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Ricci and Casorati Principal Directions of Wintgen Ideal Submanifolds

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Abstract. We show that *for Wintgen ideal submanifolds in real space forms* the (intrinsic) *Ricci principal directions* and the (extrinsic) *Casorati principal directions coincide*.

1. Wintgen Ideal Submanifolds of Real Space Forms

Let M^n be an n-dimensional Riemannian submanifold of an $(n+m)$–dimensional *real space form* $\tilde{M}^{n+m}(c)$ of constant sectional curvature c and let g, ∇ and $\tilde{g}, \tilde{\nabla}$ be the *Riemannian metric* and the corresponding *Levi–Civita connection* on M^n and on $\tilde{M}^{n+m}(c)$, respectively. *Tangent vector fields* on M^n will be written as $X, Y, \dots$ and *normal vector fields* on M^n in $\tilde{M}^{n+m}(c)$ will be written as $\xi, \eta, \dots$. The *formulae of Gauss and Weingarten*, concerning the decomposition of the vector fields $\tilde{\nabla}_X Y$ and $\tilde{\nabla}_X \xi$, respectively, into their tangential and normal components along M^n in $\tilde{M}^{n+m}(c)$, are given by $\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y)$ and $\tilde{\nabla}_X \xi = -A_\xi(X) + \nabla^\perp_X \xi$, respectively, whereby h is the *second fundamental form* and A_ξ is the *shape operator* or *Weingarten map* of M^n with respect to the normal vector field ξ, such that $\tilde{g}(h(X,Y),\xi) = g(A_\xi(X),Y)$, and $\nabla^\perp$ is the *connection in the normal bundle*.

The *mean curvature vector field* $\vec{H}$ is defined by $\vec{H} = \frac{1}{n} tr\, h$ and its length $\|\vec{H}\| = H$ is the extrinsic *mean curvature* of M^n in $\tilde{M}^{n+m}(c)$. A submanifold M^n in $\tilde{M}^{n+m}(c)$ is *totally geodesic* when $h = 0$, *totally umbilical* when $h = g\vec{H}$, *minimal* when $H = 0$ and *pseudo–umbilical* when $\vec{H}$ is an umbilical normal direction.

Let $\{E_1, \dots, E_n, \xi_1, \dots, \xi_m\}$ be any *adapted orthonormal* local frame field on the submanifold M^n in $\tilde{M}^{n+m}(c)$, denoted for short also as $\{E_i, \xi_\alpha\}$, whereby $i, j, \dots \in \{1, 2, \dots, n\}$ and $\alpha, \beta, \dots \in \{1, 2, \dots, m\}$.

By the *equation of Gauss*, the $(0,4)$ *Riemann–Christoffel curvature tensor* of a submanifold M^n in $\tilde{M}^{n+m}(c)$ is given by $R(X,Y,Z,W) = \tilde{g}(h(Y,Z),h(X,W)) - \tilde{g}(h(X,Z),h(Y,W)) + c\big\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big\}$.

The $(0,2)$ *Ricci curvature tensor* of M^n is defined by $S(X,Y) = \sum_i R(X,E_i,E_i,Y)$ and the metrically corresponding $(1,1)$ tensor or *Ricci operator* will also be denoted by S: $g(S(X),Y) = S(X,Y)$. Since S is *symmetric*

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there exists on M^n an *orthonormal set of eigenvector fields* $R_1, \ldots, R_n$ which determine *the intrinsic, Ricci principal directions* of the Riemannian manifold M^n, and the corresponding *eigenfunctions* $Ric_1, \ldots, Ric_n$ are the *Ricci curvatures* of M^n: $S(R_i) = Ric_i R_i$. A Riemannian manifold M^n is an *Einstein space* when $S = Ric\, g$, or still when *all Ricci curvatures are equal* $Ric_1 = \cdots = Ric_n = Ric$, M^n is a *quasi–Einstein space* when *it has a Ricci curvature of multiplicity* $\geq n-1$ and M^n is a 2–*quasi–Einstein space* when *it has a Ricci curvature of multiplicity* $\geq n-2$. The *scalar curvature* of a Riemannian manifold M^n is defined by $\tau = \sum_{i<j} K(E_i \wedge E_j)$ whereby $K(E_i \wedge E_j) = R(E_i, E_j, E_j, E_i)$ is the *sectional curvature* for the plane section $\pi = E_i \wedge E_j, (i \neq j)$, and the *normalized scalar curvature function* of M^n is defined by $\rho = [2/n(n-1)]\tau$. By the *equation of Ricci*, the *normal curvature tensor* of a submanifold M^n in $\tilde{M}^{n+m}(c)$ is given by $R^\perp(X, Y, \xi, \eta) = g([A_\xi, A_\eta](X), Y)$, whereby $[A_\xi, A_\eta] = A_\xi A_\eta - A_\eta A_\xi$, which, as already observed by Cartan [1], implies that *the normal connection is flat or trivial* if and only if *all shape operators* A_ξ *are simultaneously diagonalisable*. The *normal scalar curvature* of a submanifold M^n is defined by $\tau^\perp = \left\{\sum_{i<j}\sum_{\alpha<\beta} R^\perp(E_i, E_j, \xi_\alpha, \xi_\beta)^2\right\}^{1/2}$ and the *normalized normal scalar curvature* of M^n is defined by $\rho^\perp = [2/n(n-1)]\tau^\perp$.

For surfaces M^2 in E^3, the *Euler inequality* $K \leq H^2$, whereby K is the intrinsic *Gauss curvature* of M^2 at once follows from the fact that that $K = k_1 k_2$ and $H = \frac{1}{2}(k_1 + k_2)$, whereby k_1 and k_2 are the *principal curvatures* of M^2 in E^3, and $K = H^2$ if and only if M^2 is *totally umbilical*, i.e. if $k_1 = k_2$, or still, by a Theorem of Meusnier, if M^2 is (part of) a *plane* E^2 or of a *round sphere* S^2 in E^3. For surfaces M^2 in E^4, in 1979 Wintgen [21] proved that the *Gauss curvature* $K = \tau$ and the *squared mean curvature* H^2 and the extrinsic *normal scalar curvature* $K^\perp = \tau^\perp$ always satisfy the *inequality* $K \leq H^2 - K^\perp$, and that in this weak inequality actually the *equality* holds, $K = H^2 - K^\perp$, if and only if *the curvature ellipses* $\mathcal{E} = \{h(U, U) \mid U \in TM$ and $\|U\| = 1\}$ in the normal planes of M^2 in E^4 *are circles*. These results of Wintgen were extended to all surfaces M^2 in E^{2+m}, regardless their co–dimensions m by Rouxel [19] and Guadalupe, Rodriguez [12]. In 1999, De Smet, Dillen, Vrancken, one of the authors [7] proved the generalized *Wintgen inequality*

$$\rho \leq H^2 - \rho^\perp + c, \tag{$*$}$$

for all n–dimensional submanifolds M^n with co–dimension $m = 2$ in real space forms $\tilde{M}^{n+2}(c)$, gave *a characterization of the equality situation* in terms of an explicit description of the second fundamental form and *conjectured* ($*$) to hold for all n–dimensional submanifolds M^n with arbitrary co–dimensions m in real space forms $\tilde{M}^{n+m}(c)$. Recently, Choi and Lu [6], Lu [16] and Ge–Tang [11] proved that indeed ($*$) *holds in full generality* for all submanifolds M^n in $\tilde{M}^{n+m}(c)$ and gave a *characterization of the equality situation* in terms of an explicit description of the second fundamental form, thus establishing the following.

Theorem A. *Let M^n be a submanifold in a real space form $\tilde{M}^{n+m}$. Then the soft inequality ($*$) holds and in ($*$) actually the equality holds if and only if, with respect to a suitable adapted orthonormal frame $\{E_i, \xi_\alpha\}$ on M^n in $\tilde{M}^{n+m}$, the shape operators of the submanifold take the following forms:*

$$A_1 = \begin{bmatrix} \lambda_1 & \mu & 0 & \cdots & 0 \\ \mu & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \lambda_2 + \mu & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 - \mu & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_2 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} \lambda_3 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_3 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_3 \end{bmatrix},$$

$$A_4 = A_5 = \cdots = A_m = 0,$$

whereby λ_1, λ_2, λ_3 and μ are real functions on M^n.

The submanifolds M^n in $\tilde{M}^{n+m}(c)$ for which

$$\rho = H^2 - \rho^{\perp} + c \tag{$**$}$$

are called *Wintgen ideal submanifolds*; for many examples and for geometrical properties of such submanifolds, see e.g. [3, 6, 7, 9–12, 14, 16, 18, 19]. A motivation for this terminology might go as follows: for all possible isometric immersions of a Riemannian manifold M^n into a real space form $\tilde{M}^{n+m}(c)$, by $(*)$ *the value of the intrinsic normalised scalar curvature* ρ of M^n puts *a lower bound* to *the possible values of the extrinsic "stress"* $H^2 - \rho^{\perp} + c$ that M^n in any case cannot avoid "to undergo" as a submanifold in an ambient space $\tilde{M}^{n+m}(c)$, and, from this point of view, every *Wintgen ideal submanifold* M^n actually realises a particular shape in $\tilde{M}^{n+m}(c)$ such that this extrinsic stress does everywhere assume its theoretically smallest possible value as given by ρ. A frame $\{E_1, \ldots, E_n, \xi_1, \ldots, \xi_m\}$ with the corresponding shape operators A_α as stated in Theorem A is called *a Choi–Lu frame* on M^n in $\tilde{M}^{n+m}(c)$ and its distinguished tangent plane $E_1 \wedge E_2$ is called *the Choi–Lu plane* of the Wintgen ideal submanifolds concerned [9, 10].

2. The Casorati Principal Directions of Submanifolds

For any submanifold M^n in some ambient Riemannian manifold $\tilde{M}^{n+m}$, the $(1,1)$ tensor field $A^C = \sum_\alpha A_\alpha^2$ is called its *Casorati operator* and the *Casorati curvature (as such)* of M^n in $\tilde{M}^{n+m}$ is defined by $C = \frac{1}{n} tr\, A^C = \frac{1}{n}\|h\|^2$. The Casorati operator being *symmetric* there exists on M^n an *orthonormal set of eigenvector fields* $F_1, \ldots, F_n$ which determine *the extrinsic, Casorati principal directions* of the submanifold M^n in $\tilde{M}^{n+m}$, and the corresponding eigenfunctions $c_1, \ldots, c_n$, (all ≥ 0), are its *extrinsic principal curvatures* or *the Casorati principal curvatures* of M^n in $\tilde{M}^{n+m}$; $A^C(F_i) = c_i F_i$. For the *geometrical meanings* of these notions, which essentially go back to Jordan and Casorati, see [2, 8, 13, 15, 20].

A *hypersurface* M^n in a Riemannian space $\tilde{M}^{n+1}$ is called *umbilical* when *its shape operator is proportional to the identity*, i.e. *has an eigenvalue of multiplicity* n, or still, when *all its principal curvatures are equal*. A hypersurface M^n in $\tilde{M}^{n+1}$ is called *quasi–umbilical* when *its shape operator has an eigenvalue of multiplicity* $\geq n-1$, (see e.g. [4]), and is called 2*–quasi–umbilical* when *its shape operator has an eigenvalue of multiplicity* $\geq n-2$, ([5], [17]). Similarly, a general submanifold M^n in some ambient Riemannian manifold $\tilde{M}^{n+m}$ is called *Casorati umbilical* when *its Casorati operator is proportional to the identity*, i.e. *has an eigenvalue of multiplicity* n, or still, when *all its Casorati principal curvatures are equal*. A submanifold M^n in $\tilde{M}^{n+m}$ is called *Casorati quasi–umbilical* when *its Casorati operator has an eigenvalue of multiplicity* $\geq n-1$, and is called *Casorati* 2*–quasi–umbilical* when *its Casorati operator has an eigenvalue of multiplicity* $\geq n-2$.

From Theorem A it follows that *the Casorati operator of the Wintgen ideal submanifolds M^n in real space forms $\tilde{M}^{n+m}(c)$* is given by

$$A^C = \begin{bmatrix} L + 2\lambda_2\mu + 2\mu^2 & 2\lambda_1\mu & 0 & \cdots & 0 \\ 2\lambda_1\mu & L + 2\mu^2 - 2\lambda_2\mu & 0 & \cdots & 0 \\ 0 & 0 & L & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & L \end{bmatrix},$$

whereby $L = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$. Its *eigenvalues* are $c_1 = L + 2\mu^2 + 2\mu(\lambda_1^2 + \lambda_2^2)^{1/2}$, $c_2 = L + 2\mu^2 - 2\mu(\lambda_1^2 + \lambda_2^2)^{1/2}$, $c_3 = \cdots = c_n = L$, and, in terms of the basic vector fields E_1 and E_2 of the Choi–Lu frame along M^n in $\tilde{M}^{n+m}(c)$, the vector fields $\tilde{F}_1 = \{\lambda_2 + (\lambda_1^2+\lambda_2^2)^{1/2}\}E_1 + \lambda_1 E_2$ and $\tilde{F}_2 = \{\lambda_2 - (\lambda_1^2+\lambda_2^2)^{1/2}\}E_1 + \lambda_1 E_2$ determine *the* 1*–dimensional eigenspaces* of A^C corresponding to c_1 and c_2, respectively, unless when $\lambda_1 = \lambda_2 = 0$ and $\mu \neq 0$, in which case *the Choi–Lu plane* itself *is a* 2*–dimensional eigenspace* of A^C, or when $\mu = 0$, in which case of course *the Casorati principal directions are undetermined*, A^C then being proportional to the identity operator, (and, even stronger, M^n then being *totally umbilical*); and, in any case, the tangent subspace $E_3 \wedge \cdots \wedge E_n$ of M^n is *an* $(n-2)$*–dimensional eigenspace of* A^C *corresponding to the Casorati curvature* L. Hence, in particular we have the following.

Theorem 2.1. *Every Wintgen ideal submanifold M^n in a real space form $\tilde{M}^{n+m}(c)$ is Casorati 2–quasi–umbilical. When M^n is not totally umbilical, then the orthogonal complement of its Choi–Lu plane is its $(n-2)$–dimensional Casorati eigenspace.*

3. The Ricci Principal Directions of Riemannian Manifolds

From Theorem A, via the Gauss equation, it follows that *the Ricci operator of the Wintgen ideal submanifolds M^n in real space forms $\tilde{M}^{n+m}(c)$* is given by

$$S = \begin{bmatrix} (n-1)\bar{c} + (n-2)\mu\lambda_2 - 2\mu^2 & (n-2)\mu\lambda_1 & 0 & \cdots & 0 \\ (n-2)\mu\lambda_1 & (n-1)\bar{c} - (n-2)\mu\lambda_2 - 2\mu^2 & 0 & \cdots & 0 \\ 0 & 0 & (n-1)\bar{c} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (n-1)\bar{c} \end{bmatrix},$$

whereby $\bar{c} = L + c$. Its *eigenvalues* are $Ric_1 = (n-1)\bar{c} - 2\mu^2 + (n-2)\mu(\lambda_1^2+\lambda_2^2)^{1/2}$, $Ric_2 = (n-1)\bar{c} - 2\mu^2 - (n-2)\mu(\lambda_1^2+\lambda_2^2)^{1/2}$, $Ric_3 = \cdots = Ric_n = (n-1)\bar{c}$, and, in terms of E_1 and E_2 the vector fields $\tilde{R}_1 = \{\lambda_2 + (\lambda_1^2+\lambda_2^2)^{1/2})\}E_1 + \lambda_1 E_2$ and $\tilde{R}_2 = \{\lambda_2 - (\lambda_1^2+\lambda_2^2)^{1/2}\}E_1 + \lambda_1 E_2$ determine *the* 1*–dimensional eigenspaces* of S corresponding to Ric_1 and Ric_2, respectively, unless when $\lambda_1 = \lambda_2 = 0$ and $\mu \neq 0$, in which case *the Choi–Lu plane* itself is a 2*–dimensional eigenspace* of S, or when $\mu = 0$, in which case of course *the Ricci principal directions are undetermined*, M^n then being an Einstein space, (and, even stronger, M^n then being *totally umbilical*, and thus being a real space form itself); and, in any case, the tangent subspace $E_3 \wedge \cdots \wedge E_n$ of M^n is *an* $(n-2)$*–dimensional eigenspace of* S *corresponding to the Ricci curvature* $(n-1)\bar{c}$. Hence, in particular, we have the following.

Theorem 3.1. *Every Wintgen ideal submanifold M^n in a real space form $\tilde{M}^{n+m}(c)$ is Ricci 2–quasi–umbilical. When M^n is not totally umbilical, then the orthogonal complement of its Choi–Lu plane is its $(n-2)$–dimensional Ricci eigenspace.*

4. Main Result

From *the extrinsic geometric point of view*, the Casorati principal directions of a submanifold M^n in a Riemannian space $\tilde{M}^{n+m}$ likely are its most important tangent directions while, from *the intrinsic geometric point of view*, for a Riemannian manifold M^n likely its most important tangent directions are its Ricci principal directions. And, from the formulae given in Sections 2 and 3, clearly following

Theorem 4.1. *On every Wintgen ideal submanifold in a real space form the Casorati and the Ricci principal directions do coincide,*

we may conclude that the particular shape any Wintgen ideal submanifold M^n does realise in ambient real space forms $\tilde{M}^{n+m}(c)$ in order to undergo the very least possible amount of extrinsic stress as allowed by its normalised intrinsic Riemannian scalar curvature, manifests the geometrical property that the principal tangent directions which are determined by this shape, namely its Casorati principal directions, are the same as the principal intrinsic tangent directions of its Riemannian structure, namely its Ricci principal directions.

References

[1] E. Cartan, *Leçons sur la géométrie des espaces de Riemann*, Ganthier-Villars, Paris, 1928.
[2] F. Casorati, *Mesure de la courbure des surfaces suivant l'idée commune*, Acta Math. 14, No 1 (1890) 95–110.
[3] B. Y. Chen, *Classification of Wintgen ideal surfaces in Euclidean* 4*–space with equal Gauss and normal curvatures*, Ann. Glob. Anal. Geom. 38(2) (2010) 145–160.
[4] B. Y. Chen, *Geometry of Submanifolds*, M. Dekker Publ. Co., New York, 1973.
[5] B. Y. Chen and L. Verstraelen, *A characterization of quasi-umbilical submanifolds and its applications*, Boll. Un. Mat. Ital. 14 (1977) 49–57.
[6] T. Choi and Z. Lu, *On the DDVV conjecture and the comass in calibrated geometry. I.*, Math. Z. 260 (2008) 409–429.
[7] P. J. De Smet, F. Dillen, L. Verstraelen and L. Vrancken, *A pointwise inequality in submanifold theory*, Arch. Math. (Brno), 35 (1999) 115–128.
[8] S. Decu, A. Pantić, M. Petrović-Torgašev and L. Verstraelen, *Ricci and Casorati principal directions of δ2 Chen ideal submanifolds*, Kragujevac J. Math 37(1) (2013) 25–31.
[9] S. Decu, M. Petrović-Torgašev, A. Šebeković and L. Verstraelen, *On the Roter type of Wintgen ideal submanifolds*, Rev. Roumaine Math. Pures Appl. 57 (2012) 75–90.
[10] R. Deszcz, M. Petrović–Torgašev, Z. Sentürk and L. Verstraelen, *Characterization of the pseudo-symmetries of ideal Wintgen submanifolds of dimension* 3, Publ. Inst. Math.(Beograd) 88(102) (2010) 57–67.
[11] J. Ge and Z. Tang, *A proof of the DDVV conjecture and its equality case*, Pacific J. Math., 237 (2008) 87–95.
[12] I. V. Guadalupe and L. Rodriguez, *Normal curvature of surfaces in space forms*, Pacific J. Math., 106 (1983) 95–103.
[13] S. Haesen, D. Kowalczyk and L. Verstraelen, *On the extrinsic principal directions of Riemannian submanifolds*, Note Mat. 29 No.2 (2009) 41–53.
[14] S. Haesen and L. Verstraelen, *Properties of a scalar curvature invariant depending on two planes*, Manuscripta mathematica, 122 (2007) 59–72.
[15] C. Jordan, *Généralisation du théoéma d'Euler sur la courbure des surfaces*, C. R. Acad. Sc. Paris, 79 (1874) 909–912.
[16] Z. Lu, *Normal Scalar Curvature Conjecture and its applications*, J. Funct. Anal. 261 (2011) 1284–1308.
[17] M. Petrović, R. Rosca and L. Verstraelen, *Exterior concurrent vector fields on Riemannian manifolds II, Examples of exterior concurrent vector fields on manifolds*, Soochow J. Math. 19(3) (1993) 357–368.
[18] M. Petrović–Torgašev and L. Verstraelen, *On Deszcz symmetries of Wintgen ideal submanifolds*, Arch. Math. (Brno) 44 (2008) 57–67.
[19] B. Rouxel, *Sur quelques propriétés anallagmatiques de l'espace euclidien E^4*, Mémoire couronné par l'Académie Royale de Belgique, 1982, (128 pp).
[20] K. Trenčevski, *Geometrical Interpretation of the Principal Directions and Principal Curvatures of Submanifolds*, Differ. Dyn. Syst. 2(1) (2000) 50–58.
[21] P. Wintgen, *Sur l'inégalité de Chen-Willmore*, C. R. Bend. Sci., Paris, 288 (1979) 993–995.