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Some New Fixed Point Results via the Concept of Measure of Noncompactness

Asadollah Aghajani[a], Alireza Mosleh Tehrani[a], Donald O'Regan[b]

[a] *School of Mathematics, Iran University of Science and Technology, Narmak, Tehran 16844-13114, Iran.*
[b] *School of Mathematics, Statistics and Applied Mathematics National University of Ireland, Galway, University Road, Galway, Ireland.*

Abstract. In this paper we present a generalization of Darbo's fixed point theorem and using this generalization we prove an ϵ-fixed point result in Banach spaces. Also, we present a generalization of Darbo and Sadovskiĭ fixed point theorem in uniformly convex Banach spaces.

1. Introduction and Preliminaries

The concept of a measure of noncompactness (MNC) was initiated by Kuratowski [21]. If A is a bounded set of a metric space, the Kuratowski MNC of A is defined as

$$\alpha(A) = \inf\{\epsilon > 0 : A \text{ can be covered with a finite number of sets of diameter smaller than } \epsilon\}.$$

In 1957, Goldenštein, Gohberg and Markus [18] introduced another MNC called the Hausdorff MNC and is defined as

$$\chi(A) = \inf\Big\{\epsilon > 0 : A \text{ has a finite } \epsilon - \text{net in } E\Big\}.$$

We refer the reader to Sadovskiĭ [24] who introduced a general concept of MNC (see also [7], [10], [11]).

In 1955, G. Darbo proved a fixed point theorem via the concept of Kuratowski MNC [15] which generalizes the classical Schauder fixed point theorem. In 1980, Banaś proved a fixed point theorem of Darbo type (see Theorem 1.5) using the axiomatic definition of MNC [11].

For applications to differential and integral equations we refer the reader to [3, 5, 6, 9, 10, 12–14, 16, 17, 19, 20, 22, 23, 25, 26].

For the rest of this section, we provide some notations, definitions and fundamental theorems which will be needed. Let E be a given Banach space with the norm $\|.\|$ and zero element θ. Denote by $\overline{X}$ and $\mathrm{Conv}X$ the closure and closed convex hull of X, respectively, where X is a nonempty, bounded subset of E. We denote by $\mathfrak{M}_E$ the family of all nonempty, bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

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Email addresses: aghajani@iust.ac.ir (Asadollah Aghajani), amtehrani@iust.ac.ir (Alireza Mosleh Tehrani), donal.oregan@nuigalway.ie (Donald O'Regan)

Definition 1.1 ([1]). *Let X and Y be normed linear spaces. A map $T : X \to Y$ is called compact if TX is contained in a compact subset of Y.*

Definition 1.2 ([1]). *Let X be a subset of a Banach space E and $F : X \to X$ a map. Given $\epsilon > 0$, a point $x \in X$ with $\|x - F(x)\| < \epsilon$ is called an $\epsilon-$ fixed point for F. We say that F has the $\epsilon-$fixed point property if for any $\epsilon > 0$, F has a $\epsilon-$ fixed point.*

Definition 1.3. *A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+ = [0, +\infty)$ is said to be an MNC in E if it satisfies the following conditions for all $X, Y \in \mathfrak{M}_E$:*

1^o *ker* $\mu = \Big\{X \in \mathfrak{M}_E : \mu(X) = 0\Big\} \neq \emptyset$ *and ker* $\mu \subseteq \mathfrak{N}_E$.

2^o $X \subseteq Y \Rightarrow \mu(X) \leqslant \mu(Y)$.

3^o $\mu(\overline{X}) = \mu(Conv X) = \mu(X)$.

4^o $\mu\Big(\lambda X + (1 - \lambda)Y\Big) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y)$ *for every* $\lambda \in [0, 1]$.

5^o *If* $X_n \in \mathfrak{M}_E$, $X_{n+1} \subseteq X_n$, $X_n = \overline{X}_n$ *for n=1, 2, 3, ... and* $\lim\limits_{n\to\infty} \mu(X_n) = 0$, *then* $X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \emptyset$.

In addition, if μ satisfies

6^o $\mu(X + Y) \leqslant \mu(X) + \mu(Y)$

then it is said to be subadditive.

The family ker μ mentioned in 1^o is called the kernel of the MNC μ. Also, notice that the intersection set X_∞ from axiom 5^o is a member of ker μ.

An elementary example of an MNC on a Banach space E is defined as follows

$$\mu(A) = \delta(A) \text{ for all } A \in \mathfrak{M}_E; \tag{1}$$

here $\delta(A) = \sup\Big\{\|x - y\| : x, y \in A\Big\}$.

Theorem 1.4 ([1]). *Let C be a nonempty, bounded, closed and convex subset of a Banach space E. Then every compact, continuous map $F : C \to C$ has at least one fixed point.*

The above fixed point theorem is known as Schauder's fixed point principle and its generalization, called the Darbo fixed point theorem, is stated next.

Theorem 1.5 ([10]). *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : C \to C$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that*

$$\mu(TA) \leq k\mu(A), \tag{2}$$

for any nonempty subset A of C, where μ is an MNC defined in E. Then T has at least a fixed point in the set C.

A celebrated generalization of Darbo's fixed point theorem is the following result, usually called the theorem of Darbo and Sadovskiĭ.

Theorem 1.6 ([8]). *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : C \to C$ be a continuous mapping. Assume that μ is an MNC defined on $\mathfrak{M}_E$ with the following additional condition*

$$\mu(A \cup B) = \max\Big\{\mu(A), \mu(B)\Big\} \text{ for all } A, B \in \mathfrak{M}_E.$$

If for any nonempty subset A of C we have

$$\mu(TA) < \mu(A),$$

then T has a fixed point in C.

In this paper, we obtain a generalization of the Darbo fixed point theorem under weaker conditions than [4] and using this generalization we prove an ϵ-fixed point result in Banach spaces. Under a certain condition, we give some fixed point results for mappings that have the ϵ-fixed point property. The last section of this paper is devoted to generalizing Theorem 1.6 in uniformly convex Banach spaces.

2. ϵ-fixed Point Result

Let E be a Banach space and μ be an MNC on $\mathfrak{M}_E$ such that $\{\theta\} \in \ker\mu$. Let C be a nonempty subset of E and $T : C \to C$ be a map. For any nonempty subset A of C we define, and fix hereafter, the iterative sequence $\{A_n(T)\}$, dependent on the set A and the map T, as follows

$$A_0 = A \text{ and } A_n = \text{Conv}TA_{n-1} \text{ for all } n \in \mathbb{N}. \tag{3}$$

Now, we present a simple generalization of the Darbo fixed point theorem.

Theorem 2.1. *Let C be a nonempty, bounded, closed and convex subset of E and $T : C \to C$ be a continuous mapping. Assume that there exists a nondecreasing function $\psi : [0, +\infty) \to [0, +\infty)$ such that $\psi^n(t) \to 0$ as $n \to \infty$ for each $t \geq 0$. If for all nonempty, closed and convex subsets A of C with $TA \subseteq A$ there exists a constant $m = m(A) \in \mathbb{N}$ such that*

$$\mu\Big(A_m(T)\Big) \leqslant \psi\Big(\mu(A)\Big). \tag{4}$$

Then, T has at least a fixed point in C.

Proof. Let

$$\mathfrak{B} = \Big\{B \subseteq C : B \text{ is nonempty, bounded, closed and convex with } TB \subseteq B\Big\}.$$

Notice that $C_n(T) \in \mathfrak{B}$ for all $n \in \mathbb{N}$ where C_n is defined as in (3) with $C_0 = C$. Since the sequence $\Big\{\mu\Big(C_n(T)\Big)\Big\}$ is decreasing and nonnegative, therefore $\mu\Big(C_n(T)\Big) \to r$ when $n \to \infty$, where r is a nonnegative real number. Now taking into account (4), we see there exists $k_1 \in \mathbb{N}$ such that

$$\mu\Big(C_{k_1}(T)\Big) \leqslant \psi\Big(\mu(C)\Big).$$

Having chosen $k_2, k_3, ..., k_{i-1}$, we see from our assumption that there exists $k_i \in \mathbb{N}$ such that

$$\mu\Big(\Big(C_{k_1+k_2+...+k_{i-1}}(T)\Big)_{k_i}\Big(T\Big)\Big) = \mu\Big(C_{k_1+k_2+...+k_i}(T)\Big) \leqslant \psi^i\Big(\mu(C)\Big). \tag{5}$$

If we put $n_i = \sum_{n=1}^{i} k_n$, then (5) can be rewritten as

$$\mu\Big(C_{n_i}(T)\Big) \leqslant \psi^i\Big(\mu(C)\Big). \tag{6}$$

Since $\psi^i\Big(\mu(C)\Big) \to 0$ as $i \to \infty$, (6) implies that $r = 0$ so the set $C_\infty = \cap_{n=1}^{\infty} C_n(T)$ is nonempty and compact. Since the set C_∞ is also convex and invariant under T, the classical Schauder fixed point theorem (Theorem 1.4) completes the proof. □

Now, suppose that C is a nonempty, bounded, closed and convex subset of E and $T : C \to C$ is a map. Assume that $\lambda \in [0, 1]$. Define, and fix hereafter, the family $\mathcal{A}_{T,\lambda}$ as follows

$$\mathcal{A}_{T,\lambda} := \Big\{A \subseteq C : A \neq \emptyset,\ A = \text{Conv}A,\ \lambda(TA) \subseteq A \text{ and } \frac{1}{\lambda}A \subseteq C\Big\}.$$

Notice that $C \in \mathcal{A}_{T,1}$ and if $\theta \in C$, then by the convexity of C we have $C \in \mathcal{A}_{T,\lambda}$ for any $\lambda \in [0, 1)$.

The following result gives us a sufficient condition so that a self map T has the $\epsilon-$fixed point property.

Theorem 2.2. *Let C be a nonempty, bounded, closed and convex subset of E with $\theta \in C$ and $T : C \to C$ be a continuous mapping. If there exist $\lambda_0 \in [0, 1)$ with the property that: for all $A \in \mathcal{A}_{T,\lambda}$ with $\lambda \in (\lambda_0, 1)$ there exists a nonnegative integer m such that*

$$\mu\Big(TA_m(\lambda T)\Big) \leqslant \mu(A). \tag{7}$$

Then T has the $\epsilon-$fixed point property.

Proof. Choose $\lambda_0 < \lambda < 1$ and define $G_\lambda : C \to C$ with $G_\lambda x = \lambda Tx$. If $A \in \mathcal{A}_{T,\lambda}$, then, by axiom 4^o and (7), we have

$$\mu\Big(A_{m+1}(G_\lambda)\Big) = \mu\Big(\text{Conv}\lambda T\Big(A_m(G_\lambda)\Big)\Big) \leqslant \lambda\mu\Big(TA_m(\lambda T)\Big) \leqslant \lambda\mu(A).$$

Now, Theorem 2.1 (define $\psi(x) = \lambda x$ for all $x \geqslant 0$) guarantees that there exists $x_\lambda \in C$ with

$$x_\lambda = G_\lambda x_\lambda = \lambda T x_\lambda.$$

Thus

$$\|x_\lambda - Tx_\lambda\| = (1-\lambda)\|Tx_\lambda\| \leqslant (1-\lambda)\delta(C) \to 0 \text{ as } \lambda \to 1.$$

This completes the proof. □

Now, we mention two corollaries of Theorem 2.2.

Corollary 2.3. *Let C be a nonempty, bounded, closed and convex subset of E and $T : C \to C$ be a continuous mapping and $\theta \in C$. Assume that there exists $\lambda_0 \in [0, 1)$ such that for any $A \in \mathcal{A}_{T,\lambda}$ with $\lambda \in (\lambda_0, 1)$ we have*

$$\mu(TA) \leqslant \mu(A).$$

Then T has the $\epsilon-$fixed point property.

Proof. It is just sufficient to consider $m = 0$ for all $A \in \mathcal{A}_{T,\lambda}$ with $\lambda \in (\lambda_0, 1)$ in Theorem 2.2. □

Notice that the following well-known result is a special case of Corollary 2.3.

Corollary 2.4. *Let C be a nonempty, bounded, closed and convex subset of E and $T : C \to C$ be a nonexpansive mapping and $\theta \in C$. Then T has the $\epsilon-$fixed point property.*

Proof. Consider the function δ given in (1) as an MNC on $\mathfrak{M}_E$. By the nonexpansivity of T, for all nonempty subset A of C we have

$$\|Tx - Ty\| \leqslant \|x - y\| \leqslant \delta(A) \text{ for all } x, y \in A. \tag{8}$$

Now, (8) implies that

$$\delta(TA) \leqslant \delta(A).$$

Then, by Corollary 2.3, T has the ϵ-fixed point property. □

In the following examples, we apply the above results to an old and well-known example in Hilbert space l^2 (see [8]) and the Fredholm operator.

Example 2.5. *Let U_2 be the closed unit ball in l^2. Define the operator $T : U_2 \to U_2$ by*

$$T(x) = T(x^1, x^2, x^3, ...) = (\sqrt{1 - \|x\|^2}, x^1, x^2, x^3, ...) \text{ for all } x \in U_2.$$

Then we can write $T = D + S$ where D is the one dimensional mapping

$$D(x) = D(x^1, x^2, x^3, ...) = (\sqrt{1 - \|x\|^2}, 0, 0, 0, ...) \text{ for all } x \in U_2,$$

and S is an isometry. Hence, T is continuous and for every bounded subset B of U_2 we have $\mu\Big(T(B)\Big) \leqslant \mu\Big(D(B)+S(B)\Big) \leqslant \mu\Big(D(B)\Big) + \mu\Big(S(B)\Big) \leqslant 0 + \mu(B)$, where μ is a subadditive MNC on $\mathfrak{M}_{l^2}$. So, by Theorem 2.2, T has ϵ-fixed point property. However, it is easy to show that T does not have fixed points.

Let $C[a,b]$ be the Banach space of all real-valued continuous functions on a given closed interval $[a,b]$ equipped with the sup-norm

$$\|f\| = \sup_{x\in[a,b]} |f(x)| \text{ for } f \in C[a,b],$$

for arbitrary $X \in \mathfrak{M}_{C[a,b]}$ and $\epsilon > 0$ we put

$$\omega(X,\epsilon) = \sup_{x\in X}\Big\{ \sup\{|x(t) - x(s)| : s,t \in [a,b], |s-t| \leqslant \epsilon\}\Big\}.$$

It is known that the following function ω_0 is a subadditive MNC (for more details see [11])

$$\omega_0(X) = \lim_{\epsilon\to 0} \omega(X,\epsilon).$$

Example 2.6. *Let $M > 0$ be a real number, $K : [a,b]\times[a,b]\times[-M,M] \to \mathbb{R}$ be acontinuous function and $V \in C[a,b]$. Define the self map F on $C[a,b]$ by*

$$F(f)(s) = V(s) + \mu \int_a^b K(s,t,f(t))dt \qquad f \in C[a,b]. \tag{9}$$

We know that the operator F is compact ([8]).

Assume that $G : \mathbb{R} \to \mathbb{R}$ is a continuous and bounded function. Now, Let us consider the following integral equation on $C[a,b]$

$$u(s) = G(u)(s) + F(u)(s), \tag{10}$$

where $u \in C[a,b]$ is unknown and F is defined by (9)*. If we define the self map T on $C[a;b]$ by*

$$T(f)(s) = G(f)(s) + F(f)(s) \qquad f \in C[a,b],$$

then every solution of equation (10) *corresponds to a fixed point of the operator T. Let*

$$R = \sup_{x\in\mathbb{R}} |G(x)| + \|V\| + |\mu|(b-a)\sup\Big\{|K(x,y,z)| : x,y \in [a,b], z \in [-M,M]\Big\}.$$

Suppose that the function G satisfies the following property: there exist $0 < \lambda_0 < 1$ and $\epsilon_0 > 0$ such that for all $\lambda_0 < \lambda < 1$, $0 < \epsilon < \epsilon_0$, $s,t \in [a,b]$ with $|s-t| \leqslant \epsilon$, $X \subseteq B(0,R)$ with $ConvX = X$ and $f \in X$ we have

$$\lambda\Big|G\big(f(s)\big) - G\big(f(t)\big)\Big| \leqslant \omega(X,\epsilon) \quad \text{implies that} \quad \Big|G\big(f(s)\big) - G\big(f(t)\big)\Big| \leqslant |f(s) - f(t)|. \tag{11}$$

Then T has ϵ-fixed point property (*notice that if for example G is a nonexpansive function, then it satisfies* (11)).

Obviously, T maps $B(0,R)$ into $B(0,R)$. Let $X \in \mathcal{A}_{G,\lambda}$ with $\lambda_0 < \lambda < 1$ and let $0 < \epsilon < \epsilon_0$, $s,t \in [a,b]$ with $|s-t| \leqslant \epsilon$ and $f \in X$, therefore $\lambda G(f) \in X$ and this implies that

$$\lambda\Big|G\big(f(s)\big) - G\big(f(t)\big)\Big| \leqslant \omega(X,\epsilon).$$

Thus, by (11)

$$\Big|G\big(f(s)\big) - G\big(f(t)\big)\Big| \leqslant |f(s) - f(t)|.$$

This means that

$$\omega(G(X),\epsilon) \leqslant \omega(X,\epsilon). \tag{12}$$

By taking the limit as $\epsilon \to 0$ on both sides of (12)*, we conclude that*

$$\omega_0(G(X)) \leqslant \omega_0(X).$$

Thus

$$\omega_0(T(X)) \leqslant \omega_0\Big(G(X) + F(X)\Big) \leqslant \omega_0(G(X)) + \omega_0(F(X)) \leqslant \omega_0(X) + 0 = \omega_0(X).$$

Hence, Theorem 2.2 *guarantees that T has ϵ–fixed point property. This is in particular useful for the approximation purposes.*

Let C be a nonempty and bounded subset of E. Consider a map $T : C \to C$. We define the map $\beta : \mathfrak{M}(C) \to \mathbb{R}$ by

$$\beta(A) = \sup\Big\{\|x - Tx\| : x \in A\Big\},$$

where $\mathfrak{M}(C)$ is the set of all nonempty subsets of C.

Theorem 2.7. *Let C be a nonempty, bounded, closed and convex subset of E and $T : C \to C$ be a continuous mapping. Suppose that there exists a map $\varphi : [0, +\infty) \to [0, +\infty)$ such that φ is continuous at 0, $\varphi(0) = 0$, φ is increasing on $[0, \delta]$ for some $\delta > 0$ and for each nonempty and closed subset A of C with $\beta(A) \neq 0$ we have*

$$\mu(TA) \leqslant \varphi\Big(\beta(A)\Big). \tag{13}$$

If T has the $\epsilon-$fixed point property, then T has a fixed point in the set C.

Proof. For any $n \in \mathbb{N}$ we define

$$F_n = \Big\{x \in C : \|x - Tx\| \leq \frac{1}{n}\Big\}. \tag{14}$$

By the hypothesis, the set F_n is nonempty for any $n \in \mathbb{N}$. Also each F_n is closed and $F_{n+1} \subseteq F_n$ for all $n \in \mathbb{N}$. Now (13) implies

$$\mu(TF_n) \leq \varphi\Big(\beta(F_n)\Big) \text{ for any } n \in \mathbb{N}.$$

However

$$\beta(F_n) = \sup\Big\{\|x - Tx\| : x \in F_n\Big\} \leq \frac{1}{n}.$$

Thus

$$\mu(TF_n) \leq \varphi(\frac{1}{n}) \text{ for any } n \in \mathbb{N}. \tag{15}$$

By the continuity of φ at 0 and (15) we have $\lim_{n\to\infty} \mu(TF_n) = \mu(\overline{TF_n}) = 0$, therefore

$$\bigcap_{n=1}^{\infty} \overline{TF_n} \neq \emptyset.$$

We may now consider $y \in \bigcap_{n=1}^{\infty} \overline{TF_n}$. Then, for any $n \in \mathbb{N}$ we can choose $x_n \in F_n$ such that $\|Tx_n - y\| \leq \dfrac{1}{n}$. By (14) we have

$$\|x_n - y\| \leq \|Tx_n - x_n\| + \|Tx_n - y\| \leq \frac{2}{n}.$$

Hence $x_n \to y$ when $n \to \infty$. Now the continuity of T implies that $Tx_n \to Ty$, so $Ty = y$ and the proof is complete. □

Combining Corollary 2.3 and Theorem 2.7 yields the following theorem.

Theorem 2.8. *Let C be a nonempty, bounded, closed and convex subset of E and $T : C \to C$ be a continuous mapping and $\theta \in C$. Assume that there exists a $\lambda_0 \in [0, 1)$ such that for any $A \in \mathcal{A}_{T,\lambda}$ with $\lambda \in (\lambda_0, 1]$ we have*

$$\mu(TA) \leqslant \mu(A).$$

Moreover, suppose that there exists a map $\varphi : [0, +\infty) \to [0, +\infty)$ such that φ is continuous at 0, $\varphi(0) = 0$, φ is increasing on $[0, \delta]$ for some $\delta > 0$ and for each nonempty and closed subset A of C with $\beta(A) \neq 0$ we have

$$\sigma(TA) \leq \varphi\Big(\beta(A)\Big).$$

Then, T has at least a fixed point in C. Here σ is an MNC on $\mathfrak{M}_E$ such that $\{\theta\} \in ker\sigma$.

Next we present two consequences of Theorem 2.8. We begin with the following simple corollary.

Corollary 2.9. *Let C be a nonempty, bounded, closed and convex subset of E and $T : C \to C$ be a nonexpansive mapping and $\theta \in C$. If there exists $L > 0$ such that for all $x, y \in C$ with $x \neq Tx$ and $y \neq Ty$ we have*

$$\|Tx - Ty\| \leq L \max\Big\{\|x - Tx\|, \|y - Ty\|\Big\}.$$

Then, T has a fixed point in C.

Proof. To prove Corollary 2.9, it is sufficient to consider the function δ in (1) as an MNC on $\mathfrak{M}_E$ and to check that if we define $\varphi : [0 + \infty) \to [0, +\infty)$ with $\varphi(x) = Lx$ for all $x \in [0, +\infty)$, then all the conditions in Theorem 2.8 are satisfied. □

Corollary 2.10. *Let C be a nonempty, bounded, closed and convex subset of E and $F : C \to C$ be a continuous mapping and $\theta \in C$. Let all the conditions in Theorem 2.8 for the map F be satisfied and the MNCs μ, σ be subadditive. Then for every compact, continuous map $G : C \to C$ such that $(F + G)C \subseteq C$ the map $T = F + G$ has a fixed point in C.*

Proof. For every $A \in \mathcal{A}_{T,\lambda}$ with $\lambda \in (\lambda_0, 1]$ we have

$$\mu(TA) = \mu\Big((F + G)A\Big) \leq \mu(FA + GA) \leq \mu(FA) + \mu(GA) = \mu(FA) \leq \mu(A),$$

and for every nonempty, closed subset A of C with $\beta(A) \neq 0$ we have

$$\sigma(TA) = \sigma\Big((F + G)A\Big) \leq \sigma(FA + GA) \leq \sigma(FA) + \sigma(GA) = \sigma(FA) \leq \varphi\Big(\beta(A)\Big).$$

Now, Theorem 2.8 completes the proof. □

3. A Fixed Point Result in Uniformly Convex Banach Spaces

In this section, we prove a fixed point result in uniformly convex Banach spaces under some weak conditions, which in a sense is the best generalization of Theorem 1.6. First we mention a technical lemma which will be needed in the proof of the main result of this section. The proof of this lemma can be found in [2].

Lemma 3.1. *Let I be a direct set and $\{C_\alpha\}_{\alpha \in I}$ be a decreasing net of nonempty closed convex bounded subsets of a uniformly convex Banach space E. Then $\bigcap_{\alpha \in I} C_\alpha$ is a nonempty closed convex subset of E.*

The main result of this section is the following theorem.

Theorem 3.2. *Let C be a nonempty, bounded, closed and convex subset of a uniformly convex Banach space E and $T : C \to C$ be a continuous mapping. Assume that for all nonempty, bounded, closed, convex subset B of C with $TB \subseteq B$ and $\mu(B) \neq 0$ there exists $n_0 \in \mathbb{N}$ such that*

$$\mu\Big(B_{n_0}(T)\Big) \neq \mu(B),$$

where $B_n(T)$ is defined as in (3). Then T has a fixed point in C.

Proof. Let $\mathfrak{B}$ be the family of all nonempty, bounded, closed and convex subsets B of C with $TB \subseteq B$. Set inclusion defines a partial ordering on $\mathfrak{B}$. Every chain $\mathfrak{C} \subseteq \mathfrak{B}$ has a lower bound by Lemma 3.1, namely, the intersection of all subsets of E which are elements of $\mathfrak{C}$. By Zorn's lemma, $\mathfrak{B}$ has a minimal element A. We claim that $\mu(A) = 0$. If not, in view of our hypothesis, there exists $n_0 \in \mathbb{N}$ such that $\mu\Big(A_{n_0}(T)\Big) \neq \mu(A)$. Clearly, $A_{n_0}(T) \in \mathfrak{B}$ and $A_{n_0}(T) \subseteq A$. Since A is a minimal element of $\mathfrak{B}$, then $A = A_{n_0}(T)$ and this implies that $\mu(A_{n_0}) = \mu(A)$, which is a contradiction and the proof is complete. □

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