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An Algebraic Analysis of Categorical Syllogisms by Using Carroll's Diagrams

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Abstract. In this paper, we analyze the algebraic properties of categorical syllogisms by constructing a logical calculus system called Syllogistic Logic with Carroll Diagrams (SLCD). We prove that any categorical syllogism is valid if and only if it is provable in this system. For this purpose, we explain firstly the quantitative relation between two terms by means of bilateral diagrams and we clarify premises via bilateral diagrams. Afterwards, we input the data taken from bilateral diagrams, on the trilateral diagram. With the help of the elimination method, we obtain a conclusion that is transformed from trilateral diagram to bilateral diagram. Subsequently, we study a syllogistic conclusion mapping which gives us a conclusion obtained from premises. Finally, we allege valid forms of syllogisms using algebraic methods, and we examine their algebraic properties, and also by using syllogisms, we construct algebraic structures, such as lattices, Boolean algebras, Boolean rings, and many-valued algebras (MV-algebras).

1. Introduction

The first systematic approach to categorical syllogisms was searched by the Greek philosopher Aristotle within the scope of reasoning and inference. According to Aristotle, "syllogism is a discourse in which, certain things being stated, something other than what is stated follows of necessity from their being so. I mean by the last phrase that they produce the consequence, and by this, that no further term is required from without in order to make the consequence necessary" [1]. His syllogisms consist of three propositions, referred to as statements or sentences and are called major premise, minor premise, and conclusion, respectively. This syllogistic system has some patterns for logical structures which provide us a conclusion after offering premises. Syllogisms in this system are related to categorical propositions and forms of deductions on mentioned propositions.

At the end of 1800s, Lewis Carroll firstly used an original diagrammatic scheme to analyze the Aristotelian syllogisms in his book [4]. In that book, he produced some logical diagrams which could be used to solve logic problems containing 2-terms, 3-terms, and then more. And Łukasiewicz interested with this topic profoundly scrutinized the topic in terms of mathematical foundations in the middle of the 1900s [14]. All of these constitute the bases of modern mathematical works on categorical syllogisms. Recently, the topic is studied extensively and investigated under different treatments. For instance, Stanley Burris examined traditional syllogistic logic by using Boolean Algebras [3], Senturk and Oner constructed Heyting

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Algebras on categorical syllogisms [18], and Esko Turunen represented Peterson Intermediate Syllogisms by means of MV-Algebras [20]. In addition to these, we recently witness the use of syllogisms in different areas such as computer science [13, 16], engineering [11, 15], artificial intelligence [10, 21], etc. There are also some interesting results related to syllogisms used (see, [7, 17, 19]). However, one of the important problems of all these areas is to find a mathematical model for producing conclusions mechanically from given premises. More precisely, constructing a system based on mathematical foundations which deduces conclusions from given premises. If it is successful, we can solve via a mathematical model significant problems about systematically thinking.

In this work, we show that any syllogism is valid if and only if it is provable in the calculus system SLCD. In other words, we demonstrate the completeness of the calculus system SLCD. To this end, we beforehand construct a formal system SLCD, which gives us a formal approach to logical reasoning with diagrams, for representations of the fundamental Aristotelian categorical propositions and show that they are closed under the syllogistic criterion of inference which is the deletion of middle term. Therefore, it is implemented to let the formalism comprise synchronically bilateral and trilateral diagrammatical appearance and a naive algorithmic nature. And also, there is no need specific knowledge or exclusive ability to understand as well as to use it.

In other respects, we scrutinize algebraic properties of categorical syllogisms together with a representation of syllogistic arguments by using sets in SLCD. To do this, we explain quantitative relation between two terms by means of bilateral diagrams. Thereupon, we enter the data, taken from bilateral diagrams, on the trilateral diagram. With the help of the elimination method, we obtain a conclusion which is transformed from trilateral to bilateral diagram. A categorical syllogistic system consists of 256 syllogistic moods, 15 of which are unconditionally and 9 are conditionally; in total 24 of them are valid. Those syllogisms in the conditional group are also said to be *strengthened*, or valid under *existential import*, which is an explicit assumption of existence of some S, M or P. Then we add a rule to SLCD: *Some X is X when X exists* and consequently, we obtain the formal system SLCD†.

Finally, we show that syllogism is valid if and only if it is provable in SLCD and strengthened syllogism is valid if and only if it is provable in SLCD†. This means that SLCD is sound and complete. Additionally, we examine algebraic properties of SLCD, and also by using syllogisms, we construct algebraic structures on this system such as lattices, Boolean algebras, Boolean rings, and MV-algebras.

In this paper, the fundamental target of our research is to define some operations on the bilateral and trilateral diagrams, and use them for categorical syllogistic systems. Then, we can construct a mathematical bridge between algebraic systems and categorical syllogisms, and hence mathematical interpretations are provided for categorical syllogisms.

2. Preliminaries

In this section, we briefly introduce notations and terminology used throughout the manuscript.

A categorical syllogism can be thought as a logical argument: It consists of two logical propositions called premises and a logical conclusion obtained from the premises. Each of them has a quantified relationship between two objects, which are given in Table 1. An Aristotelian categorical proposition or a syllogistic proposition specifies a quantified relationship between two objects. The objects in a proposition are related to each other in four different ways, as indicated below:

Table 1: Aristotle's Syllogistic Propositions

Symbol	Statements	Generic Term
A	All X are Y	Universal Affirmative
E	No X are Y	Universal Negative
I	Some X are Y	Particular Affirmative
O	Some X are not Y	Particular Negative

For any syllogism, the categorical propositions consist of three terms, namely, subject term, predicate term, and middle term: the subject term is the subject of the conclusion and denoted by S, the predicate term characterizes the subject in the conclusion and denoted by P; and the middle term which occurs in the two premises and links the subject and predicate terms and is denoted by M. The predicate and subject terms are found in different premises but the middle term is found in each premise. The premise which consists of the predicate and the middle term is called the *major premise*; the premise which consists of the subject term and the middle term is called the *minor premise*. We assume that the relations between M and P, and between M and S hold. If we are not in contradiction with the case that certain relation between S and P does not hold, then the syllogism is valid, otherwise, the syllogism is invalid.

Syllogisms are grouped into four distinct subgroups depending on whether the middle term is the subject or the predicate in the major and the minor premises. These are traditionally called *Figures*:

Figure I	**Figure II**
A quantity Q_1 of M are P (Major Premise)	A quantity Q_1 of P are M (Major Premise)
A quantity Q_2 of S are M (Minor Premise)	A quantity Q_2 of S are M (Minor Premise)
A quantity Q_3 of S are P (Conclusion)	A quantity Q_3 of S are P (Conclusion)

Figure III	**Figure IV**
A quantity Q_1 of M are P (Major Premise)	A quantity Q_1 of P are M (Major Premise)
A quantity Q_2 of M are S (Minor Premise)	A quantity Q_2 of M are S (Minor Premise)
A quantity Q_3 of S are P (Conclusion)	A quantity Q_3 of S are P (Conclusion)

We note that Aristotle identified only the first three figures, while the forth was discovered in the middle ages. He searched each mood and figure in terms of validity. He then obtained some common properties of these syllogisms, which are called rules of deduction. These rules are as follows:

Step 1: Relating to premises irrespective of conclusion or figure:

(a) No inference can be made from two particular premises.

(b) No inference can be made from two negative premises.

Step 2: Relating to propositions irrespective of figure:

(a) If one premise is particular, the conclusion must be particular.

(b) If one premise is negative, the conclusion must be negative.

Step 3: Relating to distribution of terms:

(a) The middle term must be distributed at least once.

(b) A predicate distributed in the conclusion must be distributed in the major premise.

(c) A subject distributed in the conclusion must be distributed in the minor premise.

We use $\vdash$ symbol for valid syllogisms. For example, the syllogism

$$\boldsymbol{A}_{MP}, \boldsymbol{A}_{SM} \vdash \boldsymbol{A}_{SP}$$

consists of from left to right *major premise*, *minor premise* and *conclusion*, respectively. Its mood is ***AAA***, and it has the first figure.

Each proposition in a syllogism has one of four relations. In categorical syllogistic system, there are 64 different syllogistic forms for each figure. These are called *moods*. Thus the categorical syllogistic system is composed of 256 possible syllogisms. Only 24 of them are valid in this system and they are divided into two groups of 15 and of 9. The syllogisms in the first group are valid *unconditionally* as given in Table 2:

Table 2: Unconditionally Valid Forms

Figure I	Figure II	Figure III	Figure IV
$\boldsymbol{A}_{MP}, \boldsymbol{A}_{SM} \vdash \boldsymbol{A}_{SP}$	$\boldsymbol{E}_{PM}, \boldsymbol{A}_{SM} \vdash \boldsymbol{E}_{SP}$	$\boldsymbol{I}_{MP}, \boldsymbol{A}_{MS} \vdash \boldsymbol{I}_{SP}$	$\boldsymbol{A}_{PM}, \boldsymbol{E}_{MS} \vdash \boldsymbol{E}_{SP}$
$\boldsymbol{E}_{MP}, \boldsymbol{A}_{SM} \vdash \boldsymbol{E}_{SP}$	$\boldsymbol{A}_{PM}, \boldsymbol{E}_{SM} \vdash \boldsymbol{E}_{SP}$	$\boldsymbol{A}_{MP}, \boldsymbol{I}_{MS} \vdash \boldsymbol{I}_{SP}$	$\boldsymbol{I}_{PM}, \boldsymbol{A}_{MS} \vdash \boldsymbol{I}_{SP}$
$\boldsymbol{A}_{MP}, \boldsymbol{I}_{SM} \vdash \boldsymbol{I}_{SP}$	$\boldsymbol{E}_{PM}, \boldsymbol{I}_{SM} \vdash \boldsymbol{O}_{SP}$	$\boldsymbol{O}_{MP}, \boldsymbol{A}_{MS} \vdash \boldsymbol{O}_{SP}$	$\boldsymbol{E}_{PM}, \boldsymbol{I}_{MS} \vdash \boldsymbol{O}_{SP}$
$\boldsymbol{E}_{MP}, \boldsymbol{I}_{SM} \vdash \boldsymbol{O}_{SP}$	$\boldsymbol{A}_{PM}, \boldsymbol{O}_{SM} \vdash \boldsymbol{O}_{SP}$	$\boldsymbol{E}_{MP}, \boldsymbol{I}_{MS} \vdash \boldsymbol{O}_{SP}$	

The syllogisms in the second group are valid *conditionally* or valid *existential import* which is an explicit supposition of being of some terms and are shown in Table 3:

Table 3: Conditionally Valid Forms

Figure I	Figure II	Figure III	Figure IV	Necessary Condition
$\boldsymbol{A}_{MP}, \boldsymbol{A}_{SM} \vdash \boldsymbol{I}_{SP}$	$\boldsymbol{A}_{PM}, \boldsymbol{E}_{SM} \vdash \boldsymbol{O}_{SP}$		$\boldsymbol{A}_{PM}, \boldsymbol{E}_{MS} \vdash \boldsymbol{O}_{SP}$	S exists
$\boldsymbol{E}_{MP}, \boldsymbol{A}_{SM} \vdash \boldsymbol{O}_{SP}$	$\boldsymbol{E}_{PM}, \boldsymbol{A}_{SM} \vdash \boldsymbol{O}_{SP}$			S exists
		$\boldsymbol{A}_{MP}, \boldsymbol{A}_{MS} \vdash \boldsymbol{I}_{SP}$	$\boldsymbol{E}_{PM}, \boldsymbol{A}_{MS} \vdash \boldsymbol{O}_{SP}$	M exists
		$\boldsymbol{E}_{MP}, \boldsymbol{A}_{MS} \vdash \boldsymbol{O}_{SP}$		M exists
			$\boldsymbol{A}_{PM}, \boldsymbol{A}_{MS} \vdash \boldsymbol{I}_{SP}$	P exists

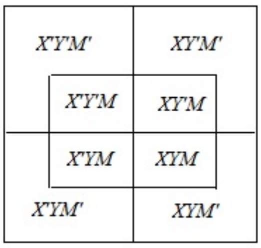
3. Carroll's Diagrams and The Elimination Method

Carroll's diagrams, thought up in 1884, are Venn-type diagrams where the universes are represented by a square. Nevertheless, it is not clear whether Carroll studied his diagrams independently or as a modification of John Venn's. Still, Carroll's scheme looks like a sophisticated method summing up several developments that have been introduced by researchers studying in this area.

Let X and Y be two terms and let X' and Y' be the complements of X and Y, respectively. For two-terms, Carroll divides the square into four cells, and he gets the so-called bilateral diagram as shown in below:

	X'	X
Y'	$X'Y'$	XY'
Y	$X'Y$	XY

Each of these four cells can have three possibilities, when we explain the relations between two terms. They can be 0 or 1 or *blank*. In this method, 0 means that there is no element intersection cell of two elements, 1 means that it is not empty, and *blank* cell means that we don't have any information about the content of the cell, therefore it could be 0 or 1. As above method, let X, Y, and M be three terms and X', Y', and M' be their respective complements. To examen all relations between three terms, Lewis Carroll added one more square in the middle of bilateral diagram which is called the trilateral diagram as the following:



Each cell in a trilateral diagram is marked with a 0, if there is no element and is marked with a **I** if it is not empty and another using of **I**, it could be on the line where the two cell is intersection. This means that at least one of these cells is not empty. So, **I** is different from 1. In addition to these,if any cell is **blank**, it has two possibilities, namely, 0 or **I**.

In order to get the conclusion of a syllogism, the data of two premises are written on a trilateral diagram. This presentation is more effective than Venn Diagram method. So, one can extract the conclusion truer and quicker from trilateral diagram. Under favour of this method, we transfer the data shown by the trilateral diagram into a bilateral diagram, involving only two terms that should occur in the conclusion and consequently eliminating the middle term.

This method can be used in accordance with the rules below [4]:

First Rule: 0 and **I** are fixed up on trilateral diagrams.

Second Rule: If the quarter of trilateral diagram contains a "**I**" in either cell, then it is certainly occupied, and one may mark the corresponding quarter of the bilateral diagram with a "1" to indicate that it is occupied.

Third Rule: If the quarter of trilateral diagram contains two "0"s, one in each cell, then it is certainly empty, and one may mark the corresponding quarter of the bilateral diagram with a "0" to indicate that it is empty.

The use of these rules gives us the obtaining conclusion. Besides, the importance of transfer method, unlike Venn's diagrams, is showing how to extract the conclusion from premises of a syllogism.

4. The Calculus System *SLCD* and Its Completeness

In this section, we correspond a set to each possible form of any syllogistic bilateral diagrams and also define universes of major and minor premises and conclusions in the categorical syllogisms. Moreover, we give a definition of a map which obtains a conclusion from two possible forms of premises. Then, we generalize it for conclusion of any two premises and also valid forms in syllogisms.

Our aim is to construct a complete bridge between *Sets* and *Aristotelian Categorical Logic*:

Table 4: *The Paradigm for the Representation of Syllogistic Arguments by using Sets*

	LOGIC	DIAGRAMS	SETS
PREMISES	Propositions	$\xrightarrow{Translate}$	Sets
			$\downarrow$
CONCLUSION	Propositions	$\xleftarrow{Translate}$	Sets

Let X and Y be two terms and their complements are denoted by X' and Y', respectively. Assume that p_i shows a possible form of any bilateral diagram such that $1 \leq j \leq k$, where k is the number of possible forms of bilateral diagram as follows:

Table 5: Bilateral diagram for a quantity relation between X and Y

p_j	X'	X
Y'	n_1	n_2
Y	n_3	n_4

where $n_1, n_2, n_3, n_4 \in \{0, 1\}$. Throughout this paper, the symbols $R_{(A)}$, $R_{(E)}$, $R_{(I)}$ and $R_{(O)}$ represent "*All*", "*No*", "*Some*" and "*Some − not*" statements, respectively.

Example 4.1. We examine *"All S are P"*; it means that there is no element in the intersection of S and P' cell. This is shown in the following bilateral diagram:

Table 6: *Bilateral diagram for "All S are P"*

$R_{(A)} =$

	P'	P
S'		
S	0	

From the Table 6, we obtain all bilateral diagrams having 0 in the intersection of S and P' cell:

Table 7: *Possible forms of "All S are P"*

p_1	P'	P
S'	0	0
S	0	0

p_2	P'	P
S'	0	0
S	0	1

p_3	P'	P
S'	0	1
S	0	0

p_4	P'	P
S'	1	0
S	0	0

p_5	P'	P
S'	0	1
S	0	1

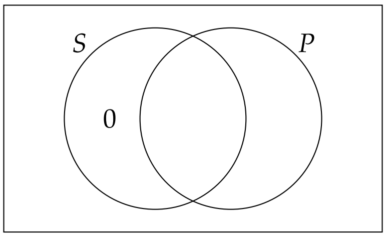
p_6	P'	P
S'	1	0
S	0	1

p_7	P'	P
s'	1	1
S	0	0

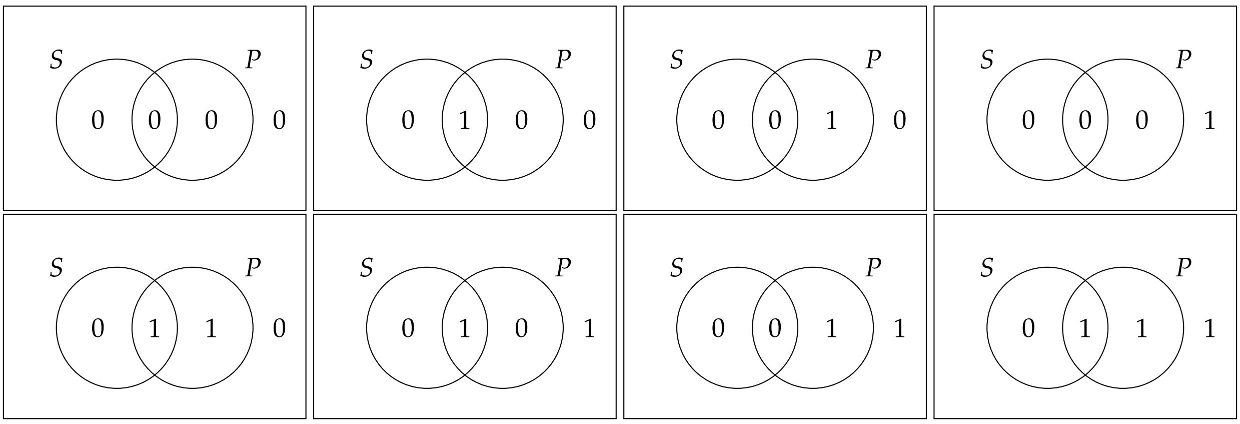
p_8	P'	P
S'	1	1
S	0	1

Table 7 shows all possible forms of *"All S are P"*.

Additionally, we can explain all possible forms of *"All S are P"* on a venn diagram. It means that there is no element in the intersection S and P'.



Then we can obtain all possible forms as follows:



Now in order to correspond bilateral diagrams to sets, let us form a set consisting of numbers which correspond to possible forms that each bilateral diagram possesses. To do this, we firstly give a mapping in which each possible bilateral diagram corresponds to exactly one value.

Note that if r_j^{val} denotes the value corresponding to a possible bilateral diagram p_j and n_i is the value that the i-th cell possesses, then the value of this possible bilateral diagram is calculated by using the formula

$$r_j^{val} = \sum_{i=1}^{4} 2^{(4-i)} n_i, \quad 1 \leq j \leq k,$$

where k is the number of all possible forms [12].

Example 4.2. Let a possible form p_j be given as following bilateral diagram:

p_j	P'	P
S'	$n_1 = 0$	$n_2 = 1$
S	$n_3 = 1$	$n_4 = 0$

Then, the value r_j^{val} which corresponds to this bilateral diagram is calculated as below

$$r_j^{val} = \sum_{i=1}^{4} 2^{(4-i)} n_i = 2^3 n_1 + 2^2 n_2 + 2^1 n_3 + 2^0 n_4 = 2^3 0 + 2^2 1 + 2^1 1 + 2^0 1 = 5.$$

As a result, we obtain $r_j^{val} = 5$.

Definition 4.3. Let R^{set} be the set of the values which correspond to all possible forms of any bilateral diagram; that is $R^{set} = \{r_j^{val} : 1 \leq j \leq k, k$ is the number of all possible forms$\}$. The set of all these R^{set}'s is denoted by $\mathcal{R}^{Set}$.

Corollary 4.4. *We obtain the set representations of all categorical propositions as follows:*

- *All X are Y: It means that the intersection of X and Y′ is empty set.*

$R_{(A)} =$

	X'	X
Y'		0
Y		

From the above table, we have all possible forms in Example 4.1. Then the set representation of "All X are Y" is $R_{(A)}^{set} = \{0, 1, 2, 3, 8, 9, 10, 11\}$.

- *No X are Y:There is no element in the intersection cell of X and Y.*

$R_{(E)} =$

	X'	X
Y'		
Y		0

By the definition of possible forms, we obtain $R_{(E)}^{set} = \{0, 2, 4, 6, 8, 10, 12, 14\}$.

- *Some X are Y: There is at least one element which belongs X and Y.*

$R_{(I)} =$

	X'	X
Y'		
Y		1

By using the possible bilateral forms of $R_{(I)}$ *diagram, we get* $R_{(I)}^{set} = \{1, 3, 5, 7, 9, 11, 13, 15\}$.

- *Some X are not Y: If some elements of X are not Y, then they have to be in Y′. So, the intersection cell of X and Y′ is not empty.*

$R_{(O)} =$

	X'	X
Y'		1
Y		

We obtain the set from the $R^{(O)}$ *bilateral diagram is* $R^{set}_{(O)} = \{4, 5, 6, 7, 12, 13, 14, 15\}$.

Example 4.5. If *"All S are M"* and *"All M are P"*, then *"All S are P"*. This syllogism, called *Barbara*, is valid. We show this truth by using elimination method from trilateral diagram to bilateral diagram.

- *All S are M*: It means that the intersection of cell S and M' is 0 without any condition. It is shown as follows:

$R_{(A)} =$

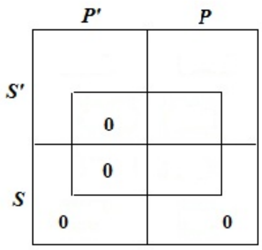
	S'	S
M'		0
M		

- *All M are P*: It means that the intersection cell of M and P' is 0 without any condition. It is also shown as follows:

$R_{(A)} =$

	P'	P
M'		
M	0	

Now, we input the data on the trilateral diagram:



By the elimination method, we obtain the relation between S and P on the bilateral diagram:

$R_{(A)} =$

	P'	P
S'		
S	0	

This means *"All S are P"*. So, we can say that *this syllogism is valid*.

Let r_i^{val} and r_j^{val} be the numbers corresponding to possible forms of bilateral diagrams which have a common term. Then we can get the relation between two other terms by using this method.

Example 4.6. Let p_i be one possible orm of the bilateral diagram having a relation between P and M, and p_j be one possible form of the bilateral diagram having a relation between S and M. Then we can obtain a relation between S and P. We take possible forms given as below:

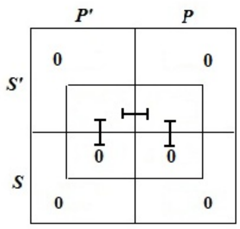
$p_i =$

	P'	P
M'	0	0
M	1	1

and $p_j =$

	S'	S
M'	0	0
M	1	0

We input the data on the trilateral diagram as follows: By using the elimination method, we can obtain a



relation between S and P as below:

$p_l =$

	P'	P
S'	1	1
S	0	0

Thus, $r_i^{val} = 2$ corresponds to possible form p_i, and $r_j^{val} = 3$ corresponds to possible form p_j, and we obtain that $r_l^{val} = 12$ corresponds to p_l that is a possible conclusion.

The first essential point is to find whether any relation between numbers correspond to possible forms or not. When we examine the conclusions, we obtained three situation about them. One of them contains only one number, the other one contains more than one, and the other is the empty set.

Example 4.7. Let p_i and p_j be possible forms of major and minor premises, respectively. Let $r_i^{val} = 7$ and $r_j^{val} = 11$ correspond to the numbers which have bilateral forms as below:

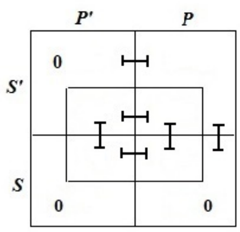
$p_i =$

	P'	P
M'	0	1
M	1	1

and $p_j =$

	S'	S
M'	1	0
M	1	1

The conclusion of these two diagrams has more than one possibility. To see them, the elimination method can be used as follows:



From this bilateral diagram, we obtain the set $\{6, 7, 13, 14, 15\}$.

Contrary to above example, $r_i^{val} = 7$ and $r_j^{val} = 2$ correspond to these possible forms. We input the data on trilateral diagram:

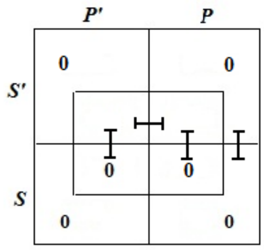
$p_i =$

	P'	P
M'	0	1
M	1	1

and $p_j =$

	S'	S
M'	0	0
M	1	0

We obtain a contradiction as below:



It means that the conclusion is the empty set.

After these examples, we try to generalize them by formula. So, we define an operation and state a theorem as follows:

Definition 4.8. The syllogistic possible conclusion mapping, denoted $*$, is a mapping which gives us the deduction set of possible forms of major and minor premises sets.

Theorem 4.9. *Let r_i^{val} and r_j^{val} correspond to the numbers of possible forms of major and minor premises, respectively. Then, $r_i^{val} * r_j^{val}$ equals the value given by the intersection of row and column numbers corresponding to r_i^{val} and r_j^{val} in Table 8.*

Proof. Assume that r_i^{val} and r_j^{val} are the numbers correspond to possible forms of major and minor premises, respectively. Then p_i and p_j are bilateral diagrams representing r_i^{val} and r_j^{val} as follows:

$p_i =$

	P'	P
M'	n_1	n_2
M	n_3	n_4

and $p_j =$

	S'	S
M'	m_1	m_2
M	m_3	m_4

Since p_i and p_j are possible forms, then $n_i, m_j \in \{0, 1\}$ for $i, j \in \{1, 2, 3, 4\}$. Firstly, we examine contradictory cases. If at least one of following conditions is satisfied, the contradiction, which is shown by empty cell in Table 8 [12], is obtained:

- $(n_4 = 1)$ and $(m_4 = m_3 = 0)$
- $(n_3 = 1)$ and $(m_4 = m_3 = 0)$
- $(n_2 = 1)$ and $(m_1 = m_2 = 0)$
- $(n_1 = 1)$ and $(m_1 = m_2 = 0)$

- $(m_4 = 1)$ and $(n_4 = n_3 = 0)$
- $(m_3 = 1)$ and $(n_4 = n_3 = 0)$
- $(m_2 = 1)$ and $(n_1 = n_2 = 0)$
- $(m_1 = 1)$ and $(n_1 = n_2 = 0)$

Now, we can handle possible conclusions obtained from possible forms of premises. We have two groups, one of them has only one conclusion, the other has more than one. Let us explain the differences between these groups. After the elimination method applies, if each cell of bilateral diagram, which has a relation between S and P, includes 0 or 1, then there is only one possible conclusion, otherwise it includes more than one possible conclusions. We define a function v which gives us the values of cells on bilateral diagram. Firstly, we examine the cell values equal 0:

- $(n_4 = 0 \text{ or } m_4 = 0)$ and $(n_2 = 0 \text{ or } m_2 = 0) \Rightarrow v(P'S') = 0$
- $(n_3 = 0 \text{ or } m_4 = 0)$ and $(n_1 = 0 \text{ or } m_2 = 0) \Rightarrow v(PS') = 0$
- $(n_2 = 0 \text{ or } m_1 = 0)$ and $(n_4 = 0 \text{ or } m_3 = 0) \Rightarrow v(P'S) = 0$
- $(n_1 = 0 \text{ or } m_1 = 0)$ and $(n_3 = 0 \text{ or } m_3 = 0) \Rightarrow v(PS) = 0$

When the following conditions are provided, the cell values equal 1:

- $\left.\begin{array}{r}(n_4 = m_4 = 1) \text{ and } (n_3 = 0 \text{ or } m_4 = 0)\\ \text{or}\\ (n_2 = m_2 = 1) \text{ and } (n_1 = 0 \text{ or } m_1 = 0)\end{array}\right\} \Rightarrow v(P'S') = 1$
- $\left.\begin{array}{r}(n_3 = m_4 = 1) \text{ and } (n_4 = 0 \text{ or } m_3 = 0)\\ \text{or}\\ (n_1 = m_2 = 1) \text{ and } (n_2 = 0 \text{ or } m_1 = 0)\end{array}\right\} \Rightarrow v(PS') = 1$
- $\left.\begin{array}{r}(n_4 = m_3 = 1) \text{ and } (n_3 = 0 \text{ or } m_4 = 0)\\ \text{or}\\ (n_2 = m_1 = 1) \text{ and } (m_2 = 0 \text{ or } n_1 = 0)\end{array}\right\} \Rightarrow v(P'S) = 1$
- $\left.\begin{array}{r}(n_3 = m_3 = 1) \text{ and } (n_4 = 0 \text{ or } m_4 = 0)\\ \text{or}\\ (n_1 = m_1 = 1) \text{ and } (n_2 = 0 \text{ or } m_2 = 0)\end{array}\right\} \Rightarrow v(PS) = 1$

We correlate the operation with the below table given by Kulinkovich [12]. In this table, the first row includes the values of all possible bilateral diagrams of major premises and the first column consists of the values of all possible bilateral diagrams of minor premises. The intersection of all rows and columns in the table gives the deduction of all the values of possible bilateral diagrams. Thus, we obtain valid syllogisms by means of set theoretical representation of Carroll's diagrams:

Table 8: Operation Table

$*$	0	1	2	3	4	8	12	5	10	6	9	7	11	13	14	15
0	0															
1		1	4	5												
2		2	8	10												
3		3	12	H												
4					1	4	5									
8					2	8	10									
12					3	12	H									
5								1	4	5	5	5	5	5	5	5
10								2	8	10	10	10	10	10	10	10
6								3	12	9	6	11	14	7	13	15
9								3	12	6	9	7	13	11	14	15
7								3	12	13	7	H_4	H'_3	7	13	H'_1
11								3	12	14	11	H_3	H'_4	11	14	H'_2
13								3	12	7	13	7	13	H_4	H'_3	H'_1
14								3	12	11	14	11	14	H_3	H'_4	H'_2
15								3	12	15	15	H_1	H_2	H_1	H_2	H

In the Table 8, considering possible conclusion mapping, as Example 4.7, some possible forms of premises have more than one possible conclusions, given as below:

$$H = \{6,7,9,11,13,14,15\},\ H_1 = \{7,11,15\},\ H'_1 = \{6,7,9,11,13,15\},$$
$$H_2 = \{13,14,15\},\ H'_2 = \{11,14,15\},\ H_3 = \{6,7,11,14,15\},$$
$$H'_3 = \{6,7,13,14,15\},\ H_4 = \{7,9,11,13,15\},\ H'_4 = \{9,11,13,14,15\}$$

Therefore, we scrutinize all possible cases between two terms and their conclusions. □

Indeed, the possible conclusion is an image of possible premises under a mapping.

Definition 4.10. Universes of values sets of major premises, minor premises, and conclusions are denoted by $\mathcal{R}^{set}_{Maj}$, $\mathcal{R}^{set}_{Min}$ and $\mathcal{R}^{set}_{Con}$, respectively.

Let $R^{set}_{(k)}$ be an element of $\mathcal{R}^{set}_{Maj}$ and $R^{set}_{(l)}$ be an element of $\mathcal{R}^{set}_{Min}$. The main problem is what the conclusion of these premises is. In syllogistic, we have some patterns which are mentioned in Table 2 and Table 3 above. Now, we explain them by using bilateral diagrams with an algebraic approach.

Definition 4.11. The syllogistic mapping, denoted by $\circledast$, is a mapping which gives us the conclusion of the major and the minor premises as below:

	P'	P			S'	S			P'	P
M'			$\circledast$	M'			$=$	S'		
M				M				S		

Theorem 4.12. *Let $R^{set}_{(k)} = \{r^{val}_{k_1}, \ldots, r^{set}_{k_n}\}$ and $R^{set}_{(l)} = \{r^{val}_{l_1}, \ldots, r^{val}_{l_t}\}$ two sets corresponding to the Major and the Minor premises. Then $\circledast : \mathcal{R}^{set}_{Maj} \times \mathcal{R}^{set}_{Min} \to \mathcal{R}^{set}_{Con}$*

$$R^{set}_{(k)} \circledast R^{set}_{(l)} := \bigcup_{j=1}^{n} \bigcup_{i=1}^{t} r^{val}_{k_j} * r^{val}_{l_i}$$

is the conclusion of the premises $R^{set}_{(k)}$ and $R^{set}_{(l)}$.

Proof. The proof includes 64 operations for each figure, that is to say, it has 256 operations. To avoid overmuch iterant operations, we give only a proof of on of the valid syllogisms in Figure I.

We assume that the major premise is "All M are P" and the minor premise is "Some S are M". In that case, we have the following bilateral diagrams:

$$R_{(A_{MP})} = \begin{array}{|c|c|c|} \hline & P' & P \\ \hline M' & & \\ \hline M & 0 & \\ \hline \end{array} \quad \text{and} \quad R_{(I_{SM})} = \begin{array}{|c|c|c|} \hline & S' & S \\ \hline M' & & \\ \hline M & & 1 \\ \hline \end{array}$$

Then we have the following values set of major and minor premises:

$$R_{(A_{MP})} = \{0, 1, 4, 5, 8, 9, 12, 13\} \textit{ and } R_{(I_{SM})} = \{1, 3, 5, 7, 9, 11, 13, 15\}$$

By the definition of $\circledast$ operation and Table 8, we obtain

$$\begin{aligned} R^{set}_{(A_{MP})} \circledast R^{set}_{(I_{SM})} &= \{0, 1, 4, 5, 8, 9, 12, 13\} \circledast \{1, 3, 5, 7, 9, 11, 13, 15\} \\ &= \{1, 3, 5, 7, 9, 11, 13, 15\}. \end{aligned}$$

This values set corresponds to the following bilateral diagram:

$$R_{(I_{SP})} = \begin{array}{|c|c|c|} \hline & P' & P \\ \hline S' & & \\ \hline S & & 1 \\ \hline \end{array}$$

Therefore, the deduction of major and minor premises is "Some S are P". As a result, we get $\boldsymbol{A}_{MP}, \boldsymbol{I}_{SM} \vdash \boldsymbol{I}_{SP}$ in Figure I.

Performing above procedure to each mood in any figure, the reader can see whether the syllogism is valid or not. Furthermore, each syllogism can be calculated by using the algorithm in [8]. □

As a result of Theorem 4.12, we obtain the following theorem.

Theorem 4.13. *A syllogism is valid if and only if it is provable in SLCD.*

Remark 4.14. For conditional valid forms, we need an addition rule which is *"Some X are X"*. We can use above Theorem by taking into consideration this rule.

Notation 4.15. Let SLCD be noted calculus system. If the rule *"Some X are X when X exists"* (i.e., $\vdash \boldsymbol{I}_{XX}$) is added to SLCD, then the calculus system SLCD is denoted by $SLCD^{\dagger}$.

Definition 4.16. Let $R_{(k)}$ be the bilateral diagram presentation of the premise. The *transposition* of a premise is the symmetric positions with respect to the main diagonal. It is shown by $Trans(R_{(k)})$.

$$\begin{aligned} Trans : \mathcal{R}^{set} &\rightarrow \mathcal{R}^{set}, \\ R^{set}_{(k)} &\rightarrow Trans(R^{set}_{(k)}) = \{r^{val}_{k^T_1}, \ldots, r^{set}_{k^T_n}\}. \end{aligned}$$

Theorem 4.17. *Let* $R^{set}_{(k)} = \{r^{val}_{k_1}, \ldots, r^{set}_{k_n}\}$ *and* $R^{set}_{(l)} = \{r^{val}_{l_1}, \ldots, r^{val}_{l_t}\}$ *be two sets to correspond to the Major and the Minor premises values sets and* $R^{set}_{(s)} = \{r^{val}_{s_1}, \ldots, r^{set}_{s_m}\}$ *be set to correspond to the constant set values which means "Some S are S", "Some M are M" and "Some P are P". Then* $\circledast^{\dagger} : \mathcal{R}^{set}_{Maj} \times \mathcal{R}^{set}_{Min} \rightarrow \mathcal{R}^{set}_{Con}$

$$R^{set}_{(k)} \circledast^{\dagger} R^{set}_{(l)} := \begin{cases} \bigcup_{j=1}^{n} \bigcup_{i=1}^{t} \bigcup_{h=1}^{m} (r^{val}_{k_j} * (r^{var}_{s_h} * r^{var}_{l^T_i})), & \textit{If S exists,} \\ \bigcup_{j=1}^{n} \bigcup_{i=1}^{t} \bigcup_{h=1}^{m} (r^{val}_{k_j} * (r^{var}_{l_i} * r^{var}_{s_h})), & \textit{If M exists,} \\ \bigcup_{j=1}^{n} \bigcup_{i=1}^{t} \bigcup_{h=1}^{m} ((r^{var}_{s_h} * r^{val}_{k^T_j}) * r^{var}_{l_i}), & \textit{If P exists.} \end{cases}$$

is the conclusion of the premises $R^{set}_{(k)}$ *and* $R^{set}_{(l)}$ *under the conditions S exists, M exists or P exists.*

Proof. Following a procedure which is similar to the proof of Theorem 4.12, we can prove Theorem 4.17. □

Theorem 4.18. *A strengthened syllogism is valid if and only if it is provable in SLCD†.*

5. Algebraic Properties of Categorical Syllogisms By Means of Set Theoretical Representation of Bilateral Diagram

In this section, we examine algebraic properties of categorical syllogism by means of set theoretical representation of bilateral diagram. At first, we define ∧ (meet) and ∨ (join) operators on the set of numbers corresponding to possible form of bilateral diagrams.

Definition 5.1. Let $R^{(k)}$ and $R^{(l)}$ be elements of $\mathcal{R}$. Then the definitions of binary join and meet operations are as follows:

$$R_{(k)} \vee R_{(l)} := R^{set}_{(k)} \cup R^{set}_{(l)}$$

$$R_{(k)} \wedge R_{(l)} := R^{set}_{(k)} \cap R^{set}_{(l)}$$

Theorem 5.2. $< \mathcal{R}, \vee, \wedge >$ *is a distributive lattice.*

Corollary 5.3. $< \mathcal{R}^{set}_{Maj}, \cup, \cap >$, $< \mathcal{R}^{set}_{Min}, \cup, \cap >$ *and* $< \mathcal{R}^{set}_{Con}, \cup, \cap >$ *are distributive lattices.*

Now, we define an order relation on $\mathcal{R}^{set}$ as follows:

$$R^{set}_{(k)} \leq R^{set}_{(l)} :\Leftrightarrow R^{set}_{(k)} \subseteq R^{set}_{(k)}.$$

Theorem 5.4. $\mathcal{R}^{set}$ *is partially ordered by the binary relation* $\leq$.

Let $(\mathcal{R}^{set}, \leq)$ be a poset. The greatest element of $\mathcal{R}^{set}$ is $\{0, 1, \ldots, 15\}$, denoted by $\mathbf{1}$ and the least element is $\emptyset$, denoted by $\mathbf{0}$. We notice again that $\mathbf{0}$ and 0 are different from each other. Let R_k be any element of $\mathcal{R}$. Then we have

$$R_{(k)} \wedge \mathbf{0} = R^{set}_{(k)} \cap \emptyset = \emptyset = \mathbf{0}$$

and

$$R_{(k)} \vee \mathbf{1} = R^{set}_{(k)} \cup \{0, 1, \ldots, 15\} = \{0, 1, \ldots, 15\} = \mathbf{1}.$$

Definition 5.5. The complement of R, denoted by R^c, $R^c = \{0, 1, 2, \ldots, 15\} \setminus R$.

Theorem 5.6. $< \mathcal{R}, \vee, \wedge, \mathbf{0}, \mathbf{1} >$ *is a bounded lattice.*

Corollary 5.7. $< \mathcal{R}^{set}_{Maj}, \cup, \cap, \mathbf{0}, \mathbf{1} >$, $< \mathcal{R}^{set}_{Min}, \cup, \cap, \mathbf{0}, \mathbf{1} >$ *and* $< \mathcal{R}^{set}_{Con}, \cup, \cap, \mathbf{0}, \mathbf{1} >$ *are bounded lattices.*

Theorem 5.8. $< \mathcal{R}, \vee, \wedge, ^c, \mathbf{0}, \mathbf{1} >$ *is an ortholattice.*

Corollary 5.9. $< \mathcal{R}^{set}_{Maj}, \cup, \cap, ^c, \mathbf{0}, \mathbf{1} >$, $< \mathcal{R}^{set}_{Min}, \cup, \cap, ^c, \mathbf{0}, \mathbf{1} >$ *and* $< \mathcal{R}^{set}_{Con}, \cup, \cap, ^c, \mathbf{0}, \mathbf{1} >$ *are ortholattices.*

Theorem 5.10. $< \mathcal{R}, \vee, \wedge, ^c, \mathbf{0}, \mathbf{1} >$ *is an orthomodular lattice.*

Corollary 5.11. $< \mathcal{R}^{set}_{Maj}, \cup, \cap, ^c, \mathbf{0}, \mathbf{1} >$, $< \mathcal{R}^{set}_{Min}, \cup, \cap, ^c, \mathbf{0}, \mathbf{1} >$ *and* $< \mathcal{R}^{set}_{Con}, \cup, \cap, ^c, \mathbf{0}, \mathbf{1} >$ *are orthomodular lattices.*

George Boole introduced his algebraic approach to Syllogistic Logic in *The Laws of Thought* in 1854 [2]. He gave a suitable uniform, however it was quite complicated to use for an algebraic treatment in syllogisms. But, with the help of this system, we can obtain algebraic structures and conclusions on syllogisms. Therefore, we can determine valid syllogisms more easily by using diagrams.

Theorem 5.12. $< \mathcal{R}, \vee, \wedge, ^c, \mathbf{0}, \mathbf{1} >$ *is a Boolean Algebra.*

Proof. We show that $< \mathcal{R}, \vee, \wedge >$ is a distributive lattice in Theorem 5.2. For all $R \in \mathcal{R}$, $R \vee \mathbf{0} = R$, $R \wedge \mathbf{0} = \mathbf{0}$, and $R \vee \mathbf{1} = \mathbf{1}$, $R \wedge \mathbf{1} = R$. Therefore, $< \mathcal{R}, \vee, \wedge, ^c, \mathbf{0}, \mathbf{1} >$ is a *Boolean Algebra*. □

Corollary 5.13. *$< \mathcal{R}^{set}_{Maj}, \cup, \cap, ^c, \mathbf{0}, \mathbf{1} >$, $< \mathcal{R}^{set}_{Min}, \cup, \cap, ^c, \mathbf{0}, \mathbf{1} >$ and $< \mathcal{R}^{set}_{Con}, \cup, \cap, ^c, \mathbf{0}, \mathbf{1} >$ are Boolean Algebra.*

Definition 5.14. Let $< \mathcal{R}, \vee, \wedge, ^c, \mathbf{0}, \mathbf{1} >$ be a Boolean Algebra. Then, we define the operations as below:

$$\begin{aligned} R_{(k)} +_B R_{(l)} &:= (R_{(k)} \wedge (R_{(l)})^c) \vee ((R_{(k)})^c \wedge R_{(l)}), \\ R_{(k)} \cdot_B R_{(l)} &:= R_{(k)} \wedge R_{(l)}, \\ -_B R_{(k)} &:= R_{(k)}. \end{aligned}$$

Theorem 5.15. *Let $< \mathcal{R}, \vee, \wedge, ^c, \mathbf{0}, \mathbf{1} >$ be a Boolean Algebra. Then, $< \mathcal{R}, +_B, \cdot_B, -_B, \mathbf{0}, \mathbf{1} >$ is a Boolean Ring.*

Proof. Let $R_{(k)}, R_{(l)}$ and $R_{(m)}$ be elements of $\mathcal{R}$. Then

(i) $R_{(k)} +_B \mathbf{0} = (R_{(k)} \wedge \mathbf{0}^c) \vee ((R_{(k)})^c \wedge \mathbf{0}) = R_{(k)} \vee \mathbf{0} = R_{(k)}$.

(ii) $R_{(k)} +_B R_{(l)} = (R_{(k)} \wedge (R_{(l)})^c) \vee ((R_{(k)})^c \wedge R_{(l)}) = (R_{(l)} \wedge (R_{(k)})^c) \vee ((R_{(l)})^c \wedge R_{(k)}) = R_{(l)} +_B R_{(k)}$.

(iii) $R_{(k)} +_B R_{(k)} = (R_{(k)} \wedge (R_{(k)})^c) \vee ((R_{(k)})^c \wedge R_{(k)}) = \mathbf{0} \vee \mathbf{0} = \mathbf{0}$.

(iv)

$$\begin{aligned} (R_{(k)} +_B R_{(l)}) +_B R_{(m)} &= ((R_{(k)} +_B R_{(l)}) \wedge (R_{(k)})^c) \vee ((R_{(k)} +_B R_{(l)})^c \wedge (R_{(k)})) \\ &= (((R_{(k)} \wedge (R_{(l)})^c) \vee ((R_{(k)})^c \wedge R_{(l)})) \wedge (R_{(k)})^c) \\ &\quad \vee (((R_{(k)} \wedge (R_{(l)})^c)^c \vee ((R_{(k)})^c \wedge R_{(l)})^c) \wedge R_{(k)}) \\ &= (R_{(k)} \wedge R_{(l)} \wedge R_{(m)}) \vee ((R_{(k)})^c \wedge R_{(l)} \wedge R_{(m)}) \vee (R_{(k)} \wedge (R_{(l)})^c \wedge R_{(m)}) \\ &\quad \vee (R_{(k)} \wedge R_{(l)} \wedge (R_{(m)})^c). \end{aligned}$$

In the last equation, we can change $R_{(k)}, R_{(l)}$ and $R_{(m)}$ orders, and by using *(ii)*, we obtain

$$(R_{(k)} +_B R_{(l)}) +_B R_{(m)} = R_{(k)} +_B (R_{(l)} +_B R_{(m)})$$

(v) $R_{(k)} \cdot_B \mathbf{1} = R_{(k)} \wedge \mathbf{1} = R_{(k)}$

(vi) $(R_{(k)} \cdot_B R_{(l)}) \cdot_B R_{(m)} = (R_{(k)} \wedge R_{(l)}) \wedge R_{(m)} = R_{(k)} \wedge (R_{(l)} \wedge R_{(m)}) = R_{(k)} \cdot_B (R_{(l)} \cdot_B R_{(m)})$.

(vii)

$$\begin{aligned} R_{(k)} \cdot_B (R_{(l)} +_B R_{(m)}) &= (R_{(k)} \wedge R_{(l)} \wedge (R_{(m)})^c) \vee (R_{(k)} \wedge (R_{(l)})^c \wedge R_{(m)}) \\ &= ((R_{(k)} \wedge R_{(l)}) \wedge ((R_{(k)})^c \vee (R_{(m)})^c)) \vee (((R_{(k)})^c \vee (R_{(l)})^c) \wedge (R_{(k)} \wedge R_{(m)})) \\ &= ((R_{(k)} \wedge R_{(l)}) \wedge (R_{(k)} \wedge R_{(m)})^c) \vee ((R_{(k)} \wedge R_{(l)})^c \wedge (R_{(k)} \wedge R_{(m)}) \\ &= (R_{(k)} \cdot_B R_{(l)}) +_B (R_{(k)} \cdot_B R_{(m)}). \end{aligned}$$

(viii) $R_{(k)} \cdot_B R_{(k)} = R_{(k)} \wedge R_{(k)} = R_{(k)}$.

Thus, $< \mathcal{R}, +_B, \cdot_B, -_B, \mathbf{0}, \mathbf{1} >$ is a *Boolean Ring*. □

Corollary 5.16. *$< \mathcal{R}^{set}_{Maj}, +_B, \cdot_B, -_B, \mathbf{0}, \mathbf{1} >$, $< \mathcal{R}^{set}_{Min}, +_B, \cdot_B, -_B, \mathbf{0}, \mathbf{1} >$ and $< \mathcal{R}^{set}_{Con}, +_B, \cdot_B, -_B, \mathbf{0}, \mathbf{1} >$ are Boolean Rings.*

Boolean algebras were found out as a result of Boole's investigations into the underlying the laws of correct reasoning. After then, they have become to axiomatic set theory, model theory, computer science, electrical engineering and other areas of science and mathematics from day to day. This many-valued algebraic approach to categorical syllogisms will be useful for these areas when they need algebraic bases in syllogistic deductions.

Here, we consider MV-algebras which were originally introduced by Chang [5] and further developed by Chang [6]. A simplified axiomatization of MV-algebras in use today can be found in the monograph Algebraic Foundations of Many-valued Reasoning [9].

Definition 5.17. ([9]) An *MV-algebra* is an algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ satisfying the following axioms where A is a nonempty set, $\oplus$ is a binary operation on A, $\neg$ is a unary operation on A, and 0 is a constant element of A:

$(MV1)$ $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
$(MV2)$ $x \oplus y = y \oplus x$,
$(MV3)$ $x \oplus 0 = x$,
$(MV4)$ $\neg\neg x = x$,
$(MV5)$ $x \oplus \neg 0 = \neg 0$,
$(MV6)$ $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

For the construction of many-valued algebraic system on categorical syllogisms, we need to define $\oplus$ and $\neg$ operations as below.

Definition 5.18. Let $R_{(k)}$ and $R_{(l)}$ be elements of $\mathcal{R}$. Then the binary operation $\oplus$ and the unary operation $\neg$ are defined by

$$R_{(k)} \oplus R_{(l)} := R^{set}_{(k)} \cup R^{set}_{(l)},$$

$$\neg R_{(k)} := (R^{set}_{(k)})^c.$$

Theorem 5.19. *$< \mathcal{R}, \oplus, \neg, \mathbf{0} >$ is an MV-algebra.*

Proof. Axioms $(MV1) - (MV5)$ follow from Definition 5.5 and Definition 5.18. Let $R_{(k)}$ and $R_{(l)} \in \mathcal{R}$. Then we get

$$\begin{aligned} \neg(\neg R_{(k)} \oplus R_{(l)}) \oplus R_{(l)} &= (\mathbf{1} \setminus ((\mathbf{1} \setminus (R^{set}_{(k)})) \cup R^{set}_{(l)})) \cup R^{set}_{(l)} \\ &= (R^{set}_{(k)} \cap (R^{set}_{(l)})^c) \cup R^{set}_{(l)} \\ &= R^{set}_{(k)} \cup R^{set}_{(l)} \end{aligned} \quad (1)$$

$$\begin{aligned} \neg(\neg R_{(l)} \oplus R_{(k)}) \oplus R_{(k)} &= (\mathbf{1} \setminus ((\mathbf{1} \setminus (R^{set}_{(l)})) \cup R^{set}_{(k)})) \cup R^{set}_{(l)} \\ &= (R^{set}_{(l)} \cap (R^{set}_{(k)})^c) \cup R^{set}_{(k)} \\ &= R^{set}_{(l)} \cup R^{set}_{(k)} \end{aligned} \quad (2)$$

From (1) and (2), $(MV6)$ is obtained. □

Corollary 5.20. *$< \mathcal{R}^{Maj}, \oplus, ^c, \mathbf{0} >$, $< \mathcal{R}^{Min}, \oplus, ^c, \mathbf{0} >$ and $< \mathcal{R}^{Con}, \oplus, ^c, \mathbf{0} >$ are MV-algebras.*

6. Conclusion

We present an analysis the algebraic properties of categorical syllogisms by establishing a calculus system SLCD. In accordance with this purpose, we explain how a given premise is stated by using a Carroll diagram and its possible bilateral diagrams are obtained. In the sequel, corresponding exactly one value to each possible bilateral diagram, we can express major and minor premises by means of sets. We decide whether the syllogism is valid or not by deducing a conclusion set from the premises sets. As a result, we prove that a syllogism is valid if and only if it is provable in SLCD and also a strengthened syllogism is valid if and only if it is provable in $SLCD^{\dagger}$. In other words, we give answer the question how the calculus system SLCD is constructed and we show the completeness of this system.

In the last part of this study, constructing different algebraic structures via premises and conclusion transferred to sets, we provide that mathematical analysis of syllogisms can be examined more easily. Therefore, we purpose that they are intended to serve to researchers getting into the act in different branches of science such as computer science, engineering, artificial intelligence etc.

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