2 to all vertices of complete graphs. Clearly $C_4 \in \mathcal{F}_1$.

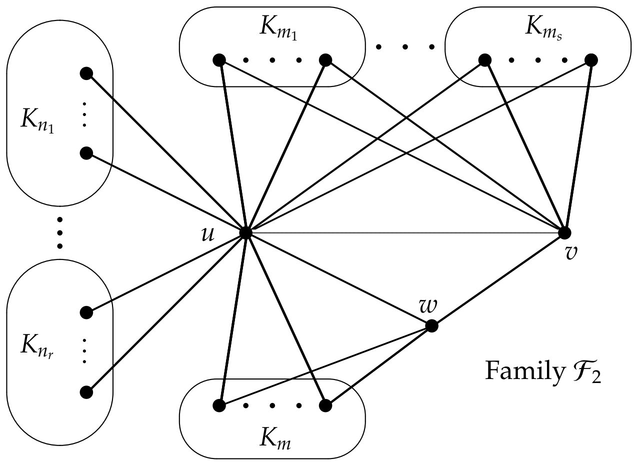
Let $\mathcal{F}_2$ be the collection of all graphs obtained from a triangle $K_3 = uvw$ and complete graphs $K_{n_1}, \ldots, K_{n_r}$ (possibly no complete graphs of this kind), $K_m, K_{m_1}, \ldots, K_{m_s}$, by joining u to all vertices of complete graphs, v to all vertices of $K_{m_1}, \ldots, K_{m_s}$ and w to the vertices of K_m.

Suppose $\mathcal{F}_3$ is the collection of all graphs obtained from $K_2 = uv$ and two classes of complete graphs $K_{n_1}, \ldots, K_{n_r}$ (possibly no complete graphs of this kind) and $K_{m_1}, \ldots, K_{m_s}$ (at least two complete graphs of this kind if there is no complete graph of the first kind), by joining u to all vertices of complete graphs and v to all vertices of $K_{m_1}, \ldots, K_{m_s}$.

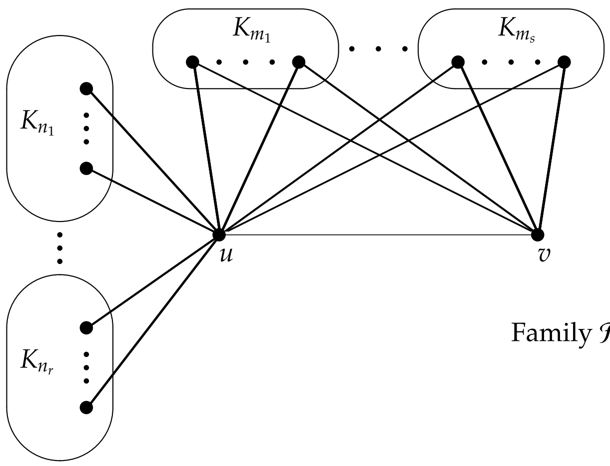
Finally assume that $\mathcal{F}_4$ is the collection of all graphs obtained from $K_2 = xy$ and complete graphs $K_{n_1}, \ldots, K_{n_r}, K_{m_1}, \ldots, K_{m_s}$ and $K_{l_1}, \ldots, K_{l_t}$ (may be no complete graph of this kind) by joining x and y to all vertices of $K_{l_1}, \ldots, K_{l_t}$ and joining x to all vertices of $K_{n_1}, \ldots, K_{n_r}$ and y to all vertices of $K_{m_1}, \ldots, K_{m_s}$.



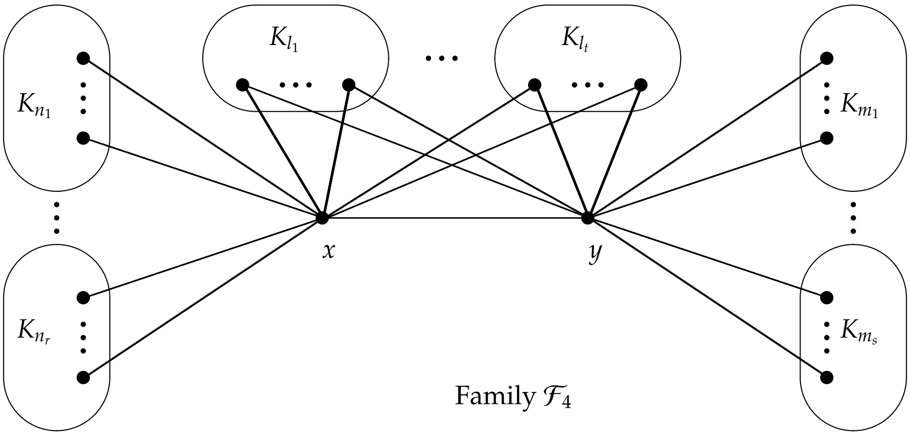
Family $\mathcal{F}_1$



Family $\mathcal{F}_2$



Family $\mathcal{F}_3$



Family $\mathcal{F}_4$

Theorem 2.2. Let G be a connected graph of order n with $\mathrm{diam}(G) = 2$. Then $g(G) = n - 2$ if and only if $G \cong C_5$ or $G \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$.

Proof. If $G = C_4$ or $G = C_5$, then clearly $g(G) = n - 2$. If $G \in \mathcal{F}_3$ then by Observation 1.1, $g(G) = n - 2$ because every vertex of $V(G) - \{u, v\}$ is a simplicial vertex of G. Now let $G \in \mathcal{F}_1 - \{C_4\}$ and S be a $g(G)$-set. By Observation 1.1, $V(G) - \{v_1, v_2, v_3, v_4\} \subseteq S$. If $v_3, v_4 \in S$, then $g(G) = |S| \geq n - 2$. If $v_3 \notin S$ (the case $v_4 \notin S$ is similar), then we must have $v_2, v_4 \in S$ implying that $g(G) = |S| \geq n - 2$. On the other hand, $V(G) - \{v_1, v_2\}$ is a geodetic set of G that implies $g(G) = n - 2$. Finally let $G \in \mathcal{F}_2$ and S be a $g(G)$-set. By Observation 1.1, $V(G) - \{u, v, w\} \subseteq S$. Since $w \notin I[w_1, w_2]$ for each $w_1, w_2 \in V(G) - \{u, v, w\}$, we deduce that $V(G) - \{u, v, w\} \subsetneq S$ and so $g(G) = |S| \geq n - 2$. On the other hand, $V(G) - \{u, v\}$ is a geodetic set of G that yields $g(G) = n - 2$.

Conversely, let G be a connected graph of order n, $\mathrm{diam}(G) = 2$ and $g(G) = n - 2$. Suppose that $S = V(G) - \{x_1, x_2\}$ is a $g(G)$-set. Since, $x_i \notin S$, there exists a $u_i - v_i$ geodesic path containing x_i for $i = 1, 2$. Further, let among the $g(G)$-sets S, the one be selected such that $|\{u_1, v_1\} \cap \{u_2, v_2\}|$ is as large as possible and $x_1x_2 \notin E(G)$ if possible. We consider two cases:

Case 1: $|\{u_1, v_1\} \cap \{u_2, v_2\}| = 2$.
We assume that $u_1 = u_2$ and $v_1 = v_2$. Then $\{x_1, x_2\} \subseteq N(u_1) \cap N(v_1)$. On the other hand, since $V(G) - (N(u_1) \cap N(v_1))$ is a geodetic set of G and since $g(G) = n - 2$, we deduce that

$$N(u_1) \cap N(v_1) = \{x_1, x_2\}. \tag{1}$$

If $n = 4$, then $G = C_4$ and hence $G \in \mathcal{F}_1$. Let $n \geq 5$. Consider the following subcases.

Subcase 1.1. $x_1x_2 \notin E(G)$.
Then $d_G(x_1, x_2) = 2$. An argument similar to that described above, we obtain

$$N(x_1) \cap N(x_2) = \{u_1, v_1\}. \tag{2}$$

Let w be an arbitrary vertex in $V(G) - \{u_1, v_1, x_1, x_2\}$. Since $g(G) = n - 2$, w must be adjacent to some vertex in $\{u_1, v_1, x_1, x_2\}$. We may assume $w \in N(u_1) \setminus \{x_1, x_2\}$. It follows from $N(u_1) \cap N(v_1) = \{x_1, x_2\}$ and the fact $d_G(w, v_1) \leq 2$ that $wv_1 \notin E(G)$ and v_1 and w have a common neighbor, say y. If $y \notin \{x_1, x_2\}$, then $V(G) - \{x_1, x_2, y\}$ is a geodetic set of G which is a contradiction. Therefore $y \in \{x_1, x_2\}$ and hence $w \in N(x_1)$ or $w \in N(x_2)$. By (2), $w \in N(x_1) \setminus N(x_2)$ or $w \in N(x_2) \setminus N(x_1)$. We claim that $N(u_1) - \{x_1, x_2\} \subseteq N(x_1) \setminus N(x_2)$ or $N(u_1) - \{x_1, x_2\} \subseteq N(x_2) \setminus N(x_1)$. Suppose $w_1 \in N(u_1) \cap (N(x_1) \setminus N(x_2))$ and $w_2 \in N(u_1) \cap (N(x_2) \setminus N(x_1))$. If $w_1w_2 \in E(G)$, then $x_2 \in I[v_1, w_2]$ and $\{u_1, w_1\} \subseteq I[x_1, w_2]$ that implies $V(G) - \{x_2, w_1, u_1\}$ is a geodetic set of G, a contradiction. Let $w_1w_2 \notin E(G)$. Then $u_1 \in I[w_1, w_2]$, $x_1 \in I[w_1, v_1]$ and $x_2 \in I[w_2, v_1]$ and so $V(G) - \{x_1, x_2, u_1\}$ is a geodetic set of G, a contradiction. Assume, without loss of generality, that

$$N(u_1) \setminus \{x_1, x_2\} \subseteq N(x_1) \setminus N(x_2). \tag{3}$$

Similarly, we have

$$N(x_1) \setminus \{u_1, v_1\} \subseteq N(u_1) \setminus N(v_1) \tag{4}$$

and

$$N(v_1) \setminus \{x_1, x_2\} \subseteq N(x_1) \setminus N(x_2) \text{ or } N(v_1) \setminus \{x_1, x_2\} \subseteq N(x_2) \setminus N(x_1). \tag{5}$$

Next we show that $\deg(v_1) = 2$. Suppose $z \in N(v_1) - \{x_1, x_2\}$. By (5), we deduce that $zx_1 \in E(G)$ and $zx_2 \notin E(G)$ or $zx_1 \notin E(G)$ and $zx_2 \in E(G)$. First let $zx_1 \in E(G)$ and $zx_2 \notin E(G)$. By (1), $w \neq z$. If $wz \in E(G)$, then $V(G) - \{x_1, x_2, w\}$ is a geodetic set of G, and if $wz \notin E(G)$, then $u_1 \in I[w, x_2]$ by (3), $x_1 \in I[w, z]$ and $v_1 \in I[x_2, z]$ and so $V(G) - \{u_1, v_1, x_1\}$ is a geodetic set of G, a contradiction. Now let $zx_1 \notin E(G)$ and $zx_2 \in E(G)$. If $wz \in E(G)$ then we get a contradiction as above. Let $wz \notin E(G)$. Then w and z have a common neighbor y not in $\{x_1, x_2\}$ and so $u_1 \in I[w, x_2]$ by (3), $y \in I[w, z]$ and $v_1 \in I[x_1, z]$. This implies that $V(G) - \{u_1, v_1, y\}$ is a geodetic set of G, a contradiction. Thus $\deg(v_1) = 2$. By symmetry, we must have $\deg(x_2) = 2$. Since $g(G) = n - 2$, we deduce from Proposition 1.4 that the components of $G[V - \{u_1, v_1, x_1, x_2\}]$ are complete graphs and hence $G \in \mathcal{F}_1$.

Subcase 1.2. $x_1x_2 \in E(G)$.
Consider the components of $G - \{x_1, x_2\}$. Since $g(G) = n - 2$, we conclude from Proposition 1.4 that the components of $G - \{x_1, x_2\}$ not containing u_1, v_1 are complete graphs.

Now let H_{u_1} and H_{v_1} be the components of $G - \{x_1, x_2\}$ containing u_1 and v_1. Clearly, $H_{u_1} \cap H_{v_1} = \emptyset$, otherwise u_1 and v_1 must have a common neighbor in H_{u_1}, a contradiction. Thus H_{u_1} and H_{v_1} are disjoint. If $g(H_{u_1}) \leq |V(H_{u_1})| - 2$ (the case $g(H_{v_1}) \leq |V(H_{v_1})| - 2$ is similar), then by Observation 1.3, we can choose a geodetic set S of H_{u_1} containing u_1 and size $|V(H_{u_1})| - 1$ such that the vertex not in S, say a, belongs to a $x - y$ geodesic path where $x, y \in S$ and $d_G(x, y) = 2$. Then obviously $V(G) - \{a, x_1, x_2\}$ is a geodetic set of G of size at most $n - 3$ which is a contradiction. Hence $g(H_{u_1}) \geq |V(H_{u_1})| - 1$ and $g(H_{v_1}) \geq |V(H_{v_1})| - 1$ implying that H_{u_1} and H_{v_1} are complete graphs or join of K_1 and at least two pairwise disjoint complete graphs where u_1 and v_1 are adjacent to all vertices of H_{u_1} and H_{v_1}, respectively (cf. [3]).

Let $w \neq u_1$ be an arbitrary vertex of H_{u_1}. Since $wv_1 \notin E(G)$ and $\operatorname{diam}(G) = 2$, we have $d_G(v_1, w) = 2$. If v_1 and w have a common neighbor z not in $\{x_1, x_2\}$, then $V(G) - \{x_1, x_2, z\}$ is a geodetic set of G which is a contradiction. Therefore, we may assume that $wx_1 \in E(G)$. We show that $V(H_{u_1}) \subset N(x_1)$. Suppose H_{u_1} has a vertex y which is not adjacent to x_1. As above we must have $x_2y \in E(G)$. If $wy \notin E(G)$, then $V(G) - \{u_1, x_1, x_2\}$ is a geodetic set of G and if $wy \in E(G)$, then $V(G) - \{u_1, w, x_2\}$ is a geodetic set of G, a contradiction. Hence $V(H_{u_1}) \subset N(x_1)$. If H_{u_1} has a vertex $z \neq u_1$ adjacent to x_2, then as above we have $V(H_{u_1}) \subset N(x_2)$. Thus either $V(H_{u_1}) - \{u_1\} \subset N(x_1) - N(x_2)$ or $V(H_{u_1}) \subset N(x_1) \cap N(x_2)$. Similarly, $V(H_{v_1}) - \{v_1\} \subset N(x_1) - N(x_2)$, $V(H_{v_1}) - \{v_1\} \subset N(x_2) - N(x_1)$ or $V(H_{v_1}) \subset N(x_1) \cap N(x_2)$.

Claim. $G - \{x_1, x_2\}$ has at most one component H of order at least 2 with $z \in V(H)$ such that $z \in N(x_1) \cap N(x_2)$ and $V(H) - \{z\} \subseteq N(x_1) - N(x_2)$ or $V(H) - \{z\} \subseteq N(x_2) - N(x_1)$.
Proof. Let H_1 and H_2 be the components of $G - \{x_1, x_2\}$ of order at least 2 with $z_i \in V(H_i)$ such that $z_i \in N(x_1) \cap N(x_2)$ and $V(H_i) - \{z_i\} \subseteq N(x_1) - N(x_2)$ or $V(H_i) - \{z_i\} \subseteq N(x_2) - N(x_1)$ for $i = 1, 2$. Let $z'_i \in V(H_i)$ for $i = 1, 2$. If $V(H_i) - \{z_i\} \subseteq N(x_1) - N(x_2)$ for $i = 1, 2$, then $x_1 \in I[z'_1, z'_2], z_1 \in I[z'_1, x_2], z_2 \in I[z'_2, x_2]$ and hence $V(G) - \{x_1, z_1, z_2\}$ is a geodetic set of G, a contradiction. If $V(H_1) - \{z_1\} \subseteq N(x_1) - N(x_2)$ and $V(H_2) - \{z_2\} \subseteq N(x_2) - N(x_1)$, then $d_G(z'_1, z'_2) = 3$ which is a contradiction again. The other cases also lead to a contradiction. ■

First assume that $G - \{x_1, x_2\}$ has no component H of order at least 2 with $z \in V(H)$ such that $z \in N(x_1) \cap N(x_2)$ and $V(H) - \{z\} \subseteq N(x_1) - N(x_2)$ or $V(H) - \{z\} \subseteq N(x_2) - N(x_1)$. Then $V(H_{u_1}) \subset N(x_1) \cap N(x_2)$ and $V(H_{v_1}) \subset N(x_1) \cap N(x_2)$. It will now be shown that H_{u_1} and H_{v_1} are complete graphs. Assume to the contrary that H_{u_1} is not complete (the other case is similar). Since $g(H_{u_1}) \geq n - 1$, we have $g(H_{u_1}) = n - 1$. Then $H_{u_1} = (K_{n_1} \cup K_{n_2} \cup \ldots K_{n_r}) + K_1$, where $r \geq 2$, $n_1, n_2, \ldots, n_r$ are positive integers with $n_1 + n_2 + \ldots + n_r = n - 1$ and u_1 is adjacent to all vertices of H_{u_1}. Let z_1 and z_2 belong to different components of $H_{u_1} - u_1$. Then $u_1 \in I[z_1, z_2]$ and $x_1, x_2 \in I[z_1, v_1]$ that implies $V(G) - \{u_1, x_1, x_2\}$ is a geodetic set of G, a contradiction. Let H be a component of $G - \{x_1, x_2\}$ not containing u_1, v_1, if any. Since $d_G(V(H), u_1) \leq 2$, we must have $zx_1 \in E(G)$

or $zx_2 \in E(G)$ for each $z \in V(H)$. Assume, without loss of generality, that H has a vertex x that is adjacent to x_1.

We show that $V(H) \subset N(x_1)$. Assume first $|V(H)| \geq 3$. Suppose H has a vertex y which is not adjacent to x_1. Since $d_G(y, u_1) = 2$, we must have $yx_2 \in E(G)$. If H has a vertex $z \neq x$ which is adjacent to x_1, then $x, z \in I[y, x_1], x_2 \in I[y, u_1]$ which leads to a contradiction. This implies that $V(H) - \{x\} \subseteq N(x_2)$. If $xx_2 \notin E(G)$, then $V(H) - \{x\} \subseteq I[x, x_2]$ and $x_1 \in I[x, v_1]$, implying that $V(G) - (\{x_1\} \cup (V(H) - \{x\}))$ is a geodetic set of G which is a contradiction. Hence $x \in N(x_1) \cap N(x_2)$ and $V(H) - \{x\} \subseteq N(x_2) - N(x_1)$ contradicting the assumption. If H has a vertex adjacent to x_2, then as above we have $V(H) \subseteq N(x_2)$. This implies that $V(H) \subset (N(x_1) - N(x_2))$ or $V(H) \subset (N(x_1) \cap N(x_2))$.

Now suppose $|V(H)| = 2$ and let $V(H) = \{x, y\}$. We have $xx_1 \in E(G)$. If $yx_1 \in E(G)$, then by assumption we must have $V(H) \subset (N(x_1) - N(x_2))$ or $V(H) \subset (N(x_1) \cap N(x_2))$. Let $yx_1 \notin E(G)$. Since $d_G(v_1, y) = 2$, we must have $yx_2 \in E(G)$. It follows from assumption that $xx_2 \notin E(G)$. Then $V(G) - \{x_1, y\}$ is a $g(G)$-set which contradicts the choice of S.

If H_1 and H_2 are components of $G - \{x_1, x_2\}$ not containing u_1, v_1, such that $V(H_1) \subset (N(x_1) - N(x_2))$ and $V(H_2) \subset (N(x_2) - N(x_1))$, then $d_G(V(H_1), V(H_2)) = 3$ which contradicts our assumption. Thus for every component K of $G - \{x_1, x_2\}$ not containing u_1, v_1, we have $V(K) \subset (N(x_1) - N(x_2))$ or $V(H) \subset (N(x_1) \cap N(x_2))$. Let $K_{n_1}, \ldots, K_{n_r}$ be the components of $G - \{x_1, x_2\}$ not containing u_1, v_1 such that $V(K_{n_j}) \subset (N(x_1) - N(x_2))$ for $1 \leq j \leq r$ and let $K_{m_1}, \ldots, K_{m_s}$ be the components of $G - \{x_1, x_2\}$ not containing u_1, v_1 such that $V(K_{m_j}) \subset (N(x_1) \cap N(x_2))$ for $1 \leq j \leq s$. It follows that $G \in \mathcal{F}_3$.

Now let $G - \{x_1, x_2\}$ has exactly one component H of order at least 2 with $z \in V(H)$ such that $z \in N(x_1) \cap N(x_2)$ and $V(H) - \{z\} \subseteq N(x_1) - N(x_2)$ or $V(H) - \{z\} \subseteq N(x_2) - N(x_1)$. We may assume, without loss of generality, that $H = H_{u_1}$ and $V(H_{u_1}) - \{u_1\} \subseteq N(x_1) - N(x_2)$. An argument similar to that described above shows that for any component $H \neq H_{u_1}$, either $V(H) \subset N(x_1) \cap N(x_2)$ or $V(H) \subset N(x_1) - N(x_2))$. Let $H_{u_1} = K_m, K_{n_1}, \ldots, K_{n_r}$ be the components of $G - \{x_1, x_2\}$ not containing u_1, v_1 such that $V(K_{n_j}) \subset (N(x_1) - N(x_2))$ for $1 \leq j \leq r$ and let $K_{m_1}, \ldots, K_{m_s}$ be the components of $G - \{x_1, x_2\}$ not containing u_1, v_1 such that $V(K_{m_j}) \subset (N(x_1) \cap N(x_2))$ for $1 \leq j \leq s$. It follows that $G \in \mathcal{F}_2$.

Case 2. $|\{u_1, v_1\} \cap \{u_2, v_2\}| = 1$.
Let $u_1 = u_2$ and $v_1 \neq v_2$.

Subcase 2.1. $x_1x_2 \notin E(G)$.
By the choice of S, we have $v_1x_2 \notin E(G)$ and $v_2x_1 \notin E(G)$. Since $\text{diam}(G) = 2$, $d(v_1, v_2) \leq 2$. If $d(v_1, v_2) = 2$ and $w \in N(v_1) \cap N(v_2)$, then $w \notin \{x_1, x_2\}$ and the set $V(G) - \{x_1, x_2, w\}$ is a geodetic set of G which contradicts $g(G) = n - 2$. Hence $v_1v_2 \in E(G)$. Since $x_1x_2 \notin E(G)$, we deduce that the cycle $(u_1x_1v_1v_2x_2)$ has no chord. We claim that $n = 5$. Suppose $n \geq 6$. Since G is connected, we may choose a vertex $w \in V(G) - \{u_1, x_1, v_1, v_2, x_2\}$ which is adjacent to a vertex in $\{u_1, x_1, v_1, v_2, x_2\}$. Suppose, without loss of generality, that $wu_1 \in E(G)$. If $v_1w \in E(G)$ or $v_2w \in E(G)$, then $V(G) - \{x_1, x_2, w\}$ is a geodetic set of G which is a contradiction. Therefore $v_1w \notin E(G)$ and $v_2w \notin E(G)$. Since $d(v_1, w) \leq 2$, w and v_1 have a common neighbor, say y. If $y \neq x_1$, then $V(G) - \{x_1, x_2, y\}$ is a geodetic set of G which contradicts $g(G) = n - 2$. Hence $N(w) \cap N(v_1) = \{x_1\}$. Similarly, we have $N(w) \cap N(v_2) = \{x_2\}$. Then $V(G) - \{v_1, u_1, w\}$ is a geodetic set of G which contradicts $g(G) = n - 2$. Thus $n = 5$ and so $G = C_5$.

Subcase 2.2. $x_1x_2 \in E(G)$.
By the choice of S, we must have $v_1x_2 \notin E(G)$ and $v_2x_1 \notin E(G)$. Also, v_1 and v_2 are adjacent, for otherwise they have a common neighbor, say w, and $V(G) - \{x_1, x_2, w\}$ is a geodetic set of G, contradicting the assumption $g(G) = n - 2$. Let $S' = V(G) - \{v_1, x_2\}$. Clearly, S' is a $g(G)$-set. Setting $u_1' = u_2' = x_1$ and $v_1' = v_2' = v_2$, we obtain $|\{u_1', v_1'\} \cap \{u_2', v_2'\}| = 2$ that contradicts the choice of S. Thus this case is impossible.

Case 3. $|\{u_1, v_1\} \cap \{u_2, v_2\}| = 0$.
By the choice of S, $u_1x_2, v_1x_2, u_2x_1, v_2x_1 \notin E(G)$. Since $\text{diam}(G) = 2$, $d(u_1, u_2) \leq 2$. If $d(u_1, u_2) = 2$ and w is a common neighbor of u_1, u_2, then $V(G) - \{x_1, x_2, w\}$ is a geodetic set of G which is a contradiction. Hence $u_1u_2 \in E(G)$. Similarly, we must have $v_1v_2 \in E(G)$. Then clearly $V(G) - \{x_1, u_2, v_2\}$ is a geodetic set of G, a contradiction. Thus this case is impossible. This completes the proof. $\square$

Theorem 2.3. Let G be a connected graph of order n with $\mathrm{diam}(G) = 3$. Then $g(G) = n - 2$ if and only if $G \in \mathcal{F}_4$.

Proof. If $G \in \mathcal{F}_4$, then clearly $g(G) = n - 2$, because every vertex of $V(G) - \{x, y\}$ is a simplicial vertex of G.

Let $g(G) = n - 2$ and let $uxyv$ be a diametral path in G. Obviously $V(G) - \{x, y\}$ is a $g(G)$-set and $N(u) \cap N(y) = \{x\}$ and $N(v) \cap N(x) = \{y\}$. We claim that all components of $G - \{x, y\}$ are complete graphs. It follows from Proposition 1.4 that the components of $G - \{x, y\}$ not containing u and v are complete graphs. Now let H_u and H_v be the component of $G - \{x, y\}$ containing u and v, respectively.

If $g(H_u) \leq |V(H_u)| - 2$ (the case $g(H_v) \leq |V(H_v)| - 2$ is similar), then by Observation 1.3, we can choose a geodetic set S of H_u containing u and size $|V(H_u)| - 1$ such that the vertex not in S, say a, belongs to a $w_1 - w_2$ geodesic path where $w_1, w_2 \in S$ and $d_G(w_1, w_2) = 2$. Then obviously $V(G) - \{a, x, y\}$ is a geodetic set of G of size at most $n - 3$ which is a contradiction. Hence $g(H_u) \geq |V(H_u)| - 1$ and $g(H_v) \geq |V(H_v)| - 1$ implying that H_u and H_v are complete graphs or join of K_1 and at least two pairwise disjoint complete graphs.

Suppose H_u is not a complete graph. Then H_u is the join of K_1 and at least two pairwise disjoint complete graphs. Then clearly u is the central vertex of H_u, otherwise the central vertex of H_u, say w, lies on some $u - w_1$ geodesic path implying that $V(G) - \{x, y, w\}$ is geodetic set of G which is a contradiction. Let z_1 and z_2 belong to different components of $H_u - \{u\}$. Then $d_G(z_1, z_2) = 2$ and $u \in I[z_1, z_2]$. On the other hand, since $d_G(u, v) = 3$, $z_1v \notin E(G)$. If $N(v) \cap N(z_1) \neq \emptyset$, then $z_1, x, y \in I[u, v]$ and so $V(G) - \{x, y, z_1\}$ is a geodetic set of G, a contradiction. Hence $N(v) \cap N(z_1) = \emptyset$. It follows that $d_G(z_1, v) = 3$. Similarly, $d_G(z_2, v) = 3$. If $z_1x'y'v$ is a diametral path in G, then obviously $u, z_2 \notin \{x', y'\}$. This implies that $V(G) - \{u, x', y'\}$ is a geodetic set of G, a contradiction. Thus H_u is a complete graph. Now we claim that each vertex of H_u is adjacent to x. Assume to the contrary that some vertex of H_u, say w, is not adjacent to x. Since $d_G(u, v) = 3$, $wv \notin E(G)$. If $N(v) \cap N(w) \neq \emptyset$, then $w, x, y \in I[u, v]$ and so $V(G) - \{x, y, w\}$ is a geodetic set of G, a contradiction. Therefore $N[v] \cap N[w] = \emptyset$ and hence $d_G(w, v) = 3$. Let $wx'y'v$ be a diametral path in G. Since $wx \notin E(G)$, we have $x \neq x'$. Then $x, y \in I[u, v]$ and $x' \in I[w, v]$ that yields $V(G) - \{x, y, x'\}$ is a geodetic set of G, a contradiction. Thus each vertex of H_u is adjacent to x. Similarly, H_v is a complete graph and every vertex of H_v is adjacent to y.

Now let H be a component of $G - \{x, y\}$ different from H_u and H_v. Since G is connected, we may assume, without loss of generality, that H has a vertex w_1 which is adjacent to x. If H has a vertex w_2 that is not adjacent to x, then $V(G) - \{x, y, w_1\}$ is a geodetic set of G, a contradiction. Thus all vertices of H are adjacent to x. Similarly, if a component of $G - \{x, y\}$ different from H_u and H_v, has a vertex that is adjacent to y, then all of its vertices must be adjacent to y.

Let $K_{n_1}, \ldots, K_{n_r}$ be the components of $G - \{x, y\}$ whose vertices are adjacent to x, $K_{m_1}, \ldots, K_{m_s}$ be the components of $G - \{x, y\}$ whose vertices are adjacent to y, and $K_{l_1}, \ldots, K_{l_t}$ (possibly there is no such component) be the components of $G - \{x, y\}$ whose vertices are adjacent to x and y. Thus $G \in \mathcal{F}_4$ and the proof is complete. □

Acknowledgements The authors would like to thank anonymous referees for their remarks and suggestions that helped improve the manuscripts.

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