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A Note on the Spectrum of Discrete Klein-Gordon s-Wave Equation with Eigenparameter Dependent Boundary Condition

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Abstract. This paper is concerned with the boundary value problem (BVP) for the discrete Klein-Gordon equation

$$\triangle\left(a_{n-1}\triangle y_{n-1}\right)+\left(v_n-\lambda\right)^2 y_n=0,\ n\in\mathbb{N}$$

and the boundary condition

$$\left(\gamma_0+\gamma_1\lambda\right)y_1+\left(\beta_0+\beta_1\lambda\right)y_0=0$$

where $(a_n), (v_n)$ are complex sequences, $\gamma_i, \beta_i\in\mathbb{C}$, $i=0,1$ and λ is a eigenparameter. The paper presents Jost solution, eigenvalues, spectral singularities and states some theorems concerning quantitative properties of the spectrum of this BVP under the condition

$$\sum_{n\in\mathbb{N}}\exp\left(\epsilon n^{\delta}\right)\left(|1-a_n|+|v_n|\right)<\infty \text{ for } \epsilon>0 \text{ and } \frac{1}{2}\le\delta\le 1.$$

1. Introduction

Spectral analysis of differential and difference operators has been a popular research field for scientists since it has a wide-ranging application area from quantum physics to engineering [2, 3].

Investigation of the spectral properties of the some basic differential operators can be traced back to Naimark [15, 16]. In particular, he studied the spectrum of the Sturm-Liouville equation considering the boundary value problem (BVP)

$$\begin{aligned} -y^{''}+q(x)y-\lambda^2 y &= 0,\ x\in\mathbb{R}_+, \\ y^{'}(0)-hy(0) &= 0, \end{aligned}$$

where $h\in\mathbb{C}$ and q is a complex valued function. He showed that the spectrum of this BVP is composed of eigenvalues, spectral singularities and continuous spectrum. He also proved that these eigenvalues and spectral singularities are of finite number with finite multiplicity under certain conditions.

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Besides Sturm-Liouville operator, spectral properties of Dirac and Schrödinger operators have been main topic of many papers in discrete form [1, 4, 8, 13, 14].

We put an emphasis on the boundary value problems including eigenparameter dependent boundary condition which is a prominent research field because they arise in models of certain physical problems such as vibration of a string, quantum mechanics and geophysics. Therefore, some issues of the spectral analysis of the boundary value problems with eigenparameter dependent boundary condition has been examined in [9, 12, 17].

Along with the spectral analysis of the operators that we have mentioned up to here, spectral analysis of Klein-Gordon operator has been considered in various studies. Spectral theory of Klein-Gordon equation with complex potential has been treated by Bairamov and Celebi [5]. In that paper, they obtained the conditions under which there exist a finite number of eigenvalues and spectral singularities with finite multiplicities. Also, some other problems of Klein-Gordon equation in terms of spectral analysis was studied in [1, 5-7, 10].

Note that the equation

$$y'' + [\lambda - p(x)]^2 y = 0, \ x \in \mathbb{R}_+,$$

is called the Klein-Gordon s-wave equation for a particle of zero mass with static potential p [6].

The present paper is motivated by the above mentioned studies. In this paper, we consider the following discrete Klein-Gordon equation with an eigenparameter dependent boundary condition

$$\triangle (a_{n-1} \triangle y_{n-1}) + (v_n - \lambda)^2 y_n = 0, \ n \in \mathbb{N}, \tag{1.1}$$

$$(\gamma_0 + \gamma_1 \lambda) y_1 + (\beta_0 + \beta_1 \lambda) y_0 = 0, \tag{1.2}$$

where (a_n), (v_n), $n \in \mathbb{N}$ are complex sequences, $a_n \neq 0$ for all $n \in \mathbb{N} \cup \{0\}$, $\gamma_0\beta_1 - \gamma_1\beta_0 \neq 0$ for $\gamma_i, \beta_i \in \mathbb{C}$, $i = 0, 1$ and λ is an eigenparameter.

The remainder of the manuscript is organized as follows. In Section 2 we present Jost solution of the BVP (1.1)-(1.2). Section 3 is concerned with the eigenvalues and spectral singularities of the BVP (1.1)-(1.2). The last section investigates the quantitative properties of the eigenvalues and spectral singularities corresponding to the BVP (1.1)-(1.2) under certain conditions.

2. Jost Solution of the BVP (1.1)-(1.2)

Determination of the Jost solution plays an important role for the spectral analysis of difference and differential operators. So, we present the structure of the Jost solution of the equation (1.1) in this section.

Let us assume that the condition

$$\sum_{n \in \mathbb{N}} n \left(|1 - a_n| + |v_n| \right) < \infty, \tag{2.1}$$

holds. Then, the equation (1.1) has the solution

$$f_n(z) = \alpha_n e^{inz} \left(1 + \sum_{m=1}^{\infty} K_{nm} e^{im\frac{z}{2}} \right) \tag{2.2}$$

for $\lambda = 2\cos\left(\frac{z}{2}\right)$, $z \in \overline{\mathbb{C}}_+$ ([1]). The expressions of K_{nm} and α_n can be written uniquely in terms of (a_n) and (v_n). In addition to this, we have the inequality

$$|K_{nm}| \leq C \sum_{r=n+\left[\left|\frac{m}{2}\right|\right]}^{\infty} \left(|1 - a_r| + |v_r| \right), \tag{2.3}$$

where $\left[\left|\frac{m}{2}\right|\right]$ is the integer part of $\frac{m}{2}$ and $C > 0$ is a constant [1]. The solution $f_n(z)$ is called Jost solution of the equation (1.1). Moreover, $f_n(z)$ is analytic with respect to z in $\mathbb{C}_+ := \{z : z \in \mathbb{C}, \operatorname{Im} z > 0\}$ and continuous in $\operatorname{Im} z = 0$.

Let us define the function A using the boundary condition (1.2) and Jost function (2.2) as

$$A(z) = (\gamma_0 + \gamma_1\lambda)\, f_1(z) + (\beta_0 + \beta_1\lambda)\, f_0(z). \tag{2.4}$$

Then, the function A is analytic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}}_+$, and $A(z) = A(z + 4\pi)$.

3. Eigenvalues and Spectral Singularities of the BVP (1.1)-(1.2)

Let us show the set of eigenvalues and spectral singularities of the BVP (1.1)-(1.2) by σ_d and σ_{ss}, respectively. We also define the semi-strips $P_0 := \{z : z \in \mathbb{C}, z = \xi + i\tau, -\pi \le \xi \le 3\pi, \tau > 0\}$ and $P := P_0 \cup [-\pi, 3\pi]$. From (2.4) and the definition of the eigenvalues and the spectral singularities, we have

$$\begin{aligned} \sigma_d &= \left\{\lambda : \lambda = 2\cos\frac{z}{2}, z \in P_0, A(z) = 0\right\}, && (3.1)\\ \sigma_{ss} &= \left\{\lambda : \lambda = 2\cos\frac{z}{2}, z \in [-\pi, 3\pi], A(z) = 0\right\} \setminus \{0, \pi, 2\pi\}. && (3.2) \end{aligned}$$

Using (2.2) and (2.4), we get

$$\begin{aligned} A(z) &= \left(\gamma_0 + \gamma_1\left(e^{i\frac{z}{2}} + e^{-i\frac{z}{2}}\right)\right)\left[\alpha_1 e^{iz} + \alpha_1 e^{iz}\sum_{m=1}^{\infty} K_{1m} e^{im\frac{z}{2}}\right] + \left(\beta_0 + \beta_1\left(e^{i\frac{z}{2}} + e^{-i\frac{z}{2}}\right)\right)\left[\alpha_0 + \alpha_0\sum_{m=1}^{\infty} K_{0m} e^{im\frac{z}{2}}\right]\\ &= \alpha_0\beta_1 e^{-i\frac{z}{2}} + \alpha_0\beta_0 + (\alpha_1\gamma_1 + \alpha_0\beta_1)\, e^{i\frac{z}{2}} + (\alpha_1\gamma_0)\, e^{iz} + (\alpha_1\gamma_1)\, e^{i\frac{3z}{2}}\\ &\quad + \sum_{m=1}^{\infty}(\alpha_0\beta_1 K_{om})\, e^{iz\left(\frac{m}{2}-\frac{1}{2}\right)} + \sum_{m=1}^{\infty}(\alpha_0\beta_0 K_{om})\, e^{iz\frac{m}{2}} + \sum_{m=1}^{\infty}(\alpha_1\gamma_1 K_{1m} + \alpha_0\beta_1 K_{0m})\, e^{iz\left(\frac{m}{2}+\frac{1}{2}\right)}\\ &\quad + \sum_{m=1}^{\infty}(\alpha_1\gamma_0 K_{1m})\, e^{iz\left(\frac{m}{2}+1\right)} + \sum_{m=1}^{\infty}(\alpha_1\gamma_1 K_{1m})\, e^{iz\left(\frac{m}{2}+\frac{3}{2}\right)}. \end{aligned}$$

We define

$$D(z) := e^{i\frac{z}{2}} A(z). \tag{3.3}$$

Then, the function D is analytic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}}_+$,

$$D(z) = D(z + 4\pi),$$

and

$$\begin{aligned} D(z) &= \alpha_0\beta_1 + (\alpha_0\beta_0)\, e^{i\frac{z}{2}} + (\alpha_1\gamma_1 + \alpha_0\beta_1)\, e^{iz} + (\alpha_1\gamma_0)\, e^{i\frac{3z}{2}} + (\alpha_1\gamma_1)\, e^{2iz}\\ &\quad + \sum_{m=1}^{\infty}(\alpha_0\beta_1 K_{om})\, e^{iz\frac{m}{2}} + \sum_{m=1}^{\infty}(\alpha_0\beta_0 K_{om})\, e^{iz\left(\frac{m}{2}+\frac{1}{2}\right)} + \sum_{m=1}^{\infty}(\alpha_1\gamma_1 K_{1m} + \alpha_0\beta_1 K_{0m})\, e^{iz\left(\frac{m}{2}+1\right)}\\ &\quad + \sum_{m=1}^{\infty}(\alpha_1\gamma_0 K_{1m})\, e^{iz\left(\frac{m}{2}+\frac{3}{2}\right)} + \sum_{m=1}^{\infty}(\alpha_1\gamma_1 K_{1m})\, e^{iz\left(\frac{m}{2}+2\right)}. \end{aligned} \tag{3.4}$$

It follows from (3.1)-(3.3) that

$$\begin{aligned} \sigma_d &= \left\{\lambda : \lambda = 2\cos\frac{z}{2}, z \in P_0, D(z) = 0\right\}, && (3.5)\\ \sigma_{ss} &= \left\{\lambda : \lambda = 2\cos\frac{z}{2}, z \in [-\pi, 3\pi], D(z) = 0\right\} \setminus \{0, \pi, 2\pi\}. && (3.6) \end{aligned}$$

Definition 3.1. The multiplicity of a zero of D in P is called the multiplicity of the corresponding eigenvalue or spectral singularity of the BVP (1.1)-(1.2).

It is seen from (3.5) and (3.6) that we need to investigate the zeros of D in P in order to study the quantitative properties of eigenvalues and spectral singularities of the BVP (1.1)-(1.2).

Let us define the sets

$$\begin{aligned} M_1 &: = \{z : z \in P_0, D(z) = 0\}, \\ M_2 &: = \{z : z \in [-\pi, 3\pi] \setminus \{0, \pi, 2\pi\}, D(z) = 0\}. \end{aligned} \tag{3.7}$$

Let us also introduce all limit points of M_1 by M_3 and the set of all zeros of D with infinite multiplicity by M_4.

It is clear from (3.5)-(3.7) that

$$\sigma_d = \left\{\lambda : \lambda = 2\cos\frac{z}{2}, z \in M_1\right\}, \tag{3.8}$$

$$\sigma_{ss} = \left\{\lambda : \lambda = 2\cos\frac{z}{2}, z \in M_2\right\}. \tag{3.9}$$

4. Quantitative Properties of Eigenvalues and Spectral Singularities

Theorem 4.1. *If (2.1) is satisfied, then*
i) M_1 is bounded and countable,
ii) $M_1 \cap M_3 = \emptyset, M_1 \cap M_4 = \emptyset$,
iii) M_2 is compact and $\mu(M_2) = 0$, where μ is Lebesgue measure in the real axis,
iv) $M_3 \subset M_2, M_4 \subset M_2; \mu(M_3) = \mu(M_4) = 0$,
v) $M_3 \subset M_4$.

Proof. If we use (2.3) and (3.4), we obtain the asymptotic

$$D(z) = \begin{cases} \alpha_0\beta_1 + O\left(e^{-\frac{\tau}{2}}\right), \ \beta_1 \neq 0, \ z \in P, \ \tau \to \infty, \\ \alpha_0\beta_0 e^{i\frac{z}{2}} + O\left(e^{-\tau}\right), \ \beta_1 = 0, \ z \in P, \ \tau \to \infty. \end{cases} \tag{4.1}$$

Boundedness of the set M_1 is obtained as a consequence of (4.1). The function D is analytic in $\mathbb{C}_+$ and is a 4π periodic function. So, M_1 has at most a countable number of elements. This proves (i). (ii)-(iv) is found from the boundary uniqueness theorems of analytic functions ([11]). From the continuity of all derivatives of D on $[-\pi, 3\pi]$, we get (v). □

As a consequence of Theorem (4.1), (3.8) and (3.9), we have the following theorem.

Theorem 4.2. *If (2.1) holds, then*
(i) the set of eigenvalues of the BVP (1.1)-(1.2) is bounded, has at most a countable number of elements, and its limit points can lie only in $[-2, 2]$.
(ii) $\sigma_{ss} \subset [-2, 2]$ *and* $\mu(\sigma_{ss}) = 0$.

Let

$$\sup_{n\in\mathbb{N}} \left[\exp(\varepsilon n)\left(|1 - a_n| + |v_n|\right)\right] < \infty, \tag{4.2}$$

for the complex sequences $(a_n), (b_n)$ and for some $\varepsilon > 0$.

Theorem 4.3. *If (4.2) holds, then the BVP (1.1)-(1.2) has a finite number of eigenvalues and spectral singularities with a finite multiplicity.*

Proof. Using (2.3), we get the inequality

$$|K_{nm}| \le C \exp\left[-\frac{\varepsilon}{3}(n+m)\right], \; n,m \in \mathbb{N},$$

where $C > 0$ is a constant. From the expression of the function $D(z)$ in (3.4), its analytic continuation can be obtained to the half plane $\operatorname{Im} z > -\frac{2\varepsilon}{3}$. Since D is 4π periodic function, the accumulation points of its zeros can not be on the interval $[-\pi, 3\pi]$. We have already got that the bounded sets M_1 and M_2 have a finite number of elements. If we use the analyticity of D in $\operatorname{Im} z > -\frac{2\varepsilon}{3}$, we get that all zeros of D in P have a finite multiplicity. Therefore, finiteness of the eigenvalues and spectral singularities of the BVP (1.1)-(1.2) is obtained. □

Let us take the condition

$$\sup_{n\in\mathbb{N}} \left[\exp\left(\varepsilon n^{\delta}\right)(|1-a_n| + |v_n|)\right] < \infty, \; \varepsilon > 0, \; \frac{1}{2} \le \delta < 1. \tag{4.3}$$

It is seen that analytic continuation of D is achieved from real axis to lower half-plane for the condition (4.2). However, this does not happen for the case condition (4.3). So, a different method to prove the finiteness of the eigenvalues and spectral singularities of the BVP (1.1)-(1.2) is necessary. We will use the following theorem.

Theorem 4.4. ([14]) *Let 4π periodic function g is analytic in $\mathbb{C}_+$, all of its derivatives are continuous in $\overline{\mathbb{C}}_+$ and*

$$\sup_{z\in P} \left|g^{(k)}(z)\right| \le \eta_k, \; k \in \mathbb{N} \cup \{0\}.$$

If the set $G \subset [-\pi, 3\pi]$ with Lebesgue measure zero is the set of all zeros the function g with infinite multiplicity in P, and if

$$\int_0^{\omega} \ln t(s) d\mu(G_s) = -\infty,$$

where $t(s) = \inf_k \frac{\eta_k s^k}{k!}$ and $\mu(G_s)$ is the Lebesgue measure of $s-$neighborhood of G and $\omega > 0$ is an arbitrary constant, then $g \equiv 0$ in $\overline{\mathbb{C}}_+$.

Obviously, the function D is analytic in $\mathbb{C}_+$ and infinitely differentiable on the real axis under the conditon (4.3). Using previous theorem, (2.3) and (3.4), we have

$$\left|D^{(k)}(z)\right| \le \eta_k, \; k \in \mathbb{N} \cup \{0\},$$

where

$$\eta_k = 3^k C \sum_{m=1}^{\infty} m^k \exp(-\varepsilon m^{\delta}).$$

The following estimate is obtained for η_k

$$\eta_k \le 3^k C \int_0^{\infty} x^k \exp(-\varepsilon x^{\delta}) dx \le E d^k k! k^{k\frac{1-\delta}{\delta}}, \tag{4.4}$$

where E and d are constants depending on C, ε and δ.

Theorem 4.5. *If (4.3) is satisfied, then $M_4 = \emptyset$.*

Proof. From Theorem (4.4), we get

$$\int_0^{\omega} \ln t(s) d\mu(G_s) > -\infty, \tag{4.5}$$

where $t(s) = \inf\limits_k \frac{\eta_k s^k}{k!}$, $k \in \mathbb{N} \cup \{0\}$, $\mu(M_{4,s})$ is the Lebesgue measure of the s-neighborhood of M_4 and η_k is defined by (4.4).

Now we have

$$t(s) \leq D \exp\left\{-\frac{1-\delta}{\delta} e^{-1} d^{-\frac{\delta}{1-\delta}} s^{-\frac{\delta}{1-\delta}}\right\}, \tag{4.6}$$

by (4.4). From (4.5) and (4.6), we get

$$\int_0^{\omega} s^{-\frac{\delta}{1-\delta}} d\mu(M_{4,s}) < \infty. \tag{4.7}$$

Since $\frac{\delta}{1-\delta} \geq 1$, (4.7) holds for arbitrary s if and only if $\mu(M_{4,s}) = 0$ or $M_4 = \emptyset$. □

Theorem 4.6. *If (4.3) holds, then the BVP (1.1)-(1.2) has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.*

Proof. We need to show that the function $D(z)$ has a finite number of zeros with a finite multiplicities in P. Theorem 4.1 and 4.5 imply that $M_3 = M_4 = \emptyset$. Thus, the bounded sets M_1 and M_2 do not have accumulation points, i.e., $D(z)$ has only finite number of zeros in P. Since $M_4 = \emptyset$, these zeros are of finite multiplicity. □

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