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Some Permanents of Hessenberg Matrices

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Abstract. In this paper, we present new relationships between the terms of sequence $\{R_n\}$ and $\{S_n\}$ and permanents of some upper Hessenberg matrices.

1. Introduction

Matrix methods are useful tools for derivation some properties of linear recurrences. Some authors obtained many connections between certain sequences and permanents of Hessenberg matrices in the literature [1, 3–5, 8–10]. The permanent of an $n \times n$ matrix $\mathbf{A_n} = \left[a_{ij}\right]$ is defined by

$$per\mathbf{A_n} = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation extends over all permutations σ of the symmetric group S_n. The permanent of a matrix is analogous to the determinant, where all of the signs used in the Laplace expansion of minors are positive.

In [9], Minc defined the $n \times n$ $(0,1)-$matrix $\mathbf{F}(n,k)$, where $k \leq n+1$, with 1 in the (i,j) position for $i-1 \leq j \leq i+k-1$ and 0 otherwise. He showed that $per\, \mathbf{F}(n,3) = T_{n+1}$, where T_n is the nth tribonacci number.

In [5], Kılıç and Taşcı defined the $n \times n$ tridiagonal Toeplitz $(0,-1,1)$-matrix $\mathbf{K_n} = \left[k_{ij}\right]$ with $k_{ii} = -1$ for $1 \leq i \leq n$, $k_{i,i+1} = k_{i+1,i} = 1$ for $1 \leq i \leq n-1$ and 0 otherwise, and the $n \times n$ tridiagonal Toeplitz $(0,-1,1)$-matrix $\mathbf{L_n} = \left[l_{ij}\right]$ with $l_{ii} = -1$ for $2 \leq i \leq n$, $l_{i,i+1} = l_{i+1,i} = 1$ for $1 \leq i \leq n-1$, $l_{11} = -\frac{1}{2}$ and 0 otherwise. They showed $per\mathbf{K_n} = F_{-(n+1)}$ and $per\mathbf{L_n} = \frac{L_{-n}}{2}$, where F_n and L_n are the nth Fibonacci and Lucas numbers, respectively.

In [6], Kılıç and Taşcı defined some Hessenberg matrices. They showed the determinants or permanents of these matrices involving the generalized Fibonacci numbers.

Moreover the authors of [7] gave the relationships between the generalized Lucas sequence and the

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permanent of some Hessenberg matrices. For example,

$$per\begin{bmatrix} a^2+3 & 1 & 0 & \dots & 0 & 0 \\ 1 & a^2+1 & 1 & \dots & \vdots & 0 \\ 1 & 1 & a^2+1 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ 1 & 1 & \dots & 1 & a^2+1 & 1 \\ 1 & 1 & 1 & \dots & 1 & a^2+1 \end{bmatrix} = a^{n-2}v_{n+2},$$

where the generalized Lucas sequence $\{v_n\}$ is defined by $v_{n+1} = av_n + v_{n-1}$, $n \geq 1$ with the initial conditions $v_0 = 2$ and $v_1 = a$.

In [2], Kalman showed that the $(n+k)-$th term of a sequence defined recursively as a linear combination of the preceding k terms:

$$R_{n+k} = c_0R_n + c_1R_{n+1} + \dots + c_{k-1}R_{n+k-1} \tag{1}$$

in which the initial terms $R_0 = \dots = R_{k-2} = 0, R_{k-1} = 1$ and $c_0, c_1, \dots, c_{k-1}$ are constants. The author showed that

$$\begin{bmatrix} R_n \\ R_{n+1} \\ \vdots \\ R_{n+k} \end{bmatrix} = \mathbf{Z}^n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

where $\mathbf{Z} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ c_0 & c_1 & c_2 & & c_{k-2} & c_{k-1} \end{bmatrix}$.

In [10], Ramírez showed some relations between the generalized Fibonacci-Narayana sequence and permanent of one type of upper Hessenberg matrix. For example,

$$per\begin{bmatrix} a & c & c & \cdots & c & & & 0 \\ 1 & a & 0 & 0 & \cdots & c & & \\ & \ddots & \ddots & \ddots & \ddots & & \ddots & \\ & & 1 & a & 0 & 0 & \cdots & c \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & 1 & a & 0 & 0 \\ & & & & & 1 & a & 0 \\ & & & & & & 1 & a \end{bmatrix} = G_{n+r-1}(a,c,r), \tag{2}$$

where the generalized Fibonacci-Narayana sequence $\{G_n(a,c,r)\}_{n\in\mathbb{N}}$ is defined as follows:

$$G_n(a,c,r) = aG_{n-1}(a,c,r) + cG_{n-r}(a,c,r),\ 2 \leq r \leq n,$$

with the initial conditions $G_0(a,c,r) = 0$, $G_i(a,c,r) = 1$, for $i = 1, 2, \dots, r-1$.

2. Some Permanents

In this section, we give some relationships between the terms of the sequences $\{R_n\}$ and $\{S_n\}$, and the permanents of some upper Hessenberg matrices.

For $n \geq 1$, define an $n \times n$ matrix $\mathbf{B_n} = \left[b_{i,j}\right]$ with $b_{i+1,i} = 1$ for $1 \leq i \leq n-1$, $b_{i,i+kt-1} = c_0 - 1$ and $b_{i,i+kt-1-m} = c_m$ for $1 \leq i \leq n$, $1 \leq m \leq k-1$, $1 \leq t \leq \left\lceil \frac{n}{k} \right\rceil$; and 0 otherwise. For example, for $k = 5$ and $n = 9$,

$$\mathbf{B_9} = \begin{bmatrix} c_4 & c_3 & c_2 & c_1 & c_0-1 & c_4 & c_3 & c_2 & c_1 \\ 1 & c_4 & c_3 & c_2 & c_1 & c_0-1 & c_4 & c_3 & c_2 \\ & 1 & c_4 & c_3 & c_2 & c_1 & c_0-1 & c_4 & c_3 \\ & & 1 & c_4 & c_3 & c_2 & c_1 & c_0-1 & c_4 \\ & & & 1 & c_4 & c_3 & c_2 & c_1 & c_0-1 \\ & & & & 1 & c_4 & c_3 & c_2 & c_1 \\ & & 0 & & & 1 & c_4 & c_3 & c_2 \\ & & & & & & 1 & c_4 & c_3 \\ & & & & & & & 1 & c_4 \end{bmatrix}. \tag{3}$$

Then we give the following theorem.

Theorem 2.1. *Let $\mathbf{B_n}$ be the matrix defined in (3). For $1 \leq n$ and $2 \leq k$*

$$per\mathbf{B_n} = R_{n+k-1} - R_{n-1}.$$

Proof. We proceed by induction on n. For $n = 1$, we have

$$per\mathbf{B_1} = c_{k-1} = R_k - R_0.$$

Suppose that the equation holds for $n-1$. Then we show that the equation holds for n. Expanding the $per\mathbf{B_n}$ with respect to the last column k times, we write

$$\begin{aligned} & per\mathbf{B_n} \\ = \; & c_{k-1}per\mathbf{B_{n-1}} + c_{k-2}per\mathbf{B_{n-2}} + \ldots + (c_0 - 1)\, per\mathbf{B}_2 + per\mathbf{B}_2. \end{aligned}$$

By our assumption, we have

$$\begin{aligned} per\mathbf{B_n} &= c_{k-1}\left(R_{n+k-2} - R_{n-2}\right) + c_{k-2}\left(R_{n+k-3} - R_{n-3}\right) + \ldots + c_0\left(R_{k+1} - R_1\right) \\ &= \left(c_{k-1}R_{n+k-2} + c_{k-2}R_{n+k-3} + \ldots + c_0R_{k+1}\right) \\ &\quad - \left(c_{k-1}R_{n-2} + c_{k-2}R_{n-3} + \ldots + c_0R_1\right) \\ &= R_{n+k-1} - R_{n-1}. \end{aligned}$$

Thus, the proof is completed. □

For $n > 1$; we define an $n \times n$ matrix $\mathbf{X_n}$ as in the compact form, by the definition of $\mathbf{B_n}$;

$$\mathbf{X_n} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & & & \\ 0 & & & \\ \vdots & & \mathbf{B_{n-1}} & \\ & & & \\ 0 & & & \end{bmatrix}. \tag{4}$$

Now, we have the following theorem.

Theorem 2.2. *Let $\mathbf{X_n}$ be the matrix defined in (4). Then, for $2 \leq n$ and $2 \leq k$*

$$per\mathbf{X_n} = \sum_{i=1}^{n-1} \left(R_{i+k-1} - R_{i-1}\right) + 1.$$

Proof. We proceed by induction on n. For $n = 2$, we have

$$per\mathbf{X_2} = c_{k-1} + 1 = \sum_{i=1}^{1} (R_{i+k-1} - R_{i-1}).$$

Suppose that the equation holds for n. Then we show that the equation holds for $n+1$. From the definitions of matrices $\mathbf{B_n}$ and $\mathbf{X_n}$, expanding the $per\mathbf{X_{n+1}}$ with respect to the first column gives us

$$per\mathbf{X_{n+1}} = per\mathbf{B_n} + per\mathbf{X_n}.$$

By our assumption and (1), we have

$$per\mathbf{X_{n+1}} = R_{n+k-1} - R_{n-1} + \sum_{i=1}^{n-1} (R_{i+k-1} - R_{i-1}) + 1 = \sum_{i=1}^{n} (R_{i+k-1} - R_{i-1}) + 1.$$

Thus the proof is obtained. □

Now, we have the generalize of the equation in (1) as follows:

$$S_{n+dk} = c_0 S_{n+d-1} + c_1 S_{n+2d-1} + ... + c_{k-1} S_{n+dk-1}, \ 1 \le d \le n \text{ and } 2 \le k, \tag{5}$$

in which the initial terms $S_0 = ... = S_{dk-2} = 0, S_{dk-1} = 1$ and $c_0, c_1, ..., c_{k-1}$ are constants. For $d = 1$, the sequence $\{S_{n+k}\}$ is reduced to the sequence $\{R_{n+k}\}$ in (1). When $k = 2$ and $c_0 = c_1 = d = 1$, the sequence $\{S_{n+2}\}$ is reduced to the Fibonacci sequence $\{F_n\}$. When $k = c_0 = 2$ and $c_1 = d = 1$, the sequence $\{S_{n+2}\}$ is reduced to the Pell sequence $\{P_n\}$.

For $n \ge 1$, define an $n \times n$ matrix $\mathbf{E_n} = \left[e_{i,j}\right]$ with $e_{i+1,i} = 1$ for $1 \le i \le n-1$, $e_{i,i} = e_{1,t} = c_{k-1}$, $e_{i,i+d} = e_{1,t+d} = c_{k-2}, ..., e_{i,i+(k-2)d} = e_{1,t+(k-2)d} = c_1$, $e_{i,i+(k-1)d} = e_{1,t+(k-1)d} = c_0$ for $2 \le t \le d \le n$, $1 \le i \le n$; and 0 otherwise. For example, for $k = d = 3$ and $n = 9$, we write

$$\mathbf{E_9} = \begin{bmatrix} c_2 & c_2 & c_2 & c_1 & c_1 & c_1 & c_0 & c_0 & c_0 \\ 1 & c_2 & 0 & 0 & c_1 & 0 & 0 & c_0 & 0 \\ & 1 & c_2 & 0 & 0 & c_1 & 0 & 0 & c_0 \\ & & 1 & c_2 & 0 & 0 & c_1 & 0 & 0 \\ & & & 1 & c_2 & 0 & 0 & c_1 & 0 \\ & & & & 1 & c_2 & 0 & 0 & c_1 \\ & & 0 & & & 1 & c_2 & 0 & 0 \\ & & & & & & 1 & c_2 & 0 \\ & & & & & & & 1 & c_2 \end{bmatrix}. \tag{6}$$

Theorem 2.3. *Let $\mathbf{E_n}$ be the matrix defined in (6). For $1 \le n$, and $2 \le k$,*

$$per\mathbf{E_n} = \sum_{i=0}^{d-1} S_{n+dk-1-i},$$

where $d = 1, 2, ..., n$.

Proof. We proceed by induction on n. For $n = 1$, we have

$$per\mathbf{E_1} = c_{k-1} = S_k = \sum_{i=0}^{0} S_{k-i}.$$

Suppose that the equation holds for $n-1$. Then we show that the equation holds for n. Expanding the $per\mathbf{E_n}$ with respect to the last column k times, we write

$$per\mathbf{E_n} = c_{k-1}per\mathbf{E_{n-1}} + c_{k-2}per\mathbf{E_{n-d-1}} + ... + c_0 per\mathbf{E_{n-d(\mathit{k}-1)-1}}.$$

By our assumption, we write

$$\begin{aligned} per\mathbf{E_n} &= c_{k-1}\sum_{i=0}^{d-1} S_{n+dk-i-2} + c_{k-2}\sum_{i=0}^{d-1} S_{n+d(k-1)-i-2} + ... + c_0\sum_{i=0}^{d-1} S_{n+d-i-2} \\ &= \sum_{i=0}^{d-1}\left(c_{k-1}S_{n+dk-i-2} + c_{k-2}S_{n+d(k-1)-i-2} + ... + c_0 S_{n+d-i-2}\right). \end{aligned}$$

Thus, by the recurrence relation in (5), we have the proof. □

3. Some Special Cases

In this section, we will give some special cases of the above theorems.

- For $d = c_0 = 1$, $k = 2$ and $c_1 = a$ in (5), the generalized Fibonacci sequence $\{U_n(a,1)\}$,

$$per\mathbf{B}_{2n} = per\begin{bmatrix} a & 0 & a & 0 & \cdots & a & 0 \\ 1 & a & 0 & a & & \vdots & a \\ & & \ddots & & \ddots & & \\ & & \ddots & \ddots & \ddots & \vdots & \\ & & & \ddots & \ddots & & \\ & 0 & & & & a & 0 \\ & & & & & 1 & a \end{bmatrix} = aU_{2n}(a,1),$$

and

$$per\mathbf{E}_n = per\begin{bmatrix} a & 1 & & & & \\ 1 & a & 1 & & 0 & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \vdots & \\ & 0 & & \ddots & a & 1 \\ & & & & 1 & a \end{bmatrix} = U_{n+1}(a,1).$$

If we take $a = 1$, we have $per\mathbf{B}_{2n} = F_{2n}$ and $per\mathbf{E}_n = F_{n+1}$.

- For $d = c_1 = 1$ and $k = c_0 = 2$ in (5), $\{J_n\}$ is the Jacobsthal sequence,

$$per\mathbf{B}_n = per\begin{bmatrix} 1 & 1 & & \cdots & & 1 & 1 \\ 1 & 1 & 1 & & & \vdots & 1 \\ & \ddots & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & \ddots & & \\ & 0 & & & 1 & 1 & 1 \\ & & & & & 1 & 1 \end{bmatrix} = J_{n+1} - J_{n-1},$$

and

$$per\mathbf{E_n} = per\begin{bmatrix} 1 & 2 & & & & \\ 1 & 1 & 2 & & 0 & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & \ddots & 1 & 2 \\ & & & & 1 & 1 \end{bmatrix} = J_{n+1}\ .$$

- For $d = c_0 = c_1 = c_2 = 1$ and $k = 3$ in (5), $\{T_n\}$ is the tribonacci sequence,

$$per\mathbf{B}_{3n} = per\begin{bmatrix} 1 & 1 & 0 & & \dots & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & & & 1 & 1 \\ & 1 & & & & & & 1 \\ & & & \ddots & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & 0 & & & & & 0 \\ & & & & & 1 & 1 & 1 \\ & & & & & & 1 & 1 \end{bmatrix} = T_{3n+1} + T_{3n},$$

and

$$per\mathbf{E_n} = per\begin{bmatrix} 1 & 1 & 1 & & & & \\ 1 & 1 & 1 & 1 & & 0 & \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & & 1 \\ & 0 & & & & 1 & 1 \\ & & & & & 1 & 1 \end{bmatrix} = T_{n+2}.$$

- For $c_0 = c_2 = d = 1$, $c_1 = 0$ and $k = 3$ in (5), $\{N_n\}$ is the Narayana sequence,

$$per\mathbf{B}_{3n} = per\begin{bmatrix} 1 & 0 & 0 & & \dots & & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & & & & 1 & 0 \\ & 1 & & & & & & & 1 \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & \ddots & \ddots & \ddots & & \vdots \\ & & & & & \ddots & \ddots & & \\ & & 0 & & & & & & 0 \\ & & & & & & & 1 & 0 \\ & & & & & & & 1 & 1 \end{bmatrix} = N_{3n+1},$$

and

$$per\mathbf{E_n} = per\begin{bmatrix} 1 & 0 & 1 & & & & \\ 1 & 1 & 0 & 1 & & 0 & \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots & 1 \\ & 0 & & & & 1 & 0 \\ & & & & & 1 & 1 \end{bmatrix} = N_{n+2}.$$

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