Filomat 29:6 (2015), 1217–1222
DOI 10.2298/FIL1506217A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Multivalued Caristi'S Type Mappings in Fuzzy Metric Spaces and a Characterization of Fuzzy Metric Completeness

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**Abstract.** In this note we show a fixed point theorem for multivalued Caristi's type mappings on a large class of complete fuzzy metric spaces. In fact, this result allows us to characterize complete fuzzy metric spaces with continuous t-norm greater than or equal to the Lukasiewicz t-norm.

## 1. Introduction and Preliminaries

Due to its applications in mathematics and other related disciplines Banach contraction principle has been generalized in many directions (for details one can see [2, 4, 5, 8, 11, 26, 33, 34]). Caristi's fixed point theorem is one of the most useful among these generalizations, which further has been extended and generalized by many authors in several directions (see e.g. [1, 11, 18–21, 28, 31, 36]).

The concept of fuzzy sets was initiated by Zadeh [35] in 1965. Kramosil and Michalek [22] introduced a notion of fuzzy metric space by using continuous t-norms, which generalizes the concept of probabilistic metric space to fuzzy situation. Fixed point theory in fuzzy metric spaces has been studied by a number of authors. For a wide survey we refer [9, 10, 14, 16, 23–25, 27, 32, 37] and the references therein.

The aim of this paper is to obtain a fixed point theorem for multivalued mappings of Caristi's type in complete fuzzy metric spaces which actually provides a characterization of fuzzy metric completeness in the case of continuous t-norms greater than or equal to the Lukasiewicz t-norm. We recall that previous and interesting versions of Caristi's fixed point theorem for fuzzy metric spaces (actually, for probabilistic Menger spaces), but with an approach different from our one, were proved by Hadžić and Pap ([17, Section 3.4]). Thus our results and the ones given in [17, Section 3.4] are of an independent value.

Next we recall some pertinent concepts and results.

Following [29], a binary operation $* : [0,1]^2 \to [0,1]$ is called a continuous t-norm if: (i) $*$ is associative and commutative; (ii) $*$ is continuous; (iii) $a * 1 = a$ for all $a \in [0,1]$; and (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$.

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2010 *Mathematics Subject Classification*. Primary 47H10; Secondary 47H04, 54A40, 54E50, 54H25
*Keywords*. Multivalued mapping, Caristi's type mapping, fuzzy metric space, complete
Received: 25 November 2013; Accepted: 17 January 2014
Communicated by Ljubomir Ćirić
The third named author thanks the support of the Ministry of Economy and Competitiveness of Spain, Grant MTM2012-37894-C02-01.
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Typical instances of continuous t-norm are $\wedge$, $\cdot$ and $*_L$, where, for all $a, b \in [0,1]$, $a \wedge b = \min\{a, b\}$, $a \cdot b = ab$, and $*_L$ is the Lukasiewicz t-norm defined by $a *_L b = \max\{a + b - 1, 0\}$.

It is easy to check that $*_L \leq \cdot \leq \wedge$. In fact $* \leq \wedge$ for all continuous t-norm $*$.

In our context we will use the following notion of a fuzzy metric space which is a slight modification to the one given by Kramosil and Michalek in [22] (it is appropriate to point out that George and Veeramani presented in [12, 13] a stronger but interesting notion of fuzzy metric completeness, which will not be explicitly considered here).

**Definition 1** (compare [22])**.** A fuzzy metric space is a triple $(X, M, *)$ such that $*$ is a continuous t-norm and $M$ is a fuzzy set in $X \times X \times [0, +\infty)$ such that for all $x, y, z \in X$ :
(i) $M(x, y, 0) = 0$;
(ii) $x = y$ if and only if $M(x, y, t) = 1$ for all $t > 0$;
(iii) $M(x, y, t) = M(y, x, t)$;
(iv) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ for all $t, s \geq 0$;
(v) $M(x, y, \_) : [0, +\infty) \to [0, 1]$ is left continuous.

In this case, the pair $(M, *)$ (or simply, $M$ if no confusion arises) is said to be a fuzzy metric on $X$.

It is well known, and easy to see, that for each $x, y \in X$, $M(x, y, \_)$ is a non-decreasing function on $[0, +\infty)$.

Each fuzzy metric $(M, *)$ on a set $X$ induces a topology $\tau_M$ on $X$ which has a base the family of open balls $\{B_M(x, \varepsilon, t) : x \in X, \varepsilon \in (0, 1), t > 0\}$, where $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$.

Observe that a sequence $(x_n)_{n \in \mathbb{N}}$ converges to $x \in X$ (with respect to $\tau_M$) if and only if $\lim_{n\to\infty} M(x, x_n, t) = 1$ for all $t > 0$.

It is also well known (see, for instance, [15]) that every fuzzy metric space $(X, M, *)$ is metrizable, i.e., there exists a metric $d$ on $X$ whose induced topology agrees with $\tau_M$.

Conversely, if $(X, d)$ is a metric space and we define $M_d : X \times X \times [0, +\infty) \to [0, 1]$ by $M(x, y, 0) = 0$ and

$$M_d(x, y, t) = \frac{t}{t + d(x, y)},$$

for all $t > 0$, then $(X, M_d, \wedge)$ is a fuzzy metric space and $(M_d, \wedge)$ is called the standard fuzzy metric of $(X, d)$ ([12]). Moreover, the topology $\tau_{M_d}$ agrees with the topology induced by $d$.

A sequence $(x_n)_{n \in \mathbb{N}}$ in a fuzzy metric space $(X, M, *)$ is said to be a Cauchy sequence if for each $t > 0$ and $\varepsilon \in (0, 1)$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.

A fuzzy metric space $(X, M, *)$ is said to be complete ([13]) if every Cauchy sequence converges.

## 2. The Results

Let $(X, d)$ be a metric space. A mapping $T : X \to X$ is called a Caristi's mapping if there is a lower semicontinuous function $\varphi : X \to [0, +\infty)$ satisfying the following condition

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx),$$

for all $x \in X$.

Caristi proved in [6] his celebrated theorem that every Caristi's mapping on a complete metric space has a fixed point.

Later on, Kirk proved the following nice characterization of metric completeness.

**Theorem 1** ([21]). *A metric space $(X, d)$ is complete if and only if every Caristi's mapping $T : X \to X$ has a fixed point.*

In the sequel, we shall denote by $C_0(X)$ the set of all non-empty closed subsets of a metric space $(X, d)$, or of a fuzzy metric space $(X, M, *)$.

There exist several multivalued generalizations of Caristi's fixed point theorem in the literature. For our purposes here we need the following .

**Theorem 2** (see e.g. [3]). *Let $(X, d)$ be a complete metric space and let $T : X \to C_0(X)$ be a multivalued mapping such that there is a lower semicontinuous function $\varphi : X \to [0, +\infty)$ satisfying the following condition: For each $x \in X$ there is $y \in Tx$ with*

$$d(x, y) \leq \varphi(x) - \varphi(y).$$

*Then $T$ has a fixed point, i.e., there is a $z \in X$ such that $z \in Tz$.*

A multivalued mapping $T$ satisfying the conditions of the preceding theorem will be called a Caristi's multivalued mapping (for $(X, d)$).

**Definition 2.** Let $(X, M, *)$ be a fuzzy metric space. We say that $T : X \to X$ is a fuzzy Caristi's mapping on $X$ if there is a lower semicontinuous function $\varphi : X \to [0, +\infty)$ satisfying the following condition:

($I_C$) $\quad \varphi(x) - \varphi(Tx) < t \Longrightarrow M(x, Tx, t) > 1 - t.$

The next example shows that every Caristi's mapping on a metric space $(X, d)$ is a Caristi's mapping on a well-known fuzzy metric space induced by $(X, d)$ in a natural way.

**Example 1.** Let $(X, d)$ be a metric space and let $T : X \to X$ be a Caristi's mapping. Consider the fuzzy metric $(M, *)$ on $X$ (where $*$ is any continuous t-norm) given by $M(x, y, t) = 1$ if $d(x, y) < t$, and $M(x, y, t) = 0$ if $d(x, y) \geq t$ (with $t \geq 0$). We show that $T$ is a fuzzy Caristi's mapping for $(X, M, *)$. Indeed, let $\varphi : X \to [0, +\infty)$ be a lower semicontinuous function for $(X, d)$ such that $d(x, Tx) \leq \varphi(x) - \varphi(Tx)$, for all $x \in X$. First observe that $\varphi$ is also lower semicontinuous for $(X, M, *)$ because the topology $\tau_M$ agrees with the topology induced by $d$. Now suppose that $\varphi(x) - \varphi(Tx) < t$ for any $t > 0$. Then $d(x, Tx) < t$, so $M(x, Tx, t) = 1$. Hence, condition (**I**$_C$) is satisfied and thus $T$ is a fuzzy Caristi's mapping for $(X, M, *)$.

**Definition 3.** Let $(X, M, *)$ be a fuzzy metric space. We say that $T : X \to C_0(X)$ is a fuzzy Caristi's multivalued mapping on $X$ if there is a lower semicontinuous function $\varphi : X \to [0, +\infty)$ satisfying the following condition:

($I_{CM}$) For each $x \in X$ there is $y_x \in Tx$ such that
$\varphi(x) - \varphi(y_x) < t \Longrightarrow M(x, y_x, t) > 1 - t.$

It is clear that every fuzzy Caristi's mapping can be considered as a fuzzy Caristi's multivalued mapping.

Consider now a fuzzy metric space $(X, M, *)$ with $* \geq *_L$. Then, Radu proved in [30] (see also [7, Remark 7.6.1]) that the function $d_M$ defined on $X \times X$ by

$$d_M(x, y) = \sup\{t \geq 0 : M(x, y, t) \leq 1 - t\}$$

for all $x, y \in X$, is a metric on $X$. Moreover, is is easy to check that for each $x, y \in X$ and $t \in (0, 1)$, we have

$$d_M(x, y) < t \Longleftrightarrow M(x, y, t) > 1 - t.$$

From this fact it is easily deduced the following.

**Proposition 1.** *Let $(X, M, *)$ be a fuzzy metric space such that $* \geq *_L$. Then $(X, M, *)$ is complete if and only if the metric space $(X, d_M)$ is complete.*

With the help of the above results and facts we can prove the following.

**Theorem 3.** *Let $(X, M, *)$ be a fuzzy metric space such that $* \geq *_L$. Then, the following are equivalent:*
(1) *$(X, M, *)$ is complete.*
(2) *Every fuzzy Caristi's multivalued mapping $T : X \to C_0(X)$ has a fixed point.*
(3) *Every fuzzy Caristi's mapping $T : X \to X$ has a fixed point.*

*Proof.* (1) ⇒ (2). Since $(X, M, *)$ is complete, then the metric space $(X, d_M)$ is complete by Proposition 1.

Let $T : X \to C_0(X)$ be a fuzzy Caristi's multivalued mapping. Then, there exists a lower semicontinuous function $\varphi : X \to [0, +\infty)$ for which condition $(I_{CM})$ holds.

Let $x \in X$. By $(I_{CM})$, there is $y_x \in Tx$ such that

$$\varphi(x) - \varphi(y_x) < t \Longrightarrow M(x, y_x, t) > 1 - t.$$

Suppose $\varphi(x) - \varphi(y_x) < d_M(x, y_x)$. Then, there exists $t_0 > 0$ such that $\varphi(x) - \varphi(y_x) < t_0 \leq 1 - M(x, y_x, t_0)$, so $M(x, y_x, t_0) \leq 1 - t_0$, which contradicts condition $(I_{CM})$. Therefore, $d_M(x, y_x) \leq \varphi(x) - \varphi(y_x)$. Hence, we can apply Theorem 2 and thus $T$ has a fixed point.

(2) ⇒ (3). Obvious.

(3) ⇒ (1). Let $T : X \to X$ be a Caristi's mapping for the metric space $(X, d_M)$. We shall prove that $T$ has a fixed point. Indeed, there exists a lower semicontinuous function $\varphi : X \to [0, +\infty)$ such that

$$d_M(x, Tx) \leq \varphi(x) - \varphi(Tx),$$

for all $x \in X$.

Suppose $\varphi(x) - \varphi(Tx) < t$. Then $d_M(x, Tx) < t$, and hence $M(x, Tx, t) > 1 - t$. We have shown that condition $(I_C)$ is satisfied, so $T$ is a fuzzy Caristi's mapping for $(X, M, *)$. By our hypothesis $T$ has a fixed point. Therefore $(X, d_M)$ is complete by Theorem 1. We conclude that $(X, M, *)$ is complete by Proposition 1.

The following natural question remains open:

**Question.** Is it possible to generalize Theorem 3 to the case that $*$ is any continuous t-norm?

We conclude the paper with two examples that illustrate Theorem 3.

**Example 2.** Let $X = (0, 1]$ and let $(M, *)$ be the fuzzy metric on $X$, with $* \geq *_L$, given by $M(x, y, 0) = 0$ for all $x, y \in X$, $M(x, x, t) = 1$ for all $x \in X$ and $t > 0$, and $M(x, y, t) = x * y$ otherwise.

First note that $x = 1$ is the unique non-isolated point of $X$, because

$$B_M(1, 1/n, 1/n) = \{y \in X : y > 1 - 1/n\},$$

for all $n \in \mathbb{N}$, while for each $x \in X \setminus \{1\}$, one has

$$B_M(x, x, 1 - x) = \{x\} \cup \{y \in X : x * y > x\} = \{x\}.$$

Consequently $(X, M, *)$ is complete. In fact, if $(x_n)_{n \in \mathbb{N}}$ is a non-eventually constant Cauchy sequence, then for each $\varepsilon \in (0, 1)$ there is $n_0 \in \mathbb{N}$ such that $x_n * x_m > 1 - \varepsilon$ for all $n, m \geq n_0$, so $(x_n)_{n \in \mathbb{N}}$ converges to 1 with respect to the Euclidean topology and hence with respect to $\tau_M$.

Define $T : X \to C_0(X)$ by $Tx = [\sqrt{x}, 1]$. We show that $T$ is a multivalued Caristi's fuzzy mapping on $(X, M, *)$. Indeed, let $\varphi : X \to [0, +\infty)$ be the lower semicontinuous function on $X$ defined by $\varphi(1) = 0$ and $\varphi(x) = 1$ otherwise. Let $x \in X$. Take $y_x = 1 \in Tx$, and suppose $\varphi(x) - \varphi(y_x) < t$. Then $t > 1$, and consequently $M(x, y_x, t) \geq 0 > 1 - t$. We have shown that condition $(\mathbf{I}_{CM})$ is satisfied, and hence $T$ is a multivalued Caristi's fuzzy mapping. Clearly 1 is a fixed point of $T$ (in fact, it is the only fixed point of $T$).

**Example 3.** Let $X = (0, 1]$ and let $(M, \wedge)$ be the fuzzy metric on $X$ given by $M(x, y, 0) = 0$ for all $x, y \in X$, $M(x, x, t) = 1$ for all $x \in X$ and $t > 0$, and $M(x, y, t) = \min\{x, y, t\}$ otherwise.

First note that $\tau_M$ is the discrete topology on $X$ because for each $x \in X$ one has

$$B_M(x, 1/2, 1/2) = \{x\} \cup \{y \in X : \min\{x, y, 1/2\} > 1/2\} = \{x\}.$$

Moreover $(X, M, *)$ is complete because for $x \neq y$ and $t \in (0, 1)$ we have $M(x, y, t) \leq t$ and hence the eventually constant sequences are the only Cauchy sequences in $(X, M, \wedge)$.

Denote by $\mathbb{Q}$ the set of rational numbers and define $T : X \to C_0(X)$ by

$$Tx = \{y \in (0, 1] \cap \mathbb{Q} : x \leq y \leq \frac{x + 1}{2}\},$$

for all $x \in X$. Note that, indeed, $Tx \in C_0(X)$ since $\tau_M$ is the discrete topology on $X$.

We show that $T$ is a multivalued Caristi's fuzzy mapping on $(X, M, *)$. Indeed, let $\varphi : X \to [0, +\infty)$ be the lower semicontinuous function on $X$ defined by $\varphi(x) = 0$ if $x \in (0, 1] \cap \mathbb{Q}$ and $\varphi(x) = 1$ otherwise. Let $x \in X$. If $x \in \mathbb{Q}$ take $y_x = x \in Tx$. Then $M(x, y_x, t) = 1$ for all $t > 0$, so condition $(\mathbf{I}_{CM})$ is trivially satisfied in this case. If $x \notin \mathbb{Q}$, take $y_x$ a rational number belonging to $(x, x + 1/2)$ and suppose $\varphi(x) - \varphi(y_x) < t$. Then $t > 1$, and consequently $M(x, y_x, t) \geq 0 > 1 - t$. Therefore $T$ is a multivalued Caristi's fuzzy mapping. Clearly $x \in Tx$ for all $x \in (0, 1] \cap \mathbb{Q}$ (in particular $T1 = \{1\}$).

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