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Coincidence Best Proximity Points in Convex Metric Spaces

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Abstract. Let $T, S : A \cup B \to A \cup B$ be mappings such that $T(A) \subseteq B, T(B) \subseteq A$ and $S(A) \subseteq A, S(B) \subseteq B$. Then the pair $(T; S)$ of mappings defined on $A \cup B$ is called cyclic-noncyclic pair, where A and B are two nonempty subsets of a metric space (X, d). A coincidence best proximity point $p \in A \cup B$ for such a pair of mappings $(T; S)$ is a point such that $d(Sp, Tp) = \text{dist}(A, B)$. In this paper, we study the existence and convergence of coincidence best proximity points in the setting of convex metric spaces. We also present an application of one of our results to an integral equation.

1. Introduction

Let (X, d) be a metric space, and let A, B be subsets of X. A mapping $T : A \cup B \to A \cup B$ is said to be *cyclic* provided that $T(A) \subseteq B$ and $T(B) \subseteq A$. The following theorem is an extension of the Banach contraction principle for such mappings.

Theorem 1.1. *([23]) Let A and B be nonempty closed subsets of a complete metric space (X, d). Suppose that T is a cyclic mapping such that*

$$d(Tx, Ty) \leq \alpha\, d(x, y),$$

for some $\alpha \in (0, 1)$ and for all $x \in A, y \in B$. Then T has a unique fixed point in $A \cap B$.

Eldred and Veeramani ([11]) introduced the class of cyclic contractions as below.

Definition 1.2. *([11]) Let A and B be nonempty subsets of a metric space X. A mapping $T : A \cup B \to A \cup B$ is said to be a cyclic contraction if T is cyclic and*

$$d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha)\text{dist}(A, B)$$

for some $\alpha \in (0, 1)$ and for all $x \in A, y \in B$, where $\text{dist}(A, B) := \inf\{d(x, y) : (x, y) \in A \times B\}$.

We recall that for a cyclic mapping $T : A \cup B \to A \cup B$, a point $x \in A \cup B$ is said to be a *best proximity point* provided that $d(x, Tx) = \text{dist}(A, B)$.

The following existence, uniqueness and convergence result of a best proximity point for cyclic contractions is the main result of [11].

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Theorem 1.3. *([11]) Let A and B be nonempty, closed and convex subsets of a uniformly convex Banach space X and let $T : A \cup B \to A \cup B$ be a cyclic contraction map. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then there exists a unique $x \in A$ such that $x_{2n} \to x$ and $\|x - Tx\| = \mathrm{dist}(A, B)$.*

This result was studied from a different and more general approach (see [1, 4–8] for more information).

Now, consider a mapping $S : A \cup B \to A \cup B$, where (A, B) is a nonempty pair of subsets of a metric space (X, d). We say that S is *noncyclic* provided that $S(A) \subseteq A$ and $S(B) \subseteq B$. A point $(x, y) \in A \times B$ is called a *best proximity pair*, whenever

$$x = Tx, \quad y = Ty, \quad \text{and } d(x, y) = \mathrm{dist}(A, B).$$

Existence of best proximity pairs was first studied in [10] by using a geometric property on a nonempty pair of subsets of a Banach space, called proximal normal structure, for noncyclic relatively nonexpansive mappings (Theorem 2.2 of [10]). Some of existence results of best proximity pairs can be found in [9, 12, 13, 15, 17–19, 21, 22, 26, 27].

In this paper, we introduce a new notion of a point, called *coincidence best proximity point* and study sufficient conditions which ensure the existence of these points for a pair of cyclic and noncyclic mappings in the setting of *convex metric spaces*. In this way, we obtain some of generalizations of best proximity point and coincidence point theorems in convex metric spaces. Finally, as an application of one of our main results, we prove the existence of a solution of an integral equation.

2. Preliminaries

In [28], Takahashi introduced the notion of convexity in metric spaces as follows.

Definition 2.1. *Let (X, d) be a metric space and $I := [0, 1]$. A mapping $\mathcal{W} : X \times X \times I \to X$ is said to be a convex structure on X provided that for each $(x, y; \lambda) \in X \times X \times I$ and $u \in X$,*

$$d(u, \mathcal{W}(x, y; \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y).$$

A metric space (X, d) together with a convex structure $\mathcal{W}$ is called a convex metric space, which is denoted by $(X, d, \mathcal{W})$. A Banach space and each of its convex subsets are convex metric spaces. But a Frechet space is not necessary a convex metric space. The other examples of convex metric spaces which are not imbedded in any Banach space can be founded in [28].

Definition 2.2. *([28]) A subset K of a convex metric space $(X, d, \mathcal{W})$ is said to be a convex set provided that $\mathcal{W}(x, y; \lambda) \in K$ for all $x, y \in K$ and $\lambda \in I$.*

Definition 2.3. *([29]) A convex metric space $(X, d, \mathcal{W})$ is said to be uniformly convex if for any $\varepsilon > 0$, there exists $\alpha = \alpha(\varepsilon)$ such that for all $r > 0$ and $x, y, z \in X$ with $d(z, x) \leq r$, $d(z, y) \leq r$ and $d(x, y) \geq r\varepsilon$,*

$$d(z, \mathcal{W}(x, y, \frac{1}{2})) \leq r(1 - \alpha) < r.$$

Clearly every uniformly convex Banach spaces are uniformly convex metric spaces.

Example 2.1.([29]) Let $\mathcal{H}$ be a Hilbert space and let X be a nonempty closed subset of $\{x \in \mathcal{H} : \|x\| = 1\}$ such that if $x, y \in X$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$, then $\frac{\alpha x + \beta y}{\|\alpha x + \beta y\|} \in X$ and $\mathrm{diam}(X) \leq \frac{\sqrt{2}}{2}$, where $\mathrm{diam}(X) := \sup\{d(x, y) : x, y \in X\}$. Let $d(x, y) := \cos^{-1}(< x, y >)$ for all $x, y \in X$, where $<, >$ is the inner product of $\mathcal{H}$. If we define the convex structure $\mathcal{W} : X \times X \times I \to X$ with $\mathcal{W}(x, y, \lambda) := \frac{\lambda x + (1-\lambda) y}{\|\lambda x + (1-\lambda) y\|}$, then (X, d) is a complete and uniformly convex metric space.

Given (A, B) a pair of nonempty subsets of a metric space (X, d), then its proximal pair is the pair (A_0, B_0) given by

$$A_0 = \{x \in A\colon d(x, y') = \mathrm{dist}(A, B), \ \text{ for some } y' \in B\},$$
$$B_0 = \{y \in B\colon d(x', y) = \mathrm{dist}(A, B), \ \text{ for some } x' \in A\}.$$

Proximal pairs may be empty but, in particular, if A and B are nonempty weakly compact and convex in a Banach space X, then (A_0, B_0) is a nonempty weakly compact convex pair in X.

Definition 2.4. *A pair of sets (A, B) is said to be proximal if $A = A_0$ and $B = B_0$.*

3. Main Results

3.1. Existence and convergence results in convex metric spaces

We begin with the following notion.

Definition 3.1. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and $(T; S)$ be a cyclic-noncyclic pair on $A \cup B$, that is, $T : A \cup B \to A \cup B$ is cyclic and $S : A \cup B \to A \cup B$ is noncyclic. A point $p \in A \cup B$ is said to be a coincidence best proximity point for $(T; S)$ provided that*

$$d(Sp, Tp) = \mathrm{dist}(A, B).$$

Note that if in above definition $S = I$, where I denotes the identity map on $A \cup B$, then $p \in A \cup B$ is a best proximity point for T. Also, if $\mathrm{dist}(A, B) = 0$, then p is called a *coincidence point* for $(T; S)$ (see [14, 16] for some information).

Definition 3.2. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and $T, S : A \cup B \to A \cup B$ be two mappings. The pair $(T; S)$ is called cyclic-noncyclic contraction pair if it satisfies the following conditions:*

(i) *(T, S) is a cyclic-noncyclic pair on $A \cup B$.*
(ii) *For some $r \in (0, 1)$ we have*

$$d(Tx, Ty) \le rd(Sx, Sy) + (1 - r)\mathrm{dist}(A, B), \quad \forall (x, y) \in A \times B.$$

Remark 3.3. *Notice that (ii) implies that $d(Tx, Ty) \le d(Sx, Sy)$ for all $(x, y) \in A \times B$. Moreover, if S is noncyclic relatively nonexpansive mapping, that is, $d(Sx, Sy) \le d(x, y)$ for all $(x, y) \in A \times B$, then T is cyclic contraction.*

The following lemma will be used in the sequel.

Lemma 3.4. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and let $(T; S)$ be a cyclic-noncyclic contraction pair defined on $A \cup B$. Suppose that $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$. Then there exists a sequence $\{x_n\}$ in X such that $Tx_n = Sx_{n+1}$ and $\{x_{2n}\}, \{x_{2n-1}\}$ are sequences in A and B respectively, and $d(Sx_{2n+1}, Sx_{2n}) \to \mathrm{dist}(A, B)$.*

Proof. Let $x_0 \in A$. Since $Tx_0 \in S(B)$, there exists $x_1 \in B$ such that $Tx_0 = Sx_1$. Again, since $Tx_1 \in S(A)$, there exists $x_2 \in A$ such that $Tx_1 = Sx_2$. Continuing this process, we obtain a sequence $\{x_n\}$ such that $\{x_{2n}\}, \{x_{2n+1}\}$ are in A and B respectively, and $Tx_n = Sx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. We now have

$$\begin{aligned}
d(Sx_{2n+1}, Sx_{2n+2}) = d(Tx_{2n}, Tx_{2n+1}) &\le rd(Sx_{2n}, Sx_{2n+1}) + (1 - r)\mathrm{dist}(A, B)\\
&= rd(Tx_{2n-1}, Tx_{2n}) + (1 - r)\mathrm{dist}(A, B)\\
&\le r[rd(Sx_{2n-1}, Sx_{2n}) + (1 - r)\mathrm{dist}(A, B)] + (1 - r)\mathrm{dist}(A, B)\\
&= r^2 d(Sx_{2n-1}, Sx_{2n}) + (1 - r^2)\mathrm{dist}(A, B)\\
&= r^2 d(Tx_{2n-2}, Tx_{2n-1}) + (1 - r^2)\mathrm{dist}(A, B)\\
&\le \ldots \le r^{2n} d(Tx_0, Tx_1) + (1 - r^{2n})\mathrm{dist}(A, B).
\end{aligned}$$

Now, if $n \to \infty$ in above relation, we conclude that $d(Sx_{2n+1}, Sx_{2n}) \to \mathrm{dist}(A, B)$. □

Theorem 3.5. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and let $(T; S)$ be a cyclic-noncyclic contraction pair defined on $A \cup B$. Suppose that $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$ and S is continuous on A. For $x_0 \in A$, define $Sx_{n+1} = Tx_n$ for each $n \geq 0$. If $\{x_{2n}\}$ has a convergent subsequence in A, then the pair (T, S) has a coincidence best proximity point in A.*

Proof. Let $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ such that $x_{2n_k} \to p \in A$. We have

$$\mathrm{dist}(A, B) \leq d(Tp, Tx_{2n_k-1}) \leq d(Sp, Sx_{2n_k-1})$$

$$\leq d(Sp, Sx_{2n_k}) + d(Sx_{2n_k}, Sx_{2n_k-1}).$$

By Lemma 3.4, if $k \to \infty$, we obtain $d(Tp, Tx_{2n_k-1}) \to \mathrm{dist}(A, B)$. Besides,

$$\mathrm{dist}(A, B) \leq d(Sp, Tp) \leq d(Sp, Tx_{2n_k-1}) + d(Tx_{2n_k-1}, Tp)$$

$$= d(Sp, Sx_{2n_k}) + d(Tp, Tx_{2n_k-1}) \to \mathrm{dist}(A, B),$$

that is, $d(Sp, Tp) = \mathrm{dist}(A, B)$. □

Lemma 3.6. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and let $(T; S)$ be a cyclic-noncyclic contraction pair defined on $A \cup B$. Suppose that $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$ and suppose T and S commute on A. For $x_0 \in A$, define $Sx_{n+1} = Tx_n$ for each $n \geq 0$. Then $\{Sx_{2n}\}$ and $\{Sx_{2n+1}\}$ are bounded sequences in A and B respectively.*

Proof. Since $d(Sx_{2n}, Sx_{2n+1}) \to \mathrm{dist}(A, B)$, it is sufficient to prove that $\{Sx_{2n}\}$ is bounded in A. Suppose the contrary. Then there exists $N_0 \in \mathbb{N}$ such that

$$d(T(Sx_1), Sx_{2N_0+1}) > M, \quad d(T(Sx_1), Sx_{2N_0-1}) \leq M,$$

where, $M > \max\{\frac{r^2}{1-r^2} d(S(Sx_0), T(Sx_1)) + \mathrm{dist}(A, B), d(T(Sx_1), Tx_0)\}$. We have

$$\begin{aligned}
\frac{M - \mathrm{dist}(A, B)}{r^2} + \mathrm{dist}(A, B) &< \frac{d(T(Sx_1), Sx_{2N_0+1}) - dist(A, B)}{r^2} + \mathrm{dist}(A, B) \\
&\leq \frac{d(T(Sx_1), Sx_{2N_0+1}) + (r^2 - 1)d(T(Sx_1), Sx_{2N_0+1})}{r^2} \\
= d(T(Sx_1), Sx_{2N_0+1}) &= d(T(Sx_1), Tx_{2N_0}) \leq rd(S(Sx_1), Sx_{2N_0}) + (1 - r)\mathrm{dist}(A, B) \\
&\leq d(S(Sx_1), Sx_{2N_0}) = d(S(Tx_0), Tx_{2N_0-1}) = d(T(Sx_0), Tx_{2N_0-1}) \\
&\leq d(S(Sx_0), Sx_{2N_0-1}) \leq d(S(Sx_0), T(Sx_1)) + d(T(Sx_1), Sx_{2N_0-1}) \\
&\leq d(S(Sx_0), T(Sx_1)) + M.
\end{aligned}$$

Thus

$$\frac{M - \mathrm{dist}(A, B)}{r^2} + \mathrm{dist}(A, B) < d(S(Sx_0), T(Sx_1)) + M$$

and so,

$$M - (1 - r^2)\mathrm{dist}(A, B) < r^2[d(S(Sx_0), T(Sx_1)) + M],$$

which implies that

$$M < \frac{r^2}{1 - r^2} d(S(Sx_0), T(Sx_1)) + \mathrm{dist}(A, B),$$

which is a contradiction. □

To establish our results, we need the following notion.

Definition 3.7. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d). A mapping $S : A\cup B \to A\cup B$ is said to be a relatively anti-Lipschitzian mapping if there exists $c > 0$ such that $d(x, y) \le cd(Sx, Sy)$ for all $(x, y) \in A \times B$.*

Next result is a straightforward consequence of Theorem 3.5 and Lemma 3.6.

Corollary 3.8. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) such that A is boundedly compact and let $(T; S)$ be a cyclic-noncyclic contraction pair defined on $A \cup B$. Suppose that $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$ and suppose T and S commute on A. If S is relatively anti-Lipschitzian and continuous on A, then the pair $(T; S)$ has a coincidence best proximity point in A.*

Let us illustrate Corollary 3.8 with the following example.

Example 3.1. Let $X := \mathbb{R}$ with the usual metric. For $A = (-\infty, -1]$ and $B = [1, +\infty)$ define $T, S : A\cup B \to A\cup B$ by

$$Tx := -x,\ \forall x \in A \cup B \quad \& \quad Sx := \begin{cases} 2x + 1 \text{ if } x \in A, \\ 2x - 1 \text{ if } x \in B. \end{cases}$$

Then $(T; S)$ is a cyclic-noncyclic contraction pair with $r = \frac{1}{2}$. Indeed, for all $(x, y) \in A \times B$ we have

$$|Tx - Ty| = (y - x) \le \frac{1}{2}(2y - 2x - 1) + \frac{1}{2}(2)$$
$$= r|Sx - Sy| + (1 - r)\mathrm{dist}(A, B).$$

Also, $T(A) = B \subseteq S(B)$ and $T(B) = A \subseteq S(A)$. Moreover, S is continuous on A and A is boundedly compact in X. Besides, S is relatively anti-Lipschitzian on $A \cup B$ with $c = 1$. In fact, for all $(x, y) \in A \times B$ we have

$$|Sx - Sy| = 2y - 2x - 1 \ge |x - y| \quad (\text{since } y - x \ge 1).$$

Finally, for each $x \in A$ we have

$$T(Sx) = T(2x + 1) = -2x - 1 = S(-x) = S(Tx),$$

that is, T and S commute on A. Thereby, the existence of coincidence best proximity point of the pair $(T; S)$ follows from Corollary 3.8. That is, there exists $p \in A$ such that $|Tp - Sp| = \mathrm{dist}(A, B) = 2$ or $-p - (2p + 1) = 2$ which implies that $p = -1$. In this case, p is a fixed point of the mapping S and so, p is a best proximity point of the cyclic mapping T. It is interesting to note that the mapping T is not cyclic contraction in the sense of Definition 1.2 and so, existence of best proximity point for T cannot be obtained from Theorem 1.3. Another observation is that whereas T is cyclic relatively nonexpansive mapping on $A \cup B$, that is, T is cyclic and $d(Tx, Ty) \le d(x, y)$ for all $(x, y) \in A \times B$, but the existence of best proximity point for T cannot be deduced from Theorem 2.1 of [10] because of unboundedness of A and B.

Lemma 3.9. *Let (A, B) be a nonempty pair of subsets of a uniformly convex metric space $(X, d; \mathcal{W})$ such that A is convex. Let $(T; S)$ be a cyclic-noncyclic contraction pair defined on $A \cup B$ such that $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$. For $x_0 \in A$ define $Sx_{n+1} := Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. Then*

$$d(Sx_{2n+2}, Sx_{2n}) \to 0 \text{ and } d(Sx_{2n+3}, Sx_{2n+1}) \to 0.$$

Proof. To prove that $d(Sx_{2n+2}, Sx_{2n}) \to 0$ suppose the contrary. Then there exists $\varepsilon_0 > 0$ such that for each $k \ge 1$, there exists $n_k \ge k$ so that $d(Sx_{2n_k+2}, Sx_{2n_k}) \ge \varepsilon_0$. Choose $0 < \gamma < 1$ such that $\frac{\varepsilon_0}{\gamma} > \mathrm{dist}(A, B)$ and choose $\varepsilon > 0$ such that

$$0 < \varepsilon < \min\{\frac{\varepsilon_0}{\gamma} - dist(A, B), \frac{dist(A, B)\alpha(\gamma)}{1 - \alpha(\gamma)}\}.$$

By Lemma 3.4, since $d(Sx_{2n_k}, Sx_{2n_k+1}) \to \operatorname{dist}(A,B)$, there exists $N \in \mathbb{N}$ such that

$$d(Sx_{2n_k}, Sx_{2n_k+1}) \le \operatorname{dist}(A,B) + \varepsilon \quad \& \quad d(Sx_{2n_k+2}, Sx_{2n_k+1}) \le \operatorname{dist}(A,B) + \varepsilon,$$

$$d(Sx_{2n_k}, Sx_{2n_k+2}) \ge \varepsilon_0 > \gamma(\operatorname{dist}(A,B) + \varepsilon).$$

It now follows from the uniformly convexity of X and the convexity of A

$$\operatorname{dist}(A,B) \le d(Sx_{2n_k+1}, \mathcal{W}(Sx_{2n_k}, Sx_{2n_k+2}, \frac{1}{2})) \le (\operatorname{dist}(A,B) + \varepsilon)(1 - \alpha(\gamma))$$

$$< \operatorname{dist}(A,B) + \frac{\operatorname{dist}(A,B)\alpha(\gamma)}{1-\alpha(\gamma)}(1 - \alpha(\gamma)) = \operatorname{dist}(A,B),$$

which is a contradiction. Similarly, we can see that $d(Sx_{2n+3}, Sx_{2n+1}) \to 0$ and this completes the proof. □

The following theorem guarantees the existence and convergence of coincidence best proximity points for cyclic-noncyclic contractions in the setting of uniformly convex metric spaces.

Theorem 3.10. *Let (A,B) be a nonempty, closed pair of subsets of a complete uniformly convex metric space $(X,d;\mathcal{W})$ such that A is convex. Let $(T;S)$ be a cyclic-noncyclic contraction pair defined on $A \cup B$ such that $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$ so that S is continuous on A and relatively anti-Lipschitzian on $A \cup B$. Then $(T;S)$ has a coincidence best proximity point in A. Further, if $x_0 \in A$ and $Sx_{n+1} := Tx_n$, then $\{x_{2n}\}$ converges to the coincidence best proximity point of $(T;S)$.*

Proof. For $x_0 \in A$ define $Sx_{n+1} := Tx_n$ for each $n \ge 0$. We prove that $\{Sx_{2n}\}$ and $\{Sx_{2n+1}\}$ are Cauchy sequences. At first, we verify that for each $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that

$$d(Sx_{2m}, Sx_{2n+1}) < \operatorname{dist}(A,B) + \varepsilon, \quad \forall m > n \ge N_0. \tag{*}$$

Assume the contrary. Then there exists $\varepsilon_0 > 0$ such that for each $k \ge 1$ there exists $m_k > n_k \ge k$ satisfying

$$d(Sx_{2m_k}, Sx_{2n_k+1}) \ge \operatorname{dist}(A,B) + \varepsilon_0 \quad \& \quad d(Sx_{2m_k-2}, Sx_{2n_k+1}) < \operatorname{dist}(A,B) + \varepsilon_0.$$

We have

$$\operatorname{dist}(A,B) + \varepsilon_0 \le d(Sx_{2m_k}, Sx_{2n_k+1})$$

$$\le d(Sx_{2m_k}, Sx_{2m_k-2}) + d(Sx_{2m_k-2}, Sx_{2n_k+1}) \le d(Sx_{2m_k}, Sx_{2m_k-2}) + \operatorname{dist}(A,B) + \varepsilon_0.$$

Letting $k \to \infty$, we obtain $d(Sx_{2m_k}, Sx_{2n_k+1}) \to \operatorname{dist}(A,B) + \varepsilon_0$. Besides,

$$\begin{aligned}
d(Sx_{2m_k}, Sx_{2n_k+1}) &\le d(Sx_{2m_k}, Sx_{2m_k+2}) + d(Sx_{2m_k+2}, Sx_{2n_k+3}) + d(Sx_{2n_k+3}, Sx_{2n_k+1}) \\
&= d(Sx_{2m_k}, Sx_{2m_k+2}) + d(Tx_{2m_k+1}, Tx_{2n_k+2}) + d(Sx_{2n_k+3}, Sx_{2n_k+1}) \\
&\le d(Sx_{2m_k}, Sx_{2m_k+2}) + d(Sx_{2m_k+1}, Sx_{2n_k+2}) + d(Sx_{2n_k+3}, Sx_{2n_k+1}) \\
&= d(Sx_{2m_k}, Sx_{2m_k+2}) + d(Tx_{2m_k}, Tx_{2n_k+1}) + d(Sx_{2n_k+3}, Sx_{2n_k+1}) \\
&\le d(Sx_{2m_k}, Sx_{2m_k+2}) + r d(Sx_{2m_k}, Sx_{2n_k+1}) + (1-r)\operatorname{dist}(A,B) + d(Sx_{2n_k+3}, Sx_{2n_k+1}) \\
&\le d(Sx_{2m_k}, Sx_{2m_k+2}) + d(Sx_{2m_k}, Sx_{2n_k+1}) + d(Sx_{2n_k+3}, Sx_{2n_k+1}).
\end{aligned}$$

Letting $k \to \infty$, we conclude that

$$\operatorname{dist}(A,B) + \varepsilon_0 \le r(\operatorname{dist}(A,B) + \varepsilon_0) + (1-r)\operatorname{dist}(A,B) \le \operatorname{dist}(A,B) + \varepsilon_0.$$

This implies that $r = 1$, which is a contradiction. That is, (*) holds. Now, suppose $\{Sx_{2n}\}$ is not Cauchy. Then there exists $\varepsilon_0 > 0$ such that for each $k \geq 1$ there exists $m_k > n_k \geq k$ so that $d(Sx_{2m_k}, Sx_{2n_k}) \geq \varepsilon_0$. Choose $0 < \gamma < 1$ such that $\frac{\varepsilon_0}{\gamma} > \mathrm{dist}(A, B)$ and choose $\varepsilon > 0$ such that

$$0 < \varepsilon < \min\{\frac{\varepsilon_0}{\gamma} - \mathrm{dist}(A, B), \frac{\mathrm{dist}(A, B)\alpha(\gamma)}{1 - \alpha(\gamma)}\}.$$

Let $N \in \mathbb{N}$ be such that

$$d(Sx_{2n_k}, Sx_{2n_k+1}) \leq \mathrm{dist}(A, B) + \varepsilon,\ \forall n_k \geq N$$

$$d(Sx_{2m_k}, Sx_{2n_k+1}) \leq \mathrm{dist}(A, B) + \varepsilon,\ \forall m_k > n_k \geq N.$$

Uniformly convexity of X deduces that

$$\mathrm{dist}(A, B) \leq d(Sx_{2n_k+1}, \mathcal{W}(Sx_{2n_k}, Sx_{2m_k}, \frac{1}{2}))$$

$$\leq (\mathrm{dist}(A, B) + \varepsilon)(1 - \alpha(\gamma)) < \mathrm{dist}(A, B),$$

which is a contradiction. Therefore, $\{Sx_{2n}\}$ is a Cauchy sequence in A. By the fact that S is relatively anti-Lipschitzian on $A \cup B$, we have

$$d(x_{2m}, x_{2n}) \leq c d(Sx_{2m}, Sx_{2n}) \to^{m,n\to\infty} 0,$$

that is, $\{x_{2n}\}$ is Cauchy. Since A is complete, there exists $p \in A$ such that $x_{2n} \to p$. Now, the result follows from the similar argument of Theorem 3.5. □

If $A = B$ in Theorem 3.10, then the existence of coincidence points for two self-mappings can be obtained under weaker conditions.

Theorem 3.11. *Let A be a nonempty and closed subset of a complete metric space (X, d). Suppose $(T; S)$ is a pair of self-mappings defined on A such that*
(i) S is continuous and unit-Lipschitzian,
(ii) $T(A) \subseteq S(A)$,
(iii) $d(Tx, Ty) \leq rd(Sx, Sy)$, for some $r \in (0, 1)$ and for all $x, y \in A$.
Then $(T; S)$ has a coincidence point.

Proof. Let $x_0 \in A$ and define $Sx_{n+1} := Tx_n$. Then we have $d(Sx_{n+2}, Sx_{n+1}) \leq r^n d(Tx_1, Tx_0)$, that is, the sequence $\{Sx_n\}$ is a Cauchy sequence in A. Since S is anti-Lipschitzian, we conclude that the sequence $\{x_n\}$ is Cauchy in A. Suppose $x_n \to p \in A$. Then $Sx_n \to Sp$ and so, $Tx_n \to Tp$. We now have

$$d(Sp, Tp) \leq d(Sp, Tx_{n-1}) + d(Tx_{n-1}, Tp) = d(Sp, Sx_n) + d(Tx_{n-1}, Tp) \to 0.$$

Hence, the point $p \in A$ is a coincidence point of the pair $(T; S)$. □

3.2. Existence results in reflexive Banach spaces

Theorem 3.12. *Let (A, B) be a nonempty weakly closed pair of subsets of a reflexive Banach space X and let $(T; S)$ be a cyclic-noncyclic contraction pair defined on $A \cup B$. Suppose that $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$ and suppose T and S commute on A. Then (A_0, B_0) is a nonempty pair.*

Proof. By Lemma 3.6, the sequences $\{Sx_{2n}\}$ and $\{Sx_{2n+1}\}$ are bounded in A and B, respectively. Since X is reflexive and (A, B) is weakly closed, we may assume that $Sx_{2n} \rightharpoonup p \in A$ and $Sx_{2n+1} \rightharpoonup q \in B$, where $\rightharpoonup$ denotes the weak convergence. Since $\|.\|$ is weakly lower semicontinuous, we obtain

$$\mathrm{dist}(A, B) \leq \|p - q\| \leq \liminf_{n\to\infty} \|Sx_{2n} - Sx_{2n+1}\| = \mathrm{dist}(A, B).$$

Thus (A_0, B_0) is a nonempty pair. □

Example 3.2. Let l^∞ be the Banach space consisting of all bounded real sequences with supremum norm and let $\{e_n\}$ be the canonical basis of l^∞. Given $r \in (0,1)$, let A and B be subsets of l^∞ defined with

$$A = \{(1 + r^{2n})e_{2n} : n \in \mathbb{N}\} \text{ and } B = \{(1 + r^{2m-1})e_{2m-1} : m \in \mathbb{N}\}.$$

Then $\operatorname{dist}(A, B) = 1$. Define $T, S : A \cup B \to A \cup B$ as below:

$$T((1 + r^{2n})e_{2n}) = (1 + r^{8n+1})e_{8n+1} \quad \& \quad T((1 + r^{2m-1})e_{2m-1}) = ((1 + r^{8m})e_{8m}),$$

$$S((1 + r^{2n})e_{2n}) = (1 + r^{4n})e_{4n} \quad \& \quad S((1 + r^{2m-1})e_{2m-1}) = ((1 + r^{4m-1})e_{4m-1}).$$

Then the pair $(T; S)$ is a cyclic-noncyclic pair. Also, if $m \leq n$, then

$$\begin{aligned} \|T((1 + r^{2n})e_{2n}) - T((1 + r^{2m-1})e_{2m-1})\|_\infty &= 1 + r^{8m} \\ &\leq 1 + r^{4m} = k(1 + r^{4m-1}) + (1 - r) \\ &= r\|S((1 + r^{2n})e_{2n}) - S((1 + r^{2m-1})e_{2m-1})\|_\infty + (1 - r)\operatorname{dist}(A, B), \end{aligned}$$

that is, $(T; S)$ is cyclic-noncyclic contraction. We note that $A_0 = B_0 = \emptyset$ because X is not reflexive.

Definition 3.13. *Let (A, B) be a nonempty pair of subsets of a normed linear space X and $(T; S)$ be a cyclic-noncyclic pair defined on $A \cup B$. We say that T satisfies the proximal property w.r.t. S provided that $x_n \rightharpoonup x \in A \cup B$ and $\|Sx_n - Tx_n\| \to \operatorname{dist}(A, B)$, then $\|Sx - Tx\| = \operatorname{dist}(A, B)$.*

Note that if $S = I$, then T satisfies the proximal property (see Definition 2 of [4]).

Theorem 3.14. *Let (A, B) be a nonempty pair of subsets of a reflexive Banach space X such that A is weakly closed and let $(T; S)$ be a cyclic-noncyclic contraction pair defined on $A \cup B$. Suppose that $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$ and suppose T and S commute on A. Let S be a relatively anti-Lipschitzian on $A \cup B$. Then $(T; S)$ has a coincidence best proximity point provided that one of the following conditions holds:*

(*i*) *T, S are weakly continuous on A.*
(*ii*) *T satisfies the proximal property w.r.t. S.*

Proof. Let $x_0 \in A$ and define $Sx_{n+1} := Tx_n$. It follows from Lemma 3.6 that $\{Sx_{2n}\}$ and $\{Sx_{2n-1}\}$ are bounded sequences in A and B respectively. We prove that $\{x_{2n}\}$ and $\{x_{2n-1}\}$ are also bounded. Since $\{Sx_{2n}\}$ is bounded, there exists $M > 0$ such that $\|Sx_{2n} - Sx_2\| \leq M$ for all $n \in \mathbb{N}$. Now, for each $f \in X^*$ and $n \in \mathbb{N}$ we have:

$$\begin{aligned} |f(x_{2n} - x_1)| &\leq \|f\|\|x_{2n} - x_1\| \leq \|f\|c\|Sx_{2n} - Sx_1\| \\ &\leq \|f\|c(\|Sx_{2n} - Sx_2\| + \|Sx_2 - Sx_1\|) \leq \|f\|c(M + \|Sx_2 - Sx_1\|). \end{aligned}$$

Therefore,

$$|f(x_{2n})| \leq \|f\|c(M + \|Sx_2 - Sx_1\|) + |f(x_2)|, \ \forall n \in \mathbb{N},$$

which concludes that $\{x_{2n}\}$ is a weakly bounded sequence in A and so, by uniform boundedness principle, is bounded. Similarly, we can see that $\{x_{2n-1}\}$ is also bounded. Since X is reflexive, we may assume that $x_{2n} \rightharpoonup p \in A$.

(*i*) If T and S are weakly continuous on A, then $Tx_{2n} \rightharpoonup Tp$ and $Sx_{2n} \rightharpoonup Sp$. Weak lower semicontinuity of norm implies that

$$\begin{aligned} \operatorname{dist}(A, B) \leq \|Sp - Tp\| &\leq \liminf_{n\to\infty} \|Sx_{2n} - Tx_{2n}\| \\ &= \liminf_{n\to\infty} \|Sx_{2n} - Sx_{2n+1}\| = \operatorname{dist}(A, B). \end{aligned}$$

(*ii*) If T satisfies the proximal property w.r.t. S, then by this reality that $\|Sx_{2n} - Tx_{2n}\| \to \operatorname{dist}(A, B)$ and $x_{2n} \rightharpoonup p$, we obtain $\|Sp - Tp\| = \operatorname{dist}(A, B)$ and the proof completes. □

Remark 3.15. *Notice that if in aforesaid results $\operatorname{dist}(A, B) = 0$, then we conclude the existence of coincidence point for the cyclic-noncyclic pair $(T; S)$.*

4. A coincidence point theorem in partially ordered metric spaces

Fixed point theory of partially ordered metric spaces was initiated by Nieto and Rodríguez-López; see the paper [25], where the authors provided some applications to ordinary differential equations as well.

Let $(X, \preceq)$ be a partially ordered set. A self mapping $T : X \to X$ is said to be *monotone nondecreasing* iff $T(x) \preceq T(y)$ whenever $x, y \in X, x \preceq y$. In [25], the authors established the following results.

Theorem 4.1. *([25]) Let $(X, \preceq)$ be a partially ordered set and let there exist a metric d in X which makes (X, d) into a complete metric space. Assume that X satisfies the condition*

$$\text{if a nondecreasing sequence } x_n \to x \in X, \text{ then } x_n \preceq x \ \forall n. \tag{1}$$

Let $T : X \to X$ be a monotone and nondecreasing mapping for which there exists $L \in [0, 1)$ such that $d(Tx, Ty) \leq Ld(x, y)$ for every $y \preceq x$. If there exists $x_0 \in X$ with $x_0 \preceq T(x_0)$, then T has a fixed point.

Recently, in [1], the authors extended Theorem 1.2 and established some theorems on the existence and convergence of fixed points, as well as, best proximity points for cyclic mappings in partially ordered sets.

In this section, we establish a new coincidence point theorem for a class of cyclic-noncyclic mappings in the setting of partially ordered metric spaces.

Definition 4.2. *([20]) A function $\varphi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following properties are satisfied:*
(i) φ is continuous and strictly increasing.
(ii) $\varphi(t) = 0 \Leftrightarrow t = 0$.

Definition 4.3. *(Compare to Definition 2.2 of [24]) Let $(X, \preceq)$ be a partially ordered set and let (A, B) be a nonempty pair of subsets of X. Suppose that $T, S : A \cup B \to A \cup B$ are two mappings such that (T, S) is a cyclic-noncyclic pair and $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$. We say that T is weakly increasing with respect to S provided that*

$$Tx \preceq Ty, \quad \forall y \in S^{-1}(Tx),$$

for all $x \in A \cup B$.

The next theorem is the main result of this section.

Theorem 4.4. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X. Suppose that (A, B) is nonempty pair of subsets of X. Let $T, S : A \cup B \to A \cup B$ be two mappings such that (T, S) is a cyclic-noncyclic pair and for each $(x, y) \in A \times B$ such that Sx and Sy are comparable we have*

$$d(Tx, Ty) \leq d(Sx, Sy) - \varphi(d(Sx, Sy)),$$

where φ is an altering distance function. Assume that the following hypotheses hold.
(i) A satisfies the condition (1).
(ii) $S(A)$ is complete.
(iii) $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$.
(iv) T is weakly increasing with respect to S.
Then $A \cap B$ is nonempty and the pair (T, S) has a coincidence point in $A \cap B$.

Proof. Let $x_0 \in A$. Since $T(A) \subseteq S(B)$, there exists an element $x_1 \in B$ such that $Tx_0 = Sx_1$. Again, since $T(B) \subseteq S(A)$, there exists $x_2 \in A$ such that $Tx_1 = Sx_2$. Continuing this process, we can find a sequence $\{x_n\}$ such that the even subsequence $\{x_{2n}\}$ and the odd subsequence $\{x_{2n-1}\}$ are in A and B, respectively and

$$Tx_n = Sx_{n+1}, \quad \forall n \in \mathbb{N} \cup \{0\}.$$

By the fact that $x_1 \in S^{-1}(Tx_0)$ and $x_2 \in S^{-1}(Tx_1)$ and that T is weakly increasing with respect to S, we have

$$Sx_1 = Tx_0 \preceq Tx_1 = Sx_2 \preceq Tx_2 = Sx_3.$$

Continuing this process, by induction, we get

$$Sx_1 \preceq Sx_2 \preceq Sx_3 \preceq \ldots \preceq Sx_n \preceq Sx_{n+1} \preceq \ldots.$$

Since $Sx_n \preceq Sx_{n+1}$ for each $n \in \mathbb{N}$, we have

$$d(Sx_{n+1}, Sx_{n+2}) = d(Tx_n, Tx_{n+1}) \leq d(Sx_n, Sx_{n+1}) - \varphi(d(Sx_n, Sx_{n+1})).$$

Now, if we put $\rho_n := d(Sx_n, Sx_{n+1})$ then $\{\rho_n\}$ is a decreasing sequence and

$$\rho_{n+1} \leq \rho_n - \varphi(\rho_n), \quad \forall n \in \mathbb{N},$$

which implies that $\{\rho_n\}$ is a decreasing sequence. Assume that $\rho_n \to \rho \geq 0$. Thus

$$\varphi(\rho) = \lim_{n\to\infty} \varphi(\rho_n) = 0.$$

Hence, $\rho = 0$. We now prove that $\{Sx_n\}$ is a Cauchy sequence. Suppose not. So there exists $\delta > 0$ such that for each $l \geq 1$ there exist $m_l > n_l \geq l$ satisfying

$$d(Sx_{m_l}, Sx_{n_l}) \geq \delta \quad \& \quad d(Sx_{m_l-1}, Sx_{n_l}) < \delta.$$

We now have

$$\begin{aligned} \delta \leq d(Sx_{m_l}, Sx_{n_l}) &\leq d(Sx_{m_l}, Sx_{m_l-1}) + d(Sx_{m_l-1}, Sx_{n_l}) \\ &< d(Sx_{m_l}, Sx_{m_l-1}) + \delta \leq d(Sx_l, Sx_{l-1}) + \delta. \end{aligned}$$

This proves that $\lim_{l\to\infty} d(Sx_{m_l}, Sx_{n_l}) = \delta$. Besides,

$$\begin{aligned} d(Sx_{m_l}, Sx_{n_l}) &\leq d(Sx_{m_l}, Sx_{m_l+1}) + d(Sx_{m_l+1}, Sx_{n_l+1}) + d(Sx_{n_l+1}, Sx_{n_l}) \\ &\leq 2d(Sx_l, Sx_{l-1}) + \varphi(d(Sx_{m_l}, Sx_{n_l})). \end{aligned}$$

It follows that $\delta \leq \varphi(\delta)$ which is a contradiction. Thereby, $\{Sx_n\}$ is a Cauchy sequence in X and so $\{Sx_{2n}\}$ is a Cauchy sequence in $S(A)$. Completeness of $S(A)$ deduces that there exists an element $p \in S(A)$ such that $Sx_{2n} \to p$. Let $x_\star \in A$ be such that $Sx_\star = p$. So, $Sx_{2n} \to Sx_\star$ which implies that $Sx_n \to Sx_\star$, because $\{Sx_n\}$ is Cauchy. Since A satisfies the condition (1) we conclude that $Sx_n \preceq Sx_\star$ for each $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} d(Tx_\star, Sx_{2n+1}) &\leq d(Tx_\star, Sx_{2n+2}) + d(Sx_{2n+2}, Sx_{2n+1}) \\ &= d(Tx_\star, Tx_{2n+1}) + d(Sx_{2n+2}, Sx_{2n+1}) \\ &\leq d(Sx_\star, Sx_{2n+1}) - \varphi(d(Sx_\star, Sx_{2n+1})) + d(Sx_{2n+2}, Sx_{2n+1}). \end{aligned}$$

If in above relation $n \to \infty$, then we must have $d(Tx_\star, Tx_{2n}) \to 0$ and hence, $Tx_{2n} \to Tx_\star$. Thus

$$d(Sx_\star, Tx_\star) = \lim_{n\to\infty} d(Sx_{2n}, Tx_{2n}) = \lim_{n\to\infty} d(Sx_{2n}, Sx_{2n+1}) = 0.$$

Therefore, $Sx_\star = Tx_\star$, that is, $x_\star \in A \cap B$ is a coincidence point of the cyclic-noncyclic pair (T, S). □

Corollary 4.5. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X. Suppose that (A, B) is nonempty pair of subsets of X. Let $T, S : A \cup B \to A \cup B$ be two mappings such that (T, S) is a cyclic-noncyclic pair and for each $(x, y) \in A \times B$ such that Sx and Sy are comparable we have*

$$d(Tx, Ty) \leq Ld(Sx, Sy),$$

for some $L \in [0, 1)$. Assume that the following hypotheses hold.
(i) A satisfies the condition (1).
(ii) $S(A)$ is complete.
(iii) $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$.
(iv) T is weakly increasing with respect to S.
Then $A \cap B$ is nonempty and the pair (T, S) has a coincidence point.

The following *new fixed point theorem* concludes of Theorem 3.2, immediately.

Corollary 4.6. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X. Suppose that (A, B) is nonempty pair of subsets of X. Let $T : A \cup B \to A \cup B$ be a cyclic mappings and for each $(x, y) \in A \times B$ such that x and y are comparable we have*

$$d(Tx, Ty) \leq \varphi(d(x, y)) - d(x, y),$$

where φ is an altering distance function. Assume that the following hypotheses hold.
(i) A satisfies the condition (1).
(ii) A is complete.
(iii) $Tx \preceq TTx, \quad \forall x \in A \cup B$.
Then $A \cap B$ is nonempty and the pair T has a fixed point in $A \cap B$.

Proof. It is sufficient to consider $S : A \cup B \to A \cup B$ with $S|_A = i_A$ and $S|_B = i_B$, where i denotes the identity mapping, then the result obtains from Theorem 3.2. □

Example 3.1. Consider $X = \mathbb{R}$ with the usual metric and ordinary partially order relation $\leq$. Suppose that

$$A = B = [0, 1].$$

Define $T, S : A \cup B \to A \cup B$ by

$$\mathcal{T}x = \begin{cases} \frac{x+1}{3}, & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{2}, & \text{if } \frac{1}{2} < x \leq 1, \end{cases} \quad \& \quad \mathcal{S}x = 1 - x, \quad \forall x \in A.$$

Suppose that $\varphi(t) := \frac{t}{2}$ for all $t \geq 0$. It is easy to check that

$$d(Tx, Ty) \leq d(Sx, Sy) - \varphi(d(Sx, Sy)), \quad \forall x, y \in A.$$

Also, T is weakly increasing with respect to S. Indeed, if $0 \leq x \leq \frac{1}{2}$, then

$$y := S^{-1}(Tx) = S^{-1}(\frac{x+1}{3}) = 1 - \frac{x+1}{3} = \frac{2-x}{3}.$$

Thus

$$Tx = \frac{x+1}{3} \leq \frac{5-x}{9} = Ty,$$

by the fact that $0 \leq x \leq \frac{1}{2}$. Also, if $\frac{1}{2} < x \leq 1$, then for $y := S^{-1}(Tx) = \frac{1}{2}$ we have $Tx = \frac{1}{2} = Ty$, that is, $Tx \leq Ty$. It now follows from Theorem 3.3 that the pair (T, S) has a coincidence point. Hence, there exists a point $x_\star \in A$ such that $Tx_\star = Sx_\star$ and this point is $x_\star = \frac{1}{2}$.

4.1. An application to integral equations

In this section we present an application of our results to an integral equation. One can refer to [2, 3, 16] for more information about the integral equations and its applications.

Theorem 4.7. *Consider the following integral equation:*

$$2 - e^{u(t)} + \int_0^t f(s, u(s))ds = 0, \tag{2}$$

where $f : [0, 1] \times \mathbb{R} \to [-1, +\infty)$ satisfies the following conditions:

(i) $f(.,.)$ is continuous on $[0, 1] \times \mathbb{R}$,

(ii) f is contraction in the second variable, that is, there exists $r \in [0, 1)$ such that

$$|f(t, x) - f(t, y)| \leq r|x - y|, \quad \forall t \in [0, 1] \text{ and } x, y \in \mathbb{R}.$$

Then the equation (2) has a solution in $C^+{}_{\mathbb{R}}([0, 1])$.

Proof. Denote by $C_{\mathbb{R}}([0,1])$, the set of all continuous functions from $[0,1]$ to $\mathbb{R}$ and define the metric $d : C_{\mathbb{R}}([0,1]) \times C_{\mathbb{R}}([0,1]) \to \mathbb{R}$ by

$$d(x,y) := \sup_{t\in[0,1]} |u(t) - v(t)|.$$

Then $(C_{\mathbb{R}}([0,1]), d)$ is a complete metric space. Suppose that $A = C^{+}{}_{\mathbb{R}}([0,1])$, where

$$C^{+}{}_{\mathbb{R}}([0,1]) := \{u : [0,1] \to \mathbb{R} \text{ continuous and } u(t) \geq 0 \text{ for all } t \in [0,1]\}.$$

Then A is a nonempty and closed subset of $C_{\mathbb{R}}([0,1])$. Define $T, S : A \to A$ as follows:

$$T(v(t)) = \int_0^t f(s, v(s))ds + 1 \quad \& \quad S(u(t)) = e^{u(t)} - 1,$$

Thus S is continuous on A and we have

$$\|Su - Sv\|_\infty \geq |S(u(t)) - S(v(t))| = |e^{u(t)} - e^{v(t)}|$$

$$= e^{u(t)}|1 - e^{v(t)-u(t)}| \geq e^{|u(t)-v(t)|} - 1, \quad \forall t \in [0,1],$$

and so, $\|Su - Sv\|_\infty \geq e^{\|u-v\|_\infty} - 1$. Besides, since $t \leq e^t - 1$ for each $t \geq 0$, we conclude that $\|Su - Sv\|_\infty \geq \|u - v\|_\infty$, that is, S is anti-Lipschitzian on A with the constant $c = 1$. On the other hand, for all $t \in [0,1]$

$$|T(u(t)) - T(v(t))| = |\int_0^t f(s, u(s))ds - \int_0^t f(s, v(s))ds|$$

$$\leq \int_0^t |f(s, u(s)) - f(s, v(s))|ds \leq \int_0^t r|u(s) - v(s)|ds$$

$$\leq r \int_0^t \|u - v\|_\infty ds \leq r\|u - v\|_\infty.$$

Therefore,

$$\|Tu - Tv\|_\infty \leq r\|u - v\|_\infty \leq r\|Su - Sv\|_\infty.$$

Moreover, $T(A) \subseteq S(A)$. Indeed, if $u \in A$ and we define $v(t) := \ln(u(t) + 1)$, then $v \in A$ and

$$S(v(t)) = e^{\ln(u(t)+1)} - 1 = u(t).$$

This implies that $S(A) = A$ and so, $T(A) \subseteq A = S(A)$. Thereby, all the assumptions of Theorem 3.11 hold which implies that the pair $(T; S)$ has a coincidence point which is a solution of the problem (2). □

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