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On Local Spectral Properties of Hamilton Operators

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Abstract. This paper deals with local spectral properties of Hamilton type operators. The strongly decomposability, Weyl type theorems and hyperinvariant subspace problem of them and the similar properties with their adjoint operators are studied. As corollaries, some local spectral properties of Hamilton operators are obtained.

1. Introduction

The Hamiltonian system is an important branch in dynamical systems, and has various applications in our daily life. While infinite dimensional Hamiltonian operators come from the corresponding infinite dimensional Hamiltonian systems, and have deep mechanical background, their spectral theory is the theoretical foundation of the separation of the variables method solving mechanical problems, and plays a significant role in elasticity mechanics and other related fields[6,8,10].

Recently, the various results on infinite dimensional Hamiltonian operators frequently appear. In [2], the authors study the symmetry with respect to imaginary axis of the spectrum of infinite dimensional Hamiltonian operators; in the proof process, some properties between operators and their adjoint operators are applied. In[7], the decomposability, Weyl type theorems and invariant subspace problem of Hamilton operators and the similar properties with their adjoint operators are studied. In this paper, the strongly decomposability, Weyl type theorems and hyperinvariant subspace problem of Hamilton operators and the similar properties with their adjoint operators are given.

This paper is organized as follows. In section 2, we state some definitions and notations. The main results and examples of this paper, together with their proofs, are presented in section 3.

2. Preliminaries

Let X be an infinite dimensional Hilbert space. Throughout this paper, an operator is always linear bounded. According to [7], bounded Hamilton type operators and bounded Hamilton operators can be defined as follows.

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Definition 2.1. *Let $H : X \times X \longrightarrow X \times X$ be a bounded operator. If $(JH)^* = JH$, then H is called an infinite dimensional Hamilton operator, where $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ with I being the identity operator on X, 0 the zero operator on X, and $(JH)^*$ the adjoint operator of JH.*

Remark 2.2. *Evidently $J^* = -J$.*

Definition 2.3. *A bounded operator $T : X \times X \longrightarrow X \times X$ is called Hamilton type operator, provided there is an unitary operator J on $X \times X$ for which $J^2 = \begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix}$ and $(JT)^* = JT$. At this time, T is called Hamilton type operators with J as unitary operator.*

Definition 2.4. *We say that T satisfies*
(1) property (h) if $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \pi_{00}^a(T)$, where $\pi_{00}^a(T) = \{\lambda \in iso\sigma_a(T) : 0 < \alpha(T-\lambda) < \infty\}$.
(2) property (gh) if $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E^a(T)$, where $E^a(T) = \{\lambda \in iso\sigma_a(T) : 0 < \alpha(T-\lambda)\}$.

Remark 2.5. *The definition of $\sigma_{SF_+^-}(T), \sigma_{SBF_+^-}(T), \alpha(T-\lambda)$ is introduced in [1,3,4].*

Definition 2.6. [5]*A linear subspace Y of X is said to be T- hyperinvariant if $SY \subset Y$ for every bounded linear operator S on X that commutes with T.*

Lemma 2.7. [5] *A bounded operator T on X, is strongly decomposable if and only if T is decomposable and $X_T(F) = X_T(F \cap \overline{U}_1) + \cdots + X_T(F \cap \overline{U}_m)$ for every open cover $\{U_1, ..., U_m\}$ of an arbitrary closed set $F \subseteq \mathbb{C}$.*

3. Main results

Lemma 3.1. *Let T be a Hamilton type operator with J as unitary operator. Then $\sigma_{SF_+^-}(T) = -\sigma_{SF_+^-}(T^*), \sigma_{SBF_+^-}(T) = -\sigma_{SBF_+^-}(T^*)$*

Proof. If λ is not belong to $\sigma_{SF_+^-}(T^*)$, then $T^* - \lambda$ is upper semi-Weyl operator[1], i.e. $\alpha(T^* - \lambda) < \infty, R(T^* - \lambda)$ is closed and $ind(T^* - \lambda) \leq 0$. Since $T^* - \lambda = JTJ + J\lambda J = J(T + \lambda)J$, then $\alpha(T + \lambda) < \infty, R(T + \lambda)$ is closed and $ind(T + \lambda) \leq 0$, and hence $T + \lambda$ is upper semi-Weyl operator, so $-\lambda$ is not belong to $\sigma_{SF_+^-}(T)$. i.e. $-\sigma_{SF_+^-}(T) \subseteq \sigma_{SF_+^-}(T^*)$.Replacing T by T^* shows that $-\sigma_{SF_+^-}(T) \supseteq \sigma_{SF_+^-}(T^*)$. Therefore $\sigma_{SF_+^-}(T) = -\sigma_{SF_+^-}(T^*)$.

If λ is not belong to $\sigma_{SBF_+^-}(T^*)$, then $T^* - \lambda$ is upper semi B-Weyl operator[1], i.e. for some $n \geq 0$, $\alpha((T^*-\lambda)_{[n]}) < \infty, R((T^*-\lambda)_{[n]}), R((T^*-\lambda)^n)$ are closed and $ind((T^*-\lambda)_{[n]}) \leq 0$. Since $T^*-\lambda = JTJ+J\lambda J = J(T+\lambda)J$, then $\alpha((T + \lambda)_{[n]}) < \infty, R((T + \lambda)_{[n]}), R((T + \lambda)^n)$ are closed and $ind((T + \lambda)_{[n]}) \leq 0$, and hence $T + \lambda$ is upper semi B-Weyl operator, so $-\lambda$ is not belong to $\sigma_{SBF_+^-}(T)$. i.e. $-\sigma_{SBF_+^-}(T) \subseteq \sigma_{SBF_+^-}(T^*)$.Replacing T by T^* shows that $-\sigma_{SBF_+^-}(T) \supseteq \sigma_{SBF_+^-}(T^*)$. Therefore $\sigma_{SBF_+^-}(T) = -\sigma_{SBF_+^-}(T^*)$. □

In the following theorem we give a duality theorem of strongly decomposable operators. In general, the strongly decomposability of T^* is not transmitted to operator T([9]).

Theorem 3.2. *Let T be a Hamilton type operator with J as unitary operator. Then T is strongly decomposable if and only if T^* is strongly decomposable.*

Proof. If T is strongly decomposable, then T is decomposable, by Lemma 2.7. By Theorem 6.1.8 of [7], it follows that T has property (β) or property (δ). Then we know from Theorem 2.2.5 of [5] that, T^* has property (δ), so T^* is decomposable. Now we consider an arbitrary closed set $F \subseteq \mathbb{C}$ and a finite open cover $\{U_1, ..., U_m\}$ of F. Then $\{-U_1, ..., -U_m\}$ is a cover of F. Given any $x \in X_{T^*}(F)$, we have $-\sigma_T(J^*x) \subseteq F$, moreover $J^*x \in X_T(-F)$. The strong decomposability of T leads to $J^*x \in X_T((-F) \cap (-\overline{U}_1)) + \cdots + X_T((-F) \cap (-\overline{U}_1))$, it is immediate that $x \in J(X_T((-F) \cap (-\overline{U}_1)) + \cdots + X_T((-F) \cap (-\overline{U}_1))) = X_{T^*}(F \cap \overline{U}_1) + \cdots + X_{T^*}(F \cap \overline{U}_m)$, therefore $X_{T^*}(F) \subseteq X_{T^*}(F \cap \overline{U}_1) + \cdots + X_{T^*}(F \cap \overline{U}_m)$. To show the opposite inclusion, let $x \in X_{T^*}(F \cap \overline{U}_1) + \cdots + X_{T^*}(F \cap \overline{U}_m)$ be arbitrary. Then $J^*x \in J^*X_{T^*}(F \cap \overline{U}_1) + \cdots + J^*X_{T^*}(F \cap \overline{U}_m) = X_T(-(F \cap \overline{U}_1)) + \cdots + X_T(-(F \cap \overline{U}_1))$. Moreover $J^*x \in X_T((-F))$, so $x \in X_{T^*}(F)$.therefore $X_{T^*}(F \cap \overline{U}_1) + \cdots + X_{T^*}(F \cap \overline{U}_m) \subseteq X_{T^*}(F)$. Thus $X_{T^*}(F \cap \overline{U}_1) + \cdots + X_{T^*}(F \cap \overline{U}_m) = X_{T^*}(F)$. By Lemma 2.7, this establishes the strong decomposability of T^*.

For the reverse implication replace T by T^*. □

Theorem 3.3. *Let T be a Hamilton type operator with J as unitary operator. Then Y is T- hyperinvariant if and only if J^*Y is T^*- hyperinvariant.*

Proof. Let S be a bounded linear operator on X and $ST^* = T^*S$, then $JSJ^*T = TJSJ^*$. Since Y is T- hyperinvariant, we know from Definition 2.6 that $JSJ^*Y \subset Y$. Therefore $SJ^*Y \subset J^*Y$. i.e. J^*Y is T^*- hyperinvariant.

To see the converse, suppose that J^*Y is T^*- hyperinvariant. Let S be a bounded linear operator on X and $ST = TS$, then $JSJ^*T^* = T^*JSJ^*$. Since J^*Y is T^*- hyperinvariant, we know from Definition 2.6 that $JSJ^*J^*Y \subset J^*Y$. Therefore $SY \subset Y$. i.e. Y is T- hyperinvariant. □

Remark 3.4. *In general, Hamilton type operator does not satisfy property (h), and does not satisfy property (gh).*

Example 3.5. *Let $T = R \bigoplus (-R^*)$, where R be the unilateral right shift operator defined on the Hilbert space $\ell^2(\mathbb{N})$. Then T is Hamilton type operator, and does not satisfy Weyl theorem, and therefore does not satisfy a-Weyl theorem, so Hamilton type operator does not satisfy property (h), therefore does not satisfy property (gh).*

In the following theorems we give the necessary and sufficient conditions for Hamilton type operator which satisfies property (h) and (gh).

Theorem 3.6. *Let T be a Hamilton type operator with J as unitary operator. Then T satisfies property (h) if and only if T^* satisfies property (h).*

Proof. Let T satisfies property (h), then $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \pi_{00}^a(T)$. Given any $\lambda \in \sigma(T^*) \setminus \sigma_{SF_+^-}(T^*)$, we have $-\lambda \in \sigma(T) \setminus \sigma_{SF_+^-}(T)$, by Lemma 3.1 and Lemma 6.1.19 of [7]. Since T satisfies property (h), then $-\lambda \in \pi_{00}^a(T)$, and hence $\lambda \in \pi_{00}^a(T^*)$, therefore $\sigma(T^*) \setminus \sigma_{SF_+^-}(T^*) \subseteq \pi_{00}^a(T^*)$. To show the opposite inclusion, let $\lambda \in \pi_{00}^a(T^*)$, then $\lambda \in iso\sigma_a(T^*)$ and $0 < \alpha(T^* - \lambda) < \infty$, and therefore $-\lambda \in \pi_{00}^a(T)$. Since T satisfies property (h), then $-\lambda \in \sigma(T) \setminus \sigma_{SF_+^-}(T)$. We conclude from Lemma 3.1 and Lemma 6.1.19 of [7] that $\lambda \in \sigma(T^*) \setminus \sigma_{SF_+^-}(T^*)$. Hence $\pi_{00}^a(T^*) \subseteq \sigma(T^*) \setminus \sigma_{SF_+^-}(T^*)$. So T^* satisfies property (h).

A similar argument shows that T^* satisfies property (h), then T satisfies property (h). □

Theorem 3.7. *Let T be a Hamilton type operator with J as unitary operator. Then T satisfies property (gh) if and only if T^* satisfies property (gh).*

Proof. Let T satisfies property (gh), then $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E^a(T)$. Given any $\lambda \in \sigma(T^*) \setminus \sigma_{SBF_+^-}(T^*)$, we have $-\lambda \in \sigma(T) \setminus \sigma_{SBF_+^-}(T)$, by Lemma 3.1 and Lemma 6.1.19 of [7]. Since T satisfies property (gh), then $-\lambda \in E^a(T)$, and hence $\lambda \in E^a(T^*)$, therefore $\sigma(T^*) \setminus \sigma_{SBF_+^-}(T^*) \subseteq E^a(T^*)$. To show the opposite inclusion, let $\lambda \in E^a(T^*)$, then $\lambda \in iso\sigma_a(T^*)$ and $0 < \alpha(T^* - \lambda)$, and therefore $-\lambda \in E^a(T)$. Since T satisfies property (gh), then $-\lambda \in \sigma(T) \setminus \sigma_{SBF_+^-}(T)$. We conclude from Lemma 3.1 and Lemma 6.1.19 of [7] that $\lambda \in \sigma(T^*) \setminus \sigma_{SBF_+^-}(T^*)$. Hence $E^a(T^*) \subseteq \sigma(T^*) \setminus \sigma_{SBF_+^-}(T^*)$. So T^* satisfies property (gh).

A similar argument shows that T^* satisfies property (gh), then T satisfies property (gh). □

Remark 3.8. *In general, the results of Theorem 3.6 and 3.7 do not hold if we replace Hamilton type operator by bounded operator.*

Example 3.9. *Let T be defined for each $x = (x_i) \in \ell^2$ by $T(x_1, x_2, x_3, \cdots, x_n, \cdots) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, \cdots, \frac{1}{n}x_n, \cdots)$. Then $\sigma(T) = \sigma_a(T^*) = \sigma_w(T) = \sigma_{SF_+^-}(T^*) = \pi_{00}(T) = \{0\}$, $\pi_{00}^a(T^*) = \emptyset$. Hence $\sigma(T) \setminus \sigma_w(T) = \emptyset \neq \{0\} = \pi_{00}(T)$, i.e. T does not satisfy Weyl's theorem. Then T does not obey a-Weyl's theorem. Hence T does not satisfy property (h), and therefore does not satisfy property (gh). But $\sigma_a(T^*) \setminus \sigma_{SF_+^-}(T^*) = \pi_{00}^a(T^*) = \emptyset$, i.e. T^* satisfies a-Weyl's theorem. Then T^* obeys property (h) and property (gh).*

Remark 3.10. *The results of Lemma 3.1, Theorem 3.2, 3.3, 3.6 and 3.7 hold if we replace Hamilton type operator by Hamilton operator.*

References

[1] M. Amouch, M. Berkani, on the property ($g\omega$), Mediterr. J. Math. 5 (2008),371–378.
[2] Alatancang. J. Huang, X. Fan, structure of the spectrum of infinite dimenional Hamiltonian operators, Sci. China Ser. A. 51(5)(2008), 915–924.
[3] M. Berkani, J. J. Koliha, Weyl type theorems for bounded linear operators, Acta. Sci. Math. (Szeged) 69(1-2)(2003), 359–376.
[4] L. A. Coburn, Weyls theorem for nonnormal operators, Michigan Math. J. 13(1966), 285–288.
[5] K. B. Laursen, M. M. Neumann, Introduction to Local Spectral Theory, Oxford: Clarendon Press, 2000.
[6] Ma J W, Xu X S, Yang H Z, et al, Solution of spatial viscous flow based on Hamiltonian system (in Chinese), Engineering Mechanics 19(3)(2001), 1–9.
[7] Shen Junli, The local spectrum of Hamilton operator, Inner Mongolia University, 2015.(in Chinese)
[8] Sui Y F, Zhong W X, Eigenvalue problem of a large scale indefinite gyroscopic dynamic system, Appl Math Mech 27(1)(2006), 15–22.
[9] Wang shengwang, On a property ang duality theorem of strongly decomposable operators (in Chinese), Acta Mathematica Sinica 29(2)(1986), 145–156.
[10] Zhong W X, A New Systematic Method in Elasticity Theory (in Chinese), Dalian: Dalian University of Technology Press, 1995.