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Two-Geodesic-Transitive Graphs Which are Neighbor Cubic or Neighbor Tetravalent

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Abstract. A vertex triple (u, v, w) with v adjacent to both u and w is called a 2-*geodesic* if $u \neq w$ and u, w are not adjacent. A graph Γ is said to be 2-*geodesic-transitive* if its automorphism group is transitive on both arcs and 2-geodesics. In this paper, a complete classification of 2-geodesic-transitive graphs is given which are neighbor cubic or neighbor tetravalent.

1. Introduction

In this paper, all graphs are finite, simple, connected and undirected. For a graph Γ, we use $V(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ to denote its *vertex set* and *automorphism group*, respectively. For the group theoretic terminology not defined here we refer the reader to [2, 8, 22]. In a non-complete graph Γ, a vertex triple (u, v, w) with v adjacent to both u and w is called a 2-*geodesic* if $u \neq w$ and in addition u, w are not adjacent. An *arc* is an ordered pair of adjacent vertices. The graph Γ is said to be 2-*geodesic-transitive* if its automorphism group $\mathrm{Aut}(\Gamma)$ is transitive on both arcs and 2-geodesics. The family of 2-geodesic-transitive graphs is closely related to the well-known family of 2-arc-transitive graphs. A vertex triple (u, v, w) with v adjacent to both u and w is called a 2-*arc* if $u \neq w$. The graph Γ is said to be 2-*arc-transitive* if its automorphism group $\mathrm{Aut}(\Gamma)$ is transitive on both arcs and 2-arcs. Clearly, each 2-geodesic is a 2-arc, but some 2-arcs may not be 2-geodesics. If Γ has girth 3 (length of the shortest cycle is 3), then the 2-arcs contained in 3-cycles are not 2-geodesics. For instance, the complete multipartite graph $K_{3[3]}$ is 2-geodesic-transitive neighbor cubic but not 2-arc-transitive. Thus the family of non-complete 2-arc-transitive graphs is properly contained in the family of 2-geodesic-transitive graphs.

The local structure of the family of 2-geodesic-transitive graphs was determined in [4]. In [5], Devillers, Li, Praeger and the author classified 2-geodesic-transitive graphs of valency 4. Later, in [6], a reduction theorem for the family of normal 2-geodesic-transitive Cayley graphs was produced and those which are complete multipartite graphs were also classified. The family of 2-geodesic-transitive but not 2-arc-transitive graphs with prime valency was precisely determined in [7].

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For a subset U of the vertex set of Γ, we denote by $[U]$ the subgraph of Γ induced by U, and $[\Gamma(u)]$ is the subgraph induced by the neighborhood of the vertex u. Devillers, Li, Praeger and the author in [4, Theorem 1] proved that if Γ is a 2-geodesic-transitive graph of valency at least 2, then for each vertex u, either

(1) $[\Gamma(u)]$ is connected of diameter 2; or

(2) $[\Gamma(u)] \cong m\mathrm{K}_r$ for some integers $m \geq 2, r \geq 1$.

Further, Theorem 1.4 of [4] shows that there is a bijection between the family of graphs in case (2) and a certain family of partial linear spaces. In particular, if $r = 1$, then Γ is 2-arc-transitive. The first remarkable result about 2-arc-transitive graphs comes from Tutte [19, 20], and this family of graphs has been studied extensively, see [1, 9, 10, 12, 15–18, 21]. The graphs in case (1) were investigated in [14]; and in [13], the author completely determined such graphs with valency twice a prime. In this paper, we continue the investigation of the graphs in case (1).

A connected graph is said to be *neighbor cubic* or *neighbor tetravalent* if its local subgraph is connected of valency 3 or 4, respectively. For a graph Γ, its *complement* $\overline{\Gamma}$ is the graph with vertex set $V(\Gamma)$, and two vertices are adjacent if and only if they are not adjacent in Γ.

Let Γ be a 2-geodesic-transitive graph. Let $u \in V(\Gamma)$. Suppose that $[\Gamma(u)]$ is connected of valency 2. Then Γ is either the octahedron or the icosahedron, see [5, Corollary 1.4]. Thus the next natural problem is to classify the family of 2-geodesic-transitive graphs whose neighbor subgraph is connected of valency 3. Our first theorem precisely determines such graphs.

Theorem 1.1. *Let Γ be a 2-geodesic-transitive neighbor cubic graph. Then Γ is one of the following graphs:* $\mathrm{K}_{3[3]}$*,* $J(5,2)$*, complement of the triangular graph* $T(7)$*, the Conway-Smith graph, or the Hall graph.*

We denote by $\mathrm{K}_{m[b]}$ the *complete multipartite graph* with m parts, and each part has b vertices where $m \geq 3, b \geq 2$. Let $\Omega = [1, n]$ where $n \geq 3$, and let $1 \leq k \leq [\frac{n}{2}]$ where $[\frac{n}{2}]$ is the integer part of $\frac{n}{2}$. Then the *Johnson graph* $J(n,k)$ is the graph whose vertex set is the set of all k-subsets of Ω, and two k-subsets u and v are adjacent if and only if $|u \cap v| = k - 1$.

The second theorem determines the family of 2-geodesic-transitive graphs whose neighbor subgraph is connected of valency 4.

Theorem 1.2. *Let Γ be a 2-geodesic-transitive neighbor tetravalent graph. Then Γ is one of the following three graphs:* $J(6,2)$*,* $J(6,3)$ *or* $\mathrm{K}_{4[2]}$.

2. Proof of Theorem 1.1

The first lemma determines the family of 2-geodesic-transitive neighbor cubic graphs whose local subgraph is symmetric.

Lemma 2.1. *Let Γ be a 2-geodesic-transitive neighbor cubic graph. Suppose that $[\Gamma(u)]$ is arc-transitive for some $u \in V(\Gamma)$. Then Γ is one of the following graphs:* $\mathrm{K}_{3[3]}$*, complement of the triangular graph* $T(7)$*, the Conway-Smith graph, or the Hall graph.*

Proof. Let (u, v) be an arc and $A = \mathrm{Aut}(\Gamma)$. Since $\Sigma := [\Gamma(u)]$ is not an empty graph, Γ has girth 3. Further, the graph Γ is 2-geodesic-transitive, so it follows from Theorem 1.1 (1) of [4] that Σ has diameter 2 and the arc stabilizer A_{uv} is transitive on $\Sigma_2(v)$. Since Σ is arc-transitive, Σ is distance-transitive, and it is listed in [3, p.221-223]. Thus by inspecting the candidates, Σ is either the complete bipartite graph $\mathrm{K}_{3,3}$ or the Petersen graph O_3. If Σ is $\mathrm{K}_{3,3}$, then Γ is $\mathrm{K}_{3[3]}$. If Σ is O_3, then by [11, Theorem 1.1], Γ is the Conway-Smith graph, the Hall graph, or the complement of the triangular graph $T(7)$. □

Let $u, v \in V(\Gamma)$. Then the distance between u, v in Γ is denoted by $d_\Gamma(u, v)$. A graph Γ is said to be 2-*distance-transitive* if, for $i = 1, 2$ and for any two vertex pairs (u_1, v_1) and (u_2, v_2) with $d_\Gamma(u_1, v_1) = d_\Gamma(u_2, v_2) = i$, there exists $g \in \mathrm{Aut}(\Gamma)$ such that $(u_1, v_1)^g = (u_2, v_2)$. By the definition, every 2-geodesic-transitive graph is 2-distance-transitive.

Lemma 2.2. *Let Γ be a 2-distance-transitive graph. If $[\Gamma(u)] \cong \overline{C_n}$ for some $u \in V(\Gamma)$ and $n \geq 5$, then Γ is either $J(5,2)$ or the icosahedron.*

Proof. Let $\Sigma := [\Gamma(u)]$. Suppose $\Sigma \cong \overline{C_n}$ where $n \geq 5$. If Σ is arc-transitive, then $n = 5$ and $\Sigma \cong C_5$. By [5, Corollary 1.4], Γ is the icosahedron. In the remaining of this proof, we assume that Σ is not arc-transitive. Hence $n \geq 6$ and Γ has diameter 2.

Let $v \in V(\Sigma)$. Then for any $v' \in \Sigma_2(v)$, $|\Sigma(v) \cap \Sigma(v')| = n - 4$. Since $u \in \Gamma(v) \cap \Gamma(v')$, it follows that $n-3 \leq |\Gamma(v) \cap \Gamma(v')|$. Since $|\Sigma_2(v) \cap \Sigma(v')| = 1$, it follows that $|\Gamma(v) \cap \Gamma(v')| \leq n-1$, so $|\Gamma(v) \cap \Gamma(v')| = n-3, n-2$ or $n-1$.

As $\Sigma \cong \overline{C_n}$, the valency of Σ is $n-3$, so $|\Gamma(u) \cap \Gamma(v)| = n-3$. Thus $|\Gamma_2(u) \cap \Gamma(v)| = 2$, so $|\Gamma_2(v) \cap \Gamma(u)| = 2$, as Γ is arc-transitive. Hence there are $2n$ edges between $\Gamma(v)$ and $\Gamma_2(v)$. By the assumption Γ is 2-distance-transitive, the value $|\Gamma(v) \cap \Gamma(v')|$ divides $2n$. Since $|\Gamma(v) \cap \Gamma(v')| < n$, it follows that $2|\Gamma(v) \cap \Gamma(v')| < 2n$, so $3|\Gamma(v) \cap \Gamma(v')| \leq 2n$. If $|\Gamma(v) \cap \Gamma(v')| = n-3$, then $n = 6$ or 9. If $|\Gamma(v) \cap \Gamma(v')| = n-2$, then $n = 6$. If $|\Gamma(v) \cap \Gamma(v')| = n-1$, then $n \leq 3$. This is impossible because $n \geq 5$. By [3, p.224], $n \neq 9$. Thus $n = 6$.

Set $\Gamma(u) = \{v_1 = v, v_2, v_3, v_4, v_5, v_6\}$. Let (v_1, v_2, v_3) and (v_4, v_5, v_6) be two triangles and let (v_1, v_6), (v_2, v_5) and (v_3, v_4) be three arcs. Then $|\Gamma_2(u) \cap \Gamma(v_1)| = 2$, say $\Gamma_2(u) \cap \Gamma(v_1) = \{w_1, w_2\}$. Hence $\Gamma(v_1) = \{u, v_2, v_3, v_6, w_1, w_2\}$. Since $[\Gamma(v_1)] \cong \overline{C_6}$, (u, v_2, v_3) is a triangle and neither v_2 nor v_3 is adjacent to v_6, it follows that (v_6, w_1, w_2) is a triangle, v_2 is adjacent to exactly one of w_1, w_2, say w_1, and v_3 is adjacent to w_2. Set $\Gamma_2(u) \cap \Gamma(v_2) = \{w_1, w_3\}$. Then $\Gamma(v_2) = \{u, v_1, v_3, v_5, w_1, w_3\}$. Since $[\Gamma(v_2)] \cong \overline{C_6}$, it follows that (v_5, w_1, w_3) is a triangle. Thus $\Gamma(u) \cap \Gamma(w_1) = \{v_1, v_2, v_5, v_6\}$ and $\Gamma_2(u) \cap \Gamma(w_1) = \{w_2, w_3\}$. Since Γ is 2-distance-transitive, Γ has diameter 2 and is distance-transitive. By inspecting the graphs in [3, p.223], $\Gamma \cong J(5,2)$. $\square$

Lemma 2.3. *Let Γ be a 2-geodesic-transitive neighbor cubic graph. If $[\Gamma(u)]$ is not arc-transitive for some $u \in V(\Gamma)$, then Γ is $J(5,2)$.*

Proof. Suppose that $\Sigma := [\Gamma(u)]$ is not arc-transitive. Let (u, v) be an arc and $A = \mathrm{Aut}(\Gamma)$. Since Σ is not an empty graph, Γ has girth 3. Since Γ is 2-geodesic-transitive, it follows from Theorem 1.1 (1) of [4] that Σ has diameter 2 and A_{uv} is transitive on $\Sigma_2(v)$.

If Σ has girth at least 5, then for any $x, y \in \Sigma(v)$, $(\Sigma_2(v) \cap \Sigma(x)) \cap (\Sigma_2(v) \cap \Sigma(y)) = \emptyset$. Since A_{uv} is transitive on $\Sigma_2(v)$, it follows that A_{uv} is transitive on $\Sigma(v)$, contradicting that Σ is not arc-transitive. Thus Σ has girth 3 or 4.

Suppose Σ has girth 4. Then there are 6 edges between $\Sigma(v)$ and $\Sigma_2(v)$, as Σ has valency 3. Further, for any $x \in \Sigma_2(v)$, $|\Sigma(v) \cap \Sigma(x)| = 2$ or 3. Suppose $|\Sigma(v) \cap \Sigma(x)| = 3$. Since A_{uv} is transitive on $\Sigma_2(v)$ and $|\Sigma(v) \cap \Sigma(x)| = 3$, it follows that $6 = 3|\Sigma_2(v)|$, so $|\Sigma_2(v)| = 2$. Let $\Delta = \{v\} \cup \Sigma_2(v)$. Then any two vertices of Δ are non-adjacent, and every vertex of Δ is adjacent to all vertices which are not in Δ, as Σ has diameter 2. Thus Δ is a block of cardinality 3, and $\Sigma(v)$ is another block. Hence $\Sigma \cong K_{3,3}$, so A_{uv} is transitive on $\Sigma(v)$, which is a contradiction. Suppose $|\Sigma(v) \cap \Sigma(x)| = 2$. Since there are 6 edges between $\Sigma(v)$ and $\Sigma_2(v)$ and A_{uv} is transitive on $\Sigma_2(v)$, it follows that $6 = 2|\Sigma_2(v)|$, so $|\Sigma_2(v)| = 3$. Set $\Sigma(v) = \{w_1, w_2, w_3\}$ and $\Sigma_2(v) = \{x_1, x_2, x_3\}$. Let $\Sigma_2(v) \cap \Sigma(w_1) = \{x_1, x_2\}$ and $\Sigma(v) \cap \Sigma(x_1) = \{w_1, w_2\}$. If $\Sigma_2(v) \cap \Sigma(w_1) = \Sigma_2(v) \cap \Sigma(w_2)$, then $\Sigma(v) \cap \Sigma(x_3) = \{w_3\}$, contradicting the fact that $|\Sigma(v) \cap \Sigma(x_3)| = 2$. Thus $\Sigma_2(v) \cap \Sigma(w_1) \neq \Sigma_2(v) \cap \Sigma(w_2)$. Hence w_2 is adjacent to x_3. In particular, $\Sigma_2(v) = \Sigma_2(v) \cap (\Sigma(w_1) \cup \Sigma(w_2))$. Since Σ has girth 4, it follows that x_1 is not adjacent to any vertex of $\{x_2, x_3\}$, so $|\Sigma_3(v) \cap \Sigma(x_1)| = 1$, contradicting that Σ has diameter 2. Thus $|\Sigma(v) \cap \Sigma(x)| \neq 2$, and so the girth of Σ is not 4.

Therefore Σ has girth 3. Set $\Sigma(v) = \{v_1, v_2, v_3\}$. Let (v, v_1, w_1) be a 2-geodesic and let (v, v_1, v_2) be a triangle. Since Σ has valency 3 and $\Sigma(v_1) = \{v, v_2, w_1\}$, v_1 and v_3 are not adjacent. Assume that v_2, v_3 are adjacent. Then v_2 is adjacent to both v_1 and v_3. Since Σ is vertex-transitive, some vertex of $\Sigma(v_1)$ is adjacent to the remaining two vertices in $\Sigma(v_1)$. Since (v, v_1, w_1) is a 2-geodesic, v, w_1 are not adjacent, it follows that v_2 is adjacent to both v and w_1, so $\{v, v_1, v_3, w_1\} \subseteq \Sigma(v_2)$, contradicting the fact that Σ has valency 3. Thus the arc (v, v_3) is not in any triangle. Hence $|\Sigma_2(v) \cap \Sigma(v_3)| = 2$, and say $\Sigma_2(v) \cap \Sigma(v_3) = \{w, w'\}$. In particular, every vertex is in a unique triangle. Hence w and w' are adjacent.

Suppose that $|\Sigma(v) \cap \Sigma(w)| = 1$. Then $\Sigma(v) \cap \Sigma(w) = \Sigma(v) \cap \Sigma(w') = \{v_3\}$. Since Σ has diameter 2, $|\Sigma_2(v) \cap \Sigma(w)| = 2$. Since v_1 is in a unique triangle, w_1, v_2 are not adjacent. Set $\Sigma_2(v) \cap \Sigma(v_2) = \{w_2\}$. As v_2 is not adjacent to any one of $\{w, w'\}$, $w_2 \notin \{w, w'\}$, so $\Sigma_2(v) = \{w_1, w_2, w, w'\}$. Since A_{uv} is transitive on $\Sigma_2(v)$ and $|\Sigma_2(v) \cap \Sigma(w)| = 2$, it follows that $[\Sigma_2(v)]$ is a vertex-transitive graph of valency 2, so $[\Sigma_2(v)] \cong C_4$. Hence the vertex w_1 is not in any triangle, which is a contradiction. Thus $|\Sigma(v) \cap \Sigma(w)| \neq 1$.

Hence $|\Sigma(v) \cap \Sigma(w)| = 2$. Since A_{uv} is transitive on $\Sigma_2(v)$ and there are 4 edges between $\Sigma(v)$ and $\Sigma_2(v)$, it follows that $\Sigma_2(v) = \{w, w'\}$ and $|\Sigma(v) \cap \Sigma(w')| = 2$. Thus Σ is $\overline{C_6}$. It follows from Lemma 2.2 that Γ is $J(5, 2)$ or the icosahedron. The icosahedron is not neighbor cubic, so Γ is $J(5, 2)$. □

It follows from Lemmas 2.1 and 2.3 that Theorem 1.1 is true.

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 by a series of lemmas.

Lemma 3.1. *Let Γ be a tetravalent vertex-transitive graph. Let $A := \mathrm{Aut}(\Gamma)$ and $u \in V(\Gamma)$. If Γ has girth 4 and A_u is transitive on $\Gamma_2(u)$, then either Γ is symmetric or $|\Gamma(u) \cap \Gamma(w)| \neq 3$ for any $w \in \Gamma_2(u)$.*

Proof. Suppose that Γ has girth 4 and A_u is transitive on $\Gamma_2(u)$. Assume Γ is not a symmetric graph. Since Γ has both valency and girth 4, $|\Gamma_2(u) \cap \Gamma(v)| = 3$ for each $v \in \Gamma(u)$, so there are 12 edges between $\Gamma(u)$ and $\Gamma_2(u)$. Let (u, v, w) be a 2-geodesic. Assume that $|\Gamma(u) \cap \Gamma(w)| = 3$. By the assumption, A_u is transitive on $\Gamma_2(u)$, so $12 = 3|\Gamma_2(u)|$, hence $|\Gamma_2(u)| = 4$.

Set $\Gamma(u) = \{v_1 = v, v_2, v_3, v_4\}$ and $\Gamma_2(u) = \{w_1 = w, w_2, w_3, w_4\}$. Let $\Gamma_2(u) \cap \Gamma(v_1) = \{w_1, w_2, w_3\}$ and $\Gamma(u) \cap \Gamma(w_1) = \{v_1, v_2, v_3\}$. If $\Gamma_2(u) \cap \Gamma(v_1) = \Gamma_2(u) \cap \Gamma(v_2)$, then $\Gamma(u) \cap \Gamma(w_4) \subseteq \{v_3, v_4\}$, contradicting the fact that $|\Gamma(u) \cap \Gamma(w_4)| = 3$. Thus $\Gamma_2(u) \cap \Gamma(v_1) \neq \Gamma_2(u) \cap \Gamma(v_2)$. Hence v_2 is adjacent to w_4, and also adjacent to one vertex of $\{w_2, w_3\}$, say w_2. In particular, $\Gamma_2(u) = \Gamma_2(u) \cap (\Gamma(v_1) \cup \Gamma(v_2))$. Since Γ has girth 4, it follows that w_1 is not adjacent to any vertex of $\Gamma_2(u) \setminus \{w_1\}$, so $|\Gamma_3(u) \cap \Gamma(w_1)| = 1$, say $\Gamma_3(u) \cap \Gamma(w_1) = \{z\}$. Then (v_1, w_1, z) and (v_2, w_1, z) are 2-geodesics. Thus $|\Gamma(v_1) \cap \Gamma(z)| = 3 = |\Gamma(v_2) \cap \Gamma(z)|$, so $\Gamma(z) = \Gamma_2(u)$. Hence $\Gamma \cong K_{5,5} - 5K_2$, and $A \cong S_2 \times S_5$. However $A_u \cong S_4$ is transitive on $\Gamma(u)$, contradicting the assumption that Γ is not a symmetric graph. Thus $|\Gamma(u) \cap \Gamma(w)| \neq 3$. □

A permutation group G on a set Ω is said to be 2-*homogeneous*, if G is transitive on the set of 2-subsets of Ω.

Lemma 3.2. *Let Γ be a tetravalent vertex-transitive but not arc-transitive graph. Let $A := \mathrm{Aut}(\Gamma)$ and $u \in V(\Gamma)$. Suppose that A_u is transitive on $\Gamma_2(u)$. Then Γ has girth 3.*

Proof. Suppose Γ has girth at least 5. Then for any $x, y \in \Gamma(u)$, $(\Gamma_2(u) \cap \Gamma(x)) \cap (\Gamma_2(u) \cap \Gamma(y)) = \emptyset$. Since A_u is transitive on $\Gamma_2(u)$, it follows that A_u is transitive on $\Gamma(u)$, contradicting that Γ is not arc-transitive. Thus Γ has girth 3 or 4.

Assume that Γ has girth 4. Then $|\Gamma(u) \cap \Gamma(w)| = 2, 3$ or 4, for any $w \in \Gamma_2(u)$. By Lemma 3.1, $|\Gamma(u) \cap \Gamma(w)| \neq 3$. Suppose that $|\Gamma(u) \cap \Gamma(w)| = 2$. Since Γ is vertex-transitive and A_u is transitive on $\Gamma_2(u)$, each 2-arc of Γ lies in a unique 4-cycle. Thus, there is a 1-1 mapping between the unordered vertex pairs in $\Gamma(u)$ and vertices in $\Gamma_2(u)$. Again since A_u is transitive on $\Gamma_2(u)$, it follows that A_u is transitive on the set of unordered vertex pairs in $\Gamma(u)$. Hence A_u is 2-homogeneous on $\Gamma(u)$, so A_u is transitive on $\Gamma(u)$, contradicting that Γ is not arc-transitive. Suppose $|\Gamma(u) \cap \Gamma(w)| = 4$. Since Γ has girth 4, there are 12 edges between $\Gamma(u)$ and $\Gamma_2(u)$. As A_u is transitive on $\Gamma_2(u)$, Γ has diameter 2 and $|\Gamma_2(u)| = 3$. Let $\Delta = \{u\} \cup \Gamma_2(u)$. Then any two vertices of Δ are non-adjacent, and every vertex of Δ is adjacent to all vertices which are not in Δ. By the structure of Γ, Δ is a block of cardinality 4, and $\Gamma(u)$ is another block. Thus $\Gamma \cong K_{4,4}$. Hence A_u is transitive on $\Gamma(u)$, again a contradiction. Therefore Γ has girth 3. □

Lemma 3.3. *Let Γ be a tetravalent vertex-transitive but not arc-transitive graph of diameter 2. Let $A := \mathrm{Aut}(\Gamma)$ and $u \in V(\Gamma)$. Suppose that A_u is transitive on $\Gamma_2(u)$. Then $|\Gamma(u) \cap \Gamma(w)| \neq 4$ for any $w \in \Gamma_2(u)$.*

Proof. Since A_u is transitive on $\Gamma_2(u)$ but not on $\Gamma(u)$, it follows from Lemma 3.2 that Γ has girth 3. If for any $v, v' \in \Gamma(u)$ we have $\Gamma_2(u) \cap \Gamma(v) \cap \Gamma(v') = \emptyset$, then as A_u is transitive on $\Gamma_2(u)$, A_u is transitive on $\Gamma(u)$, which is a contradiction. Thus, there exist $v, v' \in \Gamma(u)$ such that $\Gamma_2(u) \cap \Gamma(v) \cap \Gamma(v') \neq \emptyset$. Set $\Gamma(u) = \{v_1, v_2, v_3, v_4\}$. Suppose that $\Gamma_2(u) \cap \Gamma(v_1) \cap \Gamma(v_3) \neq \emptyset$, and say $w_1 \in \Gamma_2(u) \cap \Gamma(v_1) \cap \Gamma(v_3)$. Then $|\Gamma(u) \cap \Gamma(w_1)| = 2, 3$ or 4. Since A_u is transitive on $\Gamma_2(u)$, for any $w \in \Gamma_2(u)$, $|\Gamma(u) \cap \Gamma(w)| = 2, 3$ or 4. In particular, $|\Gamma(u) \cap \Gamma(w)|$ divides the number of edges between $\Gamma(u)$ and $\Gamma_2(u)$.

Assume that $|\Gamma(u) \cap \Gamma(w)| = 4$. If u lies in a unique triangle (u, v_1, v_2), then there are 10 edges between $\Gamma(u)$ and $\Gamma_2(u)$, however 4 does not divide 10, which is a contradiction. Assume that u is in two triangles. Then there are 8 edges between $\Gamma(u)$ and $\Gamma_2(u)$. Hence $|\Gamma_2(u)| = 2$ and Γ has 7 vertices. Let $\Delta = \{u\} \cup \Gamma_2(u)$. Then any two vertices of Δ are non-adjacent, and every vertex of Δ is adjacent to all vertices which are not in Δ. Since Γ is vertex-transitive, Δ is a block of cardinality 3. However, 3 does not divide 7, so such a Γ does not exist. Assume that u lies in more than two triangles. Then there are $x \leq 6$ edges between $\Gamma(u)$ and $\Gamma_2(u)$. Since 4 divides x, $x = 4$, so $|\Gamma_2(u)| = 1$, Γ has 6 vertices. Let $\Delta = \{u\} \cup \Gamma_2(u)$. Then any two vertices of Δ are non-adjacent, and every vertex of Δ is adjacent to all vertices which are not in Δ. Since Γ is vertex-transitive, Δ is a block of cardinality 2. In particular, $\Gamma(u)$ contains two such blocks Δ' and Δ'', and $[\Delta' \cup \Delta''] \cong C_4$. Thus $\Gamma \cong K_{3[2]}$. However A_u is transitive on $\Gamma(u)$, contradicting that Γ is not arc-transitive. Hence $|\Gamma(u) \cap \Gamma(w)| \neq 4$. □

Let Γ_1, Γ_2 be two graphs. We use $\Gamma_1 \square \Gamma_2$ to denote the *Cartesian product* of Γ_1 and Γ_2, its vertex set is $V(\Gamma_1) \times V(\Gamma_2)$, two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $u_2 = v_2$ and u_1, v_1 are adjacent in Γ_1, or $u_1 = v_1$ and u_2, v_2 are adjacent in Γ_2.

Now we can prove the second theorem.

Proof of Theorem 1.2. Let Γ be a 2-geodesic-transitive neighbor tetravalent graph. Let (u, v) be an arc and $A = \mathrm{Aut}(\Gamma)$. Since $\Sigma := [\Gamma(u)]$ is not an empty graph, Γ has girth 3. Since Γ is 2-geodesic-transitive, it follows from Theorem 1.1 (1) of [4] that Σ has diameter 2 and A_{uv} is transitive on $\Sigma_2(v)$.

Suppose first that Σ is arc-transitive. Then Σ is distance-transitive, and it is listed in [3, p.221-223]. By inspecting the candidates, Σ is either $K_{3[2]}$ or $H(2, 3)$. If Σ is $K_{3[2]}$, then Γ is $K_{4[2]}$. If Σ is $H(2, 3)$, then Γ is $J(6, 3)$.

In the remaining, we suppose that Σ is not arc-transitive. Since A_{uv} is transitive on $\Sigma_2(v)$, it follows from Lemma 3.2 that Σ has girth 3. If for any $v', v'' \in \Sigma(v)$ we have $\Sigma_2(v) \cap \Sigma(v') \cap \Sigma(v'') = \emptyset$, then as A_{uv} is transitive on $\Sigma_2(v)$, A_{uv} is transitive on $\Sigma(v)$, which is a contradiction. Thus, there exist $v', v'' \in \Sigma(v)$ such that $\Sigma_2(v) \cap \Sigma(v') \cap \Sigma(v'') \neq \emptyset$. Set $\Sigma(v) = \{v_1, v_2, v_3, v_4\}$. Suppose that $\Sigma_2(v) \cap \Sigma(v_1) \cap \Sigma(v_3) \neq \emptyset$, and say $w_1 \in \Sigma_2(v) \cap \Sigma(v_1) \cap \Sigma(v_3)$. Then $|\Sigma(v) \cap \Sigma(w_1)| = 2, 3$ or 4. Since A_{uv} is transitive on $\Sigma_2(v)$, for any $\delta \in \Sigma_2(v)$, $|\Sigma(v) \cap \Sigma(\delta)| = 2, 3$ or 4. It follows from Lemma 3.3 that $|\Sigma(v) \cap \Sigma(\delta)| \neq 4$. In particular, $|\Sigma(v) \cap \Sigma(\delta)|$ divides the number of edges between $\Sigma(v)$ and $\Sigma_2(v)$.

Assume that $|\Sigma(v) \cap \Sigma(\delta)| = 3$. If v lies in one or two triangles, then there are 10 or 8 edges between $\Sigma(v)$ and $\Sigma_2(v)$, respectively. However 3 does not divide 8 or 10, which is a contradiction. Hence v lies in more than two triangles. Then there are $x \leq 6$ edges between $\Sigma(v)$ and $\Sigma_2(v)$. Since 3 divides x, $x = 3$ or 6, so $|\Sigma_2(v)| = 1$ or 2. Assume $|\Sigma_2(v)| = 1$, say $\Sigma_2(v) = \{w\}$. Then $|\Sigma_3(v) \cap \Sigma(w)| = 1$, contradicting the fact that Σ has diameter 2. Hence $|\Sigma_2(v)| = 2$, say $\Sigma_2(v) = \{w, w'\}$. Since Σ has diameter 2, $|\Sigma_2(v) \cap \Sigma(w)| = 1$. Thus $\overline{\Sigma}$ is a vertex-transitive graph of valency 2 with 7 vertices, so $\overline{\Sigma} \cong C_7$. Thus $\Sigma \cong \overline{C_7}$. By Lemma 2.2, Γ does not exist.

Now assume that $|\Sigma(v) \cap \Sigma(\delta)| = 2$. Since Σ has diameter 2, it follows that $|\Sigma_2(v) \cap \Sigma(\delta)| = 2$. Thus $[\Sigma_2(v)]$ is a vertex-transitive graph of valency 2. If v lies in r triangles for some $r \geq 1$, then there are $12 - 2r$ edges between $\Sigma(v)$ and $\Sigma_2(v)$. Since A_{uv} is transitive on $\Sigma_2(v)$, $|\Sigma(v) \cap \Sigma(\delta)|$ divides $12 - 2r$. It follows that $r \leq 5$. Since $|\Sigma_2(v) \cap \Sigma(\delta)| = 2$, it follows that $|\Sigma_2(v)| \geq 3$, and so there are at least 6 edges between $\Sigma(v)$ and $\Sigma_2(v)$. Hence $12 - 2r \geq 6$, so $r = 1, 2$ or 3.

If $r = 1$, then there are 10 edges between $\Sigma(v)$ and $\Sigma_2(v)$. Since $|\Sigma(v) \cap \Sigma(\delta)| = 2$ for any $\delta \in \Sigma_2(v)$, one has $|\Sigma_2(v)| = 5$. Assume that (v, v_1, v_2) is a triangle. Then v_3 is not adjacent to v_4. So, A_{uv} fixes $\{v_1, v_2\}$ and $\{v_3, v_4\}$ setwise, respectively. Therefore, A_{uv} fixes $\Sigma_2(v) \cap (\Sigma(v_1) \cup \Sigma(v_2))$ setwise. As $|\Sigma_2(v) \cap (\Sigma(v_1) \cup \Sigma(v_2))| \leq 4$, it follows that $\Sigma_2(v) \cap (\Sigma(v_1) \cup \Sigma(v_2)) \subset \Sigma_2(v)$, contradicting the fact that A_{uv} is transitive on $\Sigma_2(v)$.

If $r = 2$, then there are 8 edges between $\Sigma(v)$ and $\Sigma_2(v)$. Further, $|\Sigma_2(v)| = 4$, so $[\Sigma_2(v)] \cong C_4$. Set $\Sigma(v) = \{v_1, v_2, v_3, v_4\}$. Then $|\Sigma(v) \cap \Sigma(v_i)| = 1$ or 2. If $|\Sigma(v) \cap \Sigma(v_i)| = 1$ for each v_i, then $[\Sigma(v)] \cong 2\mathrm{K}_2$. Hence each arc lies in a unique triangle. Let (v_1, v_2) and (v_3, v_4) be two arcs. Let $\Sigma_2(v) \cap \Sigma(v_1) = \{w_1, w_2\}$. Since $[\Sigma(v_1)] \cong 2\mathrm{K}_2$, (w_1, w_2) is an arc and v_2 is not adjacent to any one of $\{w_1, w_2\}$. Set $\Sigma_2(v) \cap \Sigma(v_2) = \{w_3, w_4\}$. Since $[\Sigma(v_2)] \cong 2\mathrm{K}_2$, (w_3, w_4) is an arc. Since $[\Sigma_2(v)] \cong C_4$, it follows that (w_1, w_2, w_3, w_4) is a 4-cycle. Since $|\Sigma(v) \cap \Sigma(w_1)| = 2$ and each arc lies in a unique triangle, w_1 is adjacent to one of v_3, v_4, say v_3. Then $\Sigma(w_1) = \{v_1, v_3, w_2, w_4\}$. Since $[\Sigma(w_1)] \cong 2\mathrm{K}_2$, it follows that v_3, w_4 are adjacent. Hence v_4 is adjacent to both w_2 and w_3. Thus Σ is isomorphic to the Hamming graph $\mathrm{H}(2, 3)$. However $\mathrm{H}(2, 3)$ is arc-transitive, which is a contradiction. Thus there exists v_i such that $|\Sigma(v) \cap \Sigma(v_i)| = 2$. Without loss of generality, let $v_i = v_1$ and let $\Sigma(v) \cap \Sigma(v_1) = \{v_2, v_3\}$. Then $|\Sigma_2(v) \cap \Sigma(v_1)| = 1$, and say $\Sigma_2(v) \cap \Sigma(v_1) = \{w_1\}$. Since u lies in 2 triangles, v_1 is the unique vertex of $\Sigma(v)$ such that $|\Sigma(v) \cap \Sigma(v_1)| = 2$. Thus A_{uv} can not map w_1 to other vertices of $\Sigma_2(v)$, contradicting the fact that A_{uv} is transitive on $\Sigma_2(v)$. Hence $r \neq 2$.

Finally, assume $r = 3$. Then there are 6 edges between $\Sigma(v)$ and $\Sigma_2(v)$. Further, $|\Sigma_2(v)| = 3$, so $[\Sigma_2(v)] \cong C_3$. Set $\Sigma(v) = \{v_1, v_2, v_3, v_4\}$. Then for any v_i, $|\Sigma(v) \cap \Sigma(v_i)| \leq 3$. Since v is in 3 triangles, there exist at most one vertex v_i such that $|\Sigma(v) \cap \Sigma(v_i)| = 3$. Assume there exists a vertex, v_1, such that $|\Sigma(v) \cap \Sigma(v_1)| = 3$. Then $\Sigma(v) \cap \Sigma(v_1) = \{v_2, v_3, v_4\}$, and vertices of $\{v_2, v_3, v_4\}$ are pairwise non-adjacent. Hence $\Sigma(v_2) = \{v, v_1\} \cup (\Sigma_2(v) \cap \Sigma(v_2))$. Since there are no edges between sets $\{v, v_1\}$ and $\Sigma_2(v) \cap \Sigma(v_2)$, it follows that for any $\varphi \in \Sigma(v_2)$, $|\Sigma(v_2) \cap \Sigma(\varphi)| < 3$, so $[\Sigma(v)] \ncong [\Sigma(v_2)]$. Thus A can not map v to v_2, contradicting that Σ is vertex-transitive. Hence $|\Sigma(v) \cap \Sigma(v_i)| \leq 2$. If for any v_i, $|\Sigma(v) \cap \Sigma(v_i)| \geq 1$, then $|\Sigma_2(v) \cap \Sigma(v_i)| = 1$ or 2. Since there are 6 edges between $\Sigma(v)$ and $\Sigma_2(v)$, there exists v_i such that $|\Sigma_2(v) \cap \Sigma(v_i)| = 1$. Assume that there are x vertices in $\Sigma(v)$ that are adjacent to exactly one vertex of $\Sigma_2(v)$. Then counting the edges between $\Sigma(v)$ and $\Sigma_2(v)$, $x + 2(4 - x) = 6$, so $x = 2$. Suppose $|\Sigma_2(v) \cap \Sigma(v_1)| = |\Sigma_2(v) \cap \Sigma(v_2)| = 1$, say $\Sigma_2(v) \cap \Sigma(v_1) = \{w_1\}$ and $\Sigma_2(v) \cap \Sigma(v_2) = \{w_2\}$. Then A_{uv} can not map w_3 to any one of w_1, w_2, contradicting the fact that A_{uv} is transitive on $\Sigma_2(v)$. Thus there exists a vertex v_i such that $|\Sigma(v) \cap \Sigma(v_i)| = 0$. Since v is in 3 triangles, $[\Sigma(v) \setminus \{v_i\}] \cong C_3$. Further $\Sigma_2(v) \cap \Sigma(v_i) = \Sigma_2(v)$, and there are 3 edges between $\Sigma(v) \setminus \{v_i\}$ and $\Sigma_2(v)$. Hence for each $v_j \in \Sigma(v) \setminus \{v_i\}$, $|\Sigma_2(v) \cap \Sigma(v_j)| = 1$. Therefore $\Sigma \cong \mathrm{K}_4 \Box \mathrm{K}_2$. Then by [3, Theorem 9.1.3], Γ is $J(6, 2)$. □

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