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Optimal Orientations of Some Complete Tripartite Graphs

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Abstract. For a graph G, let $\mathscr{D}(G)$ be the set of all strong orientations of G. The *orientation number* of G is $\vec{d}(G) = min\{d(D)|D \in \mathscr{D}(G)\}$, where $d(D)$ denotes the diameter of the digraph D. In this paper, we determine the orientation number for some complete tripartite graphs.

1. Introduction

Let G be a finite undirected simple graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, the *eccentricity of* v is $e_G(v) = \max\{d_G(v, x) \mid x \in V(G)\}$, where $d_G(v, x)$ denotes the length of a shortest (v, x)-path in G. The *diameter* of G is $d(G) = \max\{e_G(v) \mid v \in V(G)\}$.

Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$ which has no loops and no two of its arcs have same tail and same head. The notions $e_D(v)$, for $v \in V(D)$, and $d(D)$ are defined as in the undirected graph.

An *orientation* of a graph G is a digraph D obtained from G by assigning a direction to each of its edge. A vertex v is *reachable* from a vertex u of a digraph D if there is a directed path in D from u to v. An orientation D of G is *strong* if any pair of vertices in D are mutually reachable in D. Robbins' one-way street theorem [7] states that a connected graph G has a strong orientation if and only if G is 2-edge-connected. For a 2-edge-connected graph G, let $\mathscr{D}(G)$ denote the set of all strong orientations of G. The *orientation number* of G is $\vec{d}(G) = \min\{d(D) \mid D \in \mathscr{D}(G)\}$. Any orientation D in $\mathscr{D}(G)$ with $d(D) = \vec{d}(G)$ is called an *optimal orientation* of G.

Given positive integers $n, p_1, p_2, \ldots, p_n$, let K_n denote the complete graph of order n, and $K(p_1, p_2, \ldots, p_n)$ denote the complete n-partite graph having p_i vertices in the i^{th} partite set, $i \in \{1, 2, \ldots, n\}$. The n partite sets of $K(p_1, p_2, \ldots, p_n)$ are denoted by $V_1, V_2, \ldots, V_n$ so that $|V_i| = p_i$, $i \in \{1, 2, \ldots, n\}$. If $p_1 = p_2 = \ldots = p_n = p$, denote $K(p_1, p_2, \ldots, p_n)$ by $K_n(p)$.

Boesch and Tindell [2] and independently Maurer [5] proved that: $\vec{d}(K_n) = 2$ if $n \geq 3$ and $n \neq 4$, and $\vec{d}(K_4) = 3$. Soltés [8] proved that $\vec{d}(K_{p,q})$ is 3 if $2 \leq p \leq q \leq \binom{p}{\lfloor \frac{p}{2} \rfloor}$ and it is 4 if $q > \binom{p}{\lfloor \frac{p}{2} \rfloor}$, where $\lfloor x \rfloor$ denotes the greatest integer not exceeding the real x.

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A pair $\{p, q\}$ of integers is called a *co-pair* if $1 \le p \le q \le \binom{p}{\lfloor \frac{p}{2} \rfloor}$ or $1 \le q \le p \le \binom{q}{\lfloor \frac{q}{2} \rfloor}$. A multiset $\{p, q, r\}$ of positive integers is called a *co-triple* if $\{p, q\}$ and $\{p, r\}$ are co-pairs.

Koh and Tan proved, in [3], that:

- if $\{p, q\}$ is a co-pair with $q \ge p \ge 2$, then $\vec{d}(K(2, p, q)) = 2$;
- if $\{p, q, r\}$ is a co-triple with $q \ge p \ge r \ge 2$, then $\vec{d}(K(2, p, q, r)) = 2$;
- if $k \ge 2$, $\{p_i, q_i\}$ is a co-pair for each $i \in \{1, 2, \ldots, k\}$ and $(k, p_1, p_2) \ne (2, 1, 1)$, then $\vec{d}(K(p_1, q_1, p_2, q_2, \ldots, p_k, q_k)) = 2$; and,
- if $k \ge 2$, $\{p_i, q_i\}$ is a co-pair for each $i \in \{1, 2, \ldots, k\}$ and $\{r, p_h\}$ is a co-pair for some $h \in \{1, 2, \ldots, k\}$, then $\vec{d}(K(p_1, q_1, p_2, q_2, \ldots, p_k, q_k, r)) = 2$;

and, in [4], that:

- $2 \le \vec{d}(K(p_1, p_2, \ldots, p_n)) \le 3$ if $n \ge 3$;
- $\vec{d}(K_n(p)) = 2$ if $n \ge 3$ and $p \ge 2$;
- $\vec{d}(K(\overbrace{p, p, \ldots, p}^{r}, q)) = 2$ if $p \ge 3$, $r \ge 3$ and $1 \le q \le 2p$; and
- with $h = \sum_{i=1}^{n} p_i$, $n \ge 3$, if $p_i > \binom{h-p_i}{\lfloor \frac{h-p_i}{2} \rfloor}$ for some $i \in \{1, 2, \ldots, n\}$, then $\vec{d}(K(p_1, p_2, \ldots, p_n)) = 3$.

In [4], Koh and Tan mentioned that the problem of determining whether a given $G = K(p_1, p_2, \ldots, p_n)$ is such that $\vec{d}(G) = 2$ or $\vec{d}(G) = 3$ is very difficult. In this paper, we shall extend the known results on the computation of $\vec{d}(K(p_1, p_2, p_3))$.

The subdigraph of a digraph D induced by $A \subseteq V(D)$ is denoted by $D[A]$. We refer to [1] for notations and terminology not described here.

2. Results

Recall that known results on $\vec{d}(K(p_1, p_2, p_3))$ are:
$2 \le \vec{d}(K(p_1, p_2, p_3)) \le 3$;
$\vec{d}(K_3(p)) = 2$;
if $\{p, q\}$ is a co-pair with $q \ge p \ge 2$, then $\vec{d}(K(2, p, q)) = 2$; and
with $h = p_1 + p_2 + p_3$, if $p_i > \binom{h-p_i}{\lfloor \frac{h-p_i}{2} \rfloor}$ for some $i \in \{1, 2, 3\}$, then $\vec{d}(K(p_1, p_2, p_3)) = 3$.

The results obtained in this paper are the following.

Theorem 2.1. *For $p \ge 2$ and $q \ge 2$, $\vec{d}(K(1, p, q)) = 3$.*

Theorem 2.2. *For $p \ge 3$, $\vec{d}(K(2, 2, p)) = 3$.*

Theorem 2.3. *For $p \ge 4$, $\vec{d}(K(2, 3, p)) = 3$.*

Theorem 2.4. *For $p \ge 4$ and $4 \le q \le 2p$, $\vec{d}(K(p, p, q)) = 2$.*

3. Proofs

Proof of Theorem 2.1. Let $V_1 = \{x\}$, $V_2 = \{y_1, y_2, \ldots, y_p\}$ and $V_3 = \{z_1, z_2, \ldots, z_q\}$ be the partite sets of $K(1, p, q)$, where $p \ge 2$ and $q \ge 2$. Suppose $K(1, p, q)$ has an orientation D with $d(D) = 2$, then we consider the following four exhaustive cases to obtain the required contradiction.

Case 1. $y_i \to z_j \to y_k$ for some $i, k \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, q\}$.

$d_D(z_j, y_i) \le 2$ implies that $z_j \to x \to y_i$, and so $d_D(y_k, z_j) \ge 3$, a contradiction.

Case 2. $z_i \to y_j \to z_k$ for some $j \in \{1, 2, \ldots, p\}$ and $i, k \in \{1, 2, \ldots, q\}$.

Similar to Case 1.

Case 3. $V_2 \rightarrow V_3$.

For any $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, q\}$, $d_D(z_j, y_i) \leq 2$ implies that $z_j \rightarrow x \rightarrow y_i$. Consequently, $d_D(y_1, y_2) \geq 3$, a contradiction.

Case 4. $V_3 \rightarrow V_2$.

Similar to Case 3.

This completes the proof. □

Proof of Theorem 2.2. Let $V_1 = \{x_1, x_2\}$, $V_2 = \{y_1, y_2\}$ and $V_3 = \{z_1, z_2, \ldots, z_p\}$ be the partite sets of $K(2, 2, p)$, where $p \geq 3$. Suppose $K(2, 2, p)$ has an orientation D with $d(D) = 2$, then we consider the following four exhaustive cases to obtain the required contradiction.

Case 1. $x_1 \rightarrow y_1 \rightarrow x_2 \rightarrow y_2 \rightarrow x_1$.

$d_D(y_1, x_1) \leq 2$, $d_D(y_2, x_2) \leq 2$, $d_D(x_1, y_2) \leq 2$, and $d_D(x_2, y_1) \leq 2$ implies, respectively, that $y_1 \rightarrow z_i \rightarrow x_1$, $y_2 \rightarrow z_j \rightarrow x_2$, $x_1 \rightarrow z_k \rightarrow y_2$, and $x_2 \rightarrow z_\ell \rightarrow y_1$ for some $i, j, k, \ell \in \{1, 2, \ldots, p\}$. Note that i may be equal to j, k may be equal to ℓ, but $\{i, j\} \cap \{k, \ell\} = \phi$. We consider three subcases.

Subcase 1.1. $i = j$ and $k = \ell$.

Suppose for some $m \in \{1, 2, \ldots, p\} \setminus \{i, k\}$, $z_m \rightarrow x_1$ holds. Then $d_D(x_1, z_m) \leq 2$ implies that $y_1 \rightarrow z_m$; and hence $d_D(z_m, z_i) \leq 2$ implies that $z_m \rightarrow y_2$. Now $d_D(y_2, z_m) \geq 3$, a contradiction.

Consequently, for every $m \in \{1, 2, \ldots, p\} \setminus \{i, k\}$, $x_1 \rightarrow z_m$. $d_D(z_m, x_1) \leq 2$ implies that $z_m \rightarrow y_2$; and hence $d_D(z_k, z_m) \leq 2$ implies that $y_1 \rightarrow z_m$. Now $d_D(z_m, y_1) \geq 3$, a contradiction.

Subcase 1.2. $i \neq j$.

$d_D(z_i, x_2) \leq 2$ implies that $z_i \rightarrow x_2$; and $d_D(y_1, z_j) \leq 2$ implies that $y_1 \rightarrow z_j$. Now $d_D(z_j, z_i) \geq 3$, a contradiction.

Subcase 1.3. $k \neq \ell$.

Similar to Subcase 1.2.

Case 2. $x_1 \rightarrow V_2$ and $y_1 \rightarrow x_2 \rightarrow y_2$.

For $i \in \{1, 2, \ldots, p\}$, $d_D(z_i, x_1) \leq 2$ and $d_D(y_2, z_i) \leq 2$ implies, respectively, that $z_i \rightarrow x_1$ and $y_2 \rightarrow z_i$. $d_D(x_2, x_1) \leq 2$ implies that $x_2 \rightarrow z_i$ for some $i \in \{1, 2, \ldots, p\}$. For any $j \in \{1, 2, \ldots, p\} \setminus \{i\}$, $d_D(z_i, z_j) \leq 2$ implies that $z_i \rightarrow y_1 \rightarrow z_j$, and therefore $d_D(z_j, z_i) \leq 2$ implies that $z_j \rightarrow x_2$. Thus for $j_1, j_2 \in \{1, 2, \ldots, p\} \setminus \{i\}$ with $j_1 \neq j_2$, $d_D(z_{j_1}, z_{j_2}) \geq 3$, a contradiction.

Case 3. $x_1 \rightarrow V_2 \rightarrow x_2$.

For $i \in \{1, 2, \ldots, p\}$, $d_D(z_i, x_1) \leq 2$ and $d_D(x_2, z_i) \leq 2$ implies, respectively, that $z_i \rightarrow x_1$ and $x_2 \rightarrow z_i$. $d_D(y_1, y_2) \leq 2$ and $d_D(y_2, y_1) \leq 2$ implies, respectively, that $y_1 \rightarrow z_i \rightarrow y_2$ and $y_2 \rightarrow z_j \rightarrow y_1$ for some $i, j \in \{1, 2, \ldots, p\}$. Clearly, $i \neq j$. For $k \in \{1, 2, \ldots, p\} \setminus \{i, j\}$, $d_D(z_k, z_j) \leq 2$ implies that $z_k \rightarrow y_2$. Now $d_D(z_i, z_k) \geq 3$, a contradiction.

Case 4. $V_1 \rightarrow V_2$.

For $i, j \in \{1, 2\}$ and $k \in \{1, 2, \ldots, p\}$, $d_D(y_j, z_k) \leq 2$ and $d_D(z_k, x_i) \leq 2$ implies, respectively, that $y_j \rightarrow z_k$ and $z_k \rightarrow x_i$. Now $d_D(y_1, y_2) \geq 3$, a contradiction.

This completes the proof. □

Proof of Theorem 2.3. Let $V_1 = \{x_1, x_2\}$, $V_2 = \{y_1, y_2, y_3\}$, $V_3 = \{z_1, z_2, \ldots, z_p\}$ be the partite sets of $K(2, 3, p)$, where $p \geq 4$. Suppose $K(2, 3, p)$ has an orientation D with $d(D) = 2$, then we consider the following exhaustive cases to obtain the required contradiction. Without loss of generality assume that one of the following holds: (1) $x_1 \rightarrow V_2$, (2) $y_1 \rightarrow x_1 \rightarrow \{y_2, y_3\}$, (3) $\{y_2, y_3\} \rightarrow x_1 \rightarrow y_1$, (4) $V_2 \rightarrow x_1$. As the subdigraphs (3) and (4) are, respectively, the converse subdigraphs of (1) and (2), we consider (1) and (2) only. In each of (1) and (2) one of the following holds: (a) $x_2 \rightarrow V_2$, (b) $y_1 \rightarrow x_2 \rightarrow \{y_2, y_3\}$, (c) $y_2 \rightarrow x_2 \rightarrow \{y_1, y_3\}$, (d) $y_3 \rightarrow x_2 \rightarrow \{y_1, y_2\}$, (e) $\{y_2, y_3\} \rightarrow x_2 \rightarrow y_1$, (f) $\{y_1, y_3\} \rightarrow x_2 \rightarrow y_2$, (g) $\{y_1, y_2\} \rightarrow x_2 \rightarrow y_3$, (h) $V_2 \rightarrow x_2$.

Case 1a. $V_1 \rightarrow V_2$.

For every $i \in \{1, 2, \ldots, p\}$, $d_D(z_i, x_1) \leq 2$ and $d_D(z_i, x_2) \leq 2$ implies, respectively, that $z_i \rightarrow x_1$ and $z_i \rightarrow x_2$. Now $d_D(x_1, x_2) \geq 3$, a contradiction.

Case 2a. $y_1 \rightarrow x_1 \rightarrow \{y_2, y_3\}$ and $x_2 \rightarrow V_2$ or *Case 2b.* $y_1 \rightarrow V_1 \rightarrow \{y_2, y_3\}$.

For every $i \in \{1, 2, \ldots, p\}$, $d_D(y_2, z_i) \leq 2$ and $d_D(y_3, z_i) \leq 2$ implies, respectively, that $y_2 \rightarrow z_i$ and

$y_3 \to z_i$. Now $d_D(y_2, y_3) \geq 3$, a contradiction.

Case 2c. $y_1 \to x_1 \to \{y_2, y_3\}$ and $y_2 \to x_2 \to \{y_1, y_3\}$.

$d_D(y_3, y_1) \leq 2$ implies that $y_3 \to z_i \to y_1$ for some $i \in \{1, 2, \ldots, p\}$. $d_D(y_1, z_i) \leq 2$ implies that $x_1 \to z_i$, $d_D(z_i, y_3) \leq 2$ implies that $z_i \to x_2$, and $d_D(z_i, y_2) \leq 2$ implies that $z_i \to y_2$. Now $d_D(y_2, z_i) \geq 3$, a contradiction.

Case 2e. $x_1 \to \{y_2, y_3\} \to x_2 \to y_1 \to x_1$.

$d_D(y_2, y_3) \leq 2$ implies that $y_2 \to z_i \to y_3$ for some $i \in \{1, 2, \ldots, p\}$. $d_D(y_3, z_i) \leq 2$ implies that $x_2 \to z_i$, $d_D(z_i, y_2) \leq 2$ implies that $z_i \to x_1$, and $d_D(z_i, y_1) \leq 2$ implies that $z_i \to y_1$. Now $d_D(y_1, z_i) \geq 3$, a contradiction.

Case 2f. $y_1 \to x_1 \to \{y_2, y_3\}$ and $\{y_1, y_3\} \to x_2 \to y_2$.

For every $i \in \{1, 2, \ldots, p\}$, $d_D(z_i, y_1) \leq 2$ implies that $z_i \to y_1$ and $d_D(y_2, z_i) \leq 2$ implies that $y_2 \to z_i$. $d_D(x_2, x_1) \leq 2$ implies that $x_2 \to z_i \to x_1$ for some $i \in \{1, 2, \ldots, p\}$. $d_D(y_3, y_1) \leq 2$ implies that $y_3 \to z_j$ for some $j \in \{1, 2, \ldots, p\}$. If $i \neq j$, then $d_D(z_j, y_3) \leq 2$ implies that $z_j \to x_1$, and $d_D(z_j, z_i) \leq 2$ implies that $z_j \to x_2$; consequently, $d_D(y_1, z_j) \geq 3$, a contradiction. Thus $i = j$. For every $k \in \{1, 2, \ldots, p\} \setminus \{i\}$, $d_D(z_i, z_k) \leq 2$ implies that $x_1 \to z_k$, $d_D(z_k, y_3) \leq 2$ implies that $z_k \to y_3$, and $d_D(y_3, z_k) \leq 2$ implies that $x_2 \to z_k$. Now $d_D(z_k, y_2) \geq 3$, a contradiction.

Case 1h. $x_1 \to V_2 \to x_2$.

For every $i \in \{1, 2, \ldots, p\}$, $d_D(x_2, z_i) \leq 2$ and $d_D(z_i, x_1) \leq 2$ implies, respectively, that $x_2 \to z_i$ and $z_i \to x_1$. $d_D(y_1, y_2) \leq 2$ and $d_D(y_2, y_1) \leq 2$ implies, respectively, that $y_1 \to z_i \to y_2$ and $y_2 \to z_j \to y_1$ for some $i, j \in \{1, 2, \ldots, p\}$ with $i \neq j$. $d_D(y_1, y_3) \leq 2$ implies that either $z_i \to y_3$ or $y_1 \to z_k \to y_3$ for some $k \in \{1, 2, \ldots, p\} \setminus \{i, j\}$.

First assume that $z_i \to y_3$. For any $k \in \{1, 2, \ldots, p\} \setminus \{i, j\}$, $d_D(z_k, z_i) \leq 2$ implies that $z_k \to y_1$, $d_D(z_j, z_k) \leq 2$ implies that $z_j \to y_3 \to z_k$, and $d_D(z_k, z_j) \leq 2$ implies that $z_k \to y_2$. Now for $k_1, k_2 \in \{1, 2, \ldots, p\} \setminus \{i, j\}$ with $k_1 \neq k_2$, $d_D(z_{k_1}, z_{k_2}) \geq 3$, a contradiction.

Next assume that $y_1 \to z_k \to y_3$ for some $k \in \{1, 2, \ldots, p\} \setminus \{i, j\}$. $d_D(z_k, z_i) \leq 2$ implies that $y_3 \to z_i$, $d_D(z_i, z_k) \leq 2$ implies that $y_2 \to z_k$, and $d_D(z_k, z_j) \leq 2$ implies that $y_3 \to z_j$. Choose $\ell \in \{1, 2, \ldots, p\} \setminus \{i, j, k\}$. $d_D(z_i, z_\ell) \leq 2$ implies that $y_2 \to z_\ell$, $d_D(z_j, z_\ell) \leq 2$ implies that $y_1 \to z_\ell$, and $d_D(z_k, z_\ell) \leq 2$ implies that $y_3 \to z_\ell$. Now $d_D(z_\ell, z_j) \geq 3$, a contradiction.

Case 2h. $y_1 \to x_1 \to \{y_2, y_3\}$ and $V_2 \to x_2$.

For every $i \in \{1, 2, \ldots, p\}$, $d_D(x_2, z_i) \leq 2$ and $d_D(z_i, y_1) \leq 2$ implies, respectively, that $x_2 \to z_i$ and $z_i \to y_1$. $d_D(x_1, y_1) \leq 2$ implies that $x_1 \to z_i$ for some $i \in \{1, 2, \ldots, p\}$. $d_D(z_i, y_2) \leq 2$ implies that $z_i \to y_2$, and $d_D(z_i, y_3) \leq 2$ implies that $z_i \to y_3$. For every $j \in \{1, 2, \ldots, p\} \setminus \{i\}$, $d_D(z_j, z_i) \leq 2$ implies that $z_j \to x_1$. $d_D(y_2, x_1) \leq 2$ and $d_D(y_3, x_1) \leq 2$ implies, respectively, that $y_2 \to z_j$ and $y_3 \to z_k$ for some $j, k \in \{1, 2, \ldots, p\} \setminus \{i\}$. If $j = k$, then for any $\ell \in \{1, 2, \ldots, p\} \setminus \{i, j\}$, $d_D(z_j, z_\ell) \geq 3$, a contradiction. Hence $j \neq k$. $d_D(z_k, z_j) \leq 2$ implies that $z_k \to y_2$ and $d_D(z_j, z_k) \leq 2$ implies that $z_j \to y_3$. For every $\ell \in \{1, 2, \ldots, p\} \setminus \{i, j, k\}$, $d_D(z_j, z_\ell) \leq 2$ implies that $y_3 \to z_\ell$. Now $d_D(z_\ell, z_k) \geq 3$, a contradiction.

Case 1b. $y_1 \to x_2 \to \{y_2, y_3\}$ and $x_1 \to V_2$.

Similar to Case 2a. Permute x_1 and x_2.

Case 1c. $y_2 \to x_2 \to \{y_1, y_3\}$ and $x_1 \to V_2$.

Similar to Case 2a. Apply the permutation $(x_1, x_2)(y_1, y_2)$.

Case 1d. $y_3 \to x_2 \to \{y_1, y_2\}$ and $x_1 \to V_2$.

Similar to Case 2a. Apply the permutation $(x_1, x_2)(y_1, y_3)$.

Case 1e. $\{y_2, y_3\} \to x_2 \to y_1$ and $x_1 \to V_2$.

Similar to Case 2h. Permute x_1 and x_2, and consider the converse digraph.

Case 1f. $\{y_1, y_3\} \to x_2 \to y_2$ and $x_1 \to V_2$.

Similar to Case 1e. Permute y_1 and y_2.

Case 1g. $\{y_1, y_2\} \to x_2 \to y_3$ and $x_1 \to V_2$.

Similar to Case 1e. Permute y_1 and y_3.

Case 2d. $y_1 \to x_1 \to \{y_2, y_3\}$ and $y_3 \to x_2 \to \{y_1, y_2\}$.

Similar to Case 2c. Permute y_2 and y_3.

Case 2g. $y_1 \to x_1 \to \{y_2, y_3\}$ and $\{y_1, y_2\} \to x_2 \to y_3$.

Similar to Case 2f. Permute y_2 and y_3.

This completes the proof. □

Proof of Theorem 2.4. Let $V_1 = \{x_1, x_2, \ldots, x_p\}$, $V_2 = \{y_1, y_2, \ldots, y_p\}$, and $V_3 = \{z_1, z_2, \ldots, z_q\}$ be the partite sets of $K(p, p, q)$, $p \geq 4$ and $4 \leq q \leq 2p$. Orient $K(p, p, q)$ as follows:
(i) For $i \in \{1, 2, \ldots, p\}$,

$$\{y_i, y_{i+1}\} \rightarrow x_i \rightarrow [V_2 \backslash \{y_i, y_{i+1}\}];$$

(ii) For $i \in \{1, 2, \ldots, q\}$ and i is odd,

$$\{x_{\frac{i+1}{2}}\} \cup [V_2 \backslash \{y_{\frac{i+1}{2}}\}] \rightarrow z_i \rightarrow \{y_{\frac{i+1}{2}}\} \cup [V_1 \setminus \{x_{\frac{i+1}{2}}\}];$$

(iii) For $i \in \{1, 2, \ldots, q\}$ and i is even,

$$\{y_{\frac{i-2}{2}}\} \cup [V_1 \setminus \{x_{\frac{i}{2}}\}] \rightarrow z_i \rightarrow \{x_{\frac{i}{2}}\} \cup [V_2 \setminus \{y_{\frac{i-2}{2}}\}];$$

where suffixes under x and y are reduced modulo p with residues $1, 2, \ldots, p-1, p$ instead of $1, 2, \ldots, p-1, 0$ and that of z are reduced modulo q with residues $1, 2, \ldots, q-1, q$ instead of $1, 2, \ldots, q-1, 0$. (Note that $x_0 = x_p$, $y_0 = y_p$ and $z_0 = z_q$.)

Let D be the resulting digraph. See Fig. 1. Now, we verify that $d(D) = 2$.

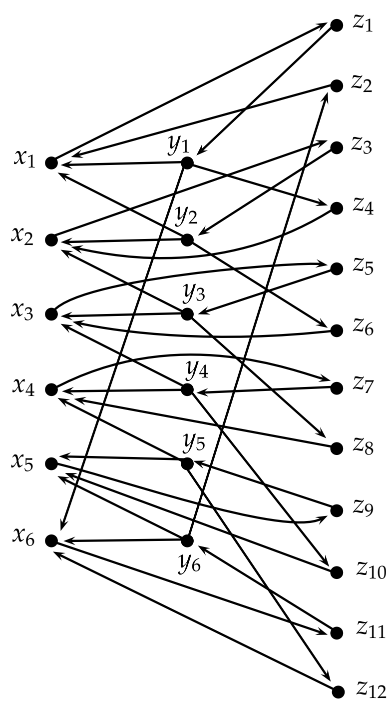

Fig. 1. The optimal orientation D described in the proof of Theorem 2.4 for $K_{6,6,12}$.
Missing arcs, of Fig. 1:
from V_1 to V_2 are of the form $x_i \rightarrow y_j$;
from V_1 to V_3 are of the form $x_i \rightarrow z_j$ when j is even and $z_j \rightarrow x_i$ when j is odd;
from V_2 to V_3 are of the form $y_i \rightarrow z_j$ when j is odd and $z_j \rightarrow y_i$ when j is even.

Claim 1. For every distinct $i, j \in \{1, 2, \ldots, p\}$, $d_D(x_i, x_j) \leq 2$.

Let $i \in \{1, 2, \ldots, p\}$ be arbitrary. The existence of the paths $x_i \rightarrow y_j \rightarrow x_j$ for $j \in \{1, 2, \ldots, p\} \backslash \{i, i+1\}$ (for $i = p$, $i + 1 = 1$), and $x_i \rightarrow y_{i+2} \rightarrow x_{i+1}$, in D, proves Claim 1.

Claim 2. For every distinct $i, j \in \{1, 2, \ldots, p\}$, $d_D(y_i, y_j) \leq 2$.

Let $i \in \{1, 2, \dots, p\}$ be arbitrary. The existence of the paths $y_i \to x_i \to y_j$ for $j \in \{1, 2, \dots, p\} \backslash \{i, i+1\}$ (for $i = p$, $i + 1 = 1$), and $y_i \to x_{i-1} \to y_{i+1}$, in D, proves Claim 2.

Claim 3. For every distinct $i, j \in \{1, 2, \dots, q\}$, $d_D(z_i, z_j) \leq 2$.

If i, j are both odd, the existence of the path $z_i \to y_{\frac{i+1}{2}} \to z_j$, in D, proves Claim 3.

If i, j are both even, the existence of the path $z_i \to x_{\frac{i}{2}} \to z_j$, in D, proves Claim 3.

If i is odd and j is even, the existence of the paths:

$z_i \to x_1 \to z_j$ for $i \neq 1$ and $j \neq 2$,
$z_1 \to x_2 \to z_j$ for $j \neq 4$,
$z_1 \to x_3 \to z_4$,
$z_i \to x_2 \to z_2$ for $i \neq 3$,
$z_3 \to x_3 \to z_2$,

in D, proves Claim 3.

If i is even and j is odd, the existence of the paths:

$z_i \to y_1 \to z_j$ for $i \neq 4$ and $j \neq 1$,
$z_4 \to y_2 \to z_j$ for $j \neq 3$,
$z_4 \to y_3 \to z_3$,
$z_i \to y_2 \to z_1$ for $i \neq 6$ and $q \geq 6$,
$z_6 \to y_3 \to z_1$ for $q \geq 6$,
$z_i \to y_2 \to z_1$ for $q \in \{4, 5\}$,

in D, proves Claim 3.

Claim 4. For every $i, j \in \{1, 2, \dots, p\}$, $d_D(x_i, y_j) \leq 2$.

Let $i \in \{1, 2, \dots, p\}$ be arbitrary. For $j \in \{1, 2, \dots, p\} \backslash \{i, i+1\}$, the existence of the arc $x_i \to y_j$, in D, together with the existence of the paths:

$x_i \to z_4 \to y_i$ for $i \notin \{1, 2\}$,
$x_1 \to z_1 \to y_1$,
$x_2 \to z_2 \to y_2$,
$x_i \to z_4 \to y_{i+1}$ for $i \notin \{2, p\}$,
$x_2 \to z_2 \to y_3$,
$x_p \to z_2 \to y_1$,

in D, proves Claim 4.

Claim 5. For every $i, j \in \{1, 2, \dots, p\}$, $d_D(y_i, x_j) \leq 2$.

The existence of the paths:

$y_i \to z_1 \to x_j$ for $i, j \in \{2, 3, \dots, p\}$,
$y_1 \to z_3 \to x_j$ for $j \neq 2$,
$y_1 \to z_4 \to x_2$,
$y_i \to z_3 \to x_1$ for $i \neq 2$,

and the arc $y_2 \to x_1$, in D, proves Claim 5.

Claim 6. For every $i \in \{1, 2, \dots, p\}$ and $j \in \{1, 2, \dots, q\}$, $d_D(x_i, z_j) \leq 2$.

Let $i \in \{1, 2, \dots, p\}$ be arbitrary.

First assume that j is odd. The existence of the paths:

$x_i \to y_1 \to z_j$ for $i \notin \{1, p\}$ and $j \neq 1$,
$x_1 \to y_3 \to z_j$ for either $q = 4$ or $q \geq 5$ and $j \neq 5$,
$x_1 \to y_4 \to z_5$ for $q \geq 5$,
$x_p \to y_2 \to z_j$ for $j \neq 3$,
$x_p \to y_3 \to z_3$,
$x_i \to y_2 \to z_1$ for $i \notin \{1, 2\}$,
$x_1 \to y_{p-1} \to z_1$, and
$x_2 \to y_4 \to z_1$,

in D, proves Claim 6.

Next assume that j is even. The existence of the arc $x_i \to z_j$ for $i \neq \frac{j}{2}$ and the existence of the path $x_{\frac{j}{2}} \to y_{\frac{j-2}{2}} \to z_j$, in D, proves Claim 6.

Claim 7. For every $i \in \{1, 2, \ldots, q\}$ and $j \in \{1, 2, \ldots, p\}$, $d_D(z_i, x_j) \leq 2$.

Let $j \in \{1, 2, \ldots, p\}$ be arbitrary.

First assume that i is odd. For $j \neq \frac{i+1}{2}$, the existence of the arc $z_i \to x_j$, in D, together with the existence of the path $z_i \to y_{\frac{i+1}{2}} \to x_{\frac{i+1}{2}}$, in D, proves Claim 7.

Next assume that i is even. For $j \neq \frac{i-2}{2}$, the existence of the path $z_i \to y_j \to x_j$, in D, together with the existence of the path $z_i \to y_{\frac{i}{2}} \to x_{\frac{i-2}{2}}$, in D, proves Claim 7.

Claim 8. For every $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, q\}$, $d_D(y_i, z_j) \leq 2$.

Let $i \in \{1, 2, \ldots, p\}$ be arbitrary.

First assume that j is odd. For $i \neq \frac{j+1}{2}$, the existence of the arc $y_i \to z_j$, in D, together with the existence of the path $y_{\frac{j+1}{2}} \to x_{\frac{j+1}{2}} \to z_j$, in D, proves Claim 8.

Next assume that j is even. For $i \neq \frac{j}{2}$, the existence of the path $y_i \to x_i \to z_j$, in D, together with the existence of the path $y_{\frac{j}{2}} \to x_{\frac{j-2}{2}} \to z_j$, in D, proves Claim 8.

Claim 9. For every $i \in \{1, 2, \ldots, q\}$ and $j \in \{1, 2, \ldots, p\}$, $d_D(z_i, y_j) \leq 2$.

Let $j \in \{1, 2, \ldots, p\}$ be arbitrary.

First assume that i is odd. For $j \mod p \notin \{\frac{i+3}{2} \mod p, \frac{i+5}{2} \mod p\}$, the existence of the path $z_i \to x_{\frac{i+3}{2}} \to y_j$, in D, together with the existence of the paths $z_i \to x_{\frac{i+7}{2}} \to y_{\frac{i+3}{2}}$ and $z_i \to x_{\frac{i+7}{2}} \to y_{\frac{i+5}{2}}$, in D, proves Claim 9.

Next assume that i is even. For $j \neq \frac{i-2}{2}$, the existence of the arc $z_i \to y_j$, in D, together with the existence of the path $z_i \to x_{\frac{i}{2}} \to y_{\frac{i-2}{2}}$, in D, proves Claim 9.

By Claims 1-9, $\vec{d}(D) = 2$. □

4. Conclusion

Based on the results of Koh and Tan, Theorems 2.2 and 2.3, and the result "for $p \geq 7$, $\vec{d}(K(2, 4, p)) = 3$" of [6], we conjecture that $\vec{d}(K(2, p, q)) = 3$ when $p \geq 5$ and $q > \binom{p}{\lfloor \frac{p}{2} \rfloor}$.

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References

[1] J. Bang-Jensen and G. Gutin, Digraphs: Theory, Algorithms and Applications, Second Edition, Springer, London, 2009.
[2] F. Boesch and R. Tindell, Robbin's theorem for mixed multigraphs, Amer. Math. Monthly 87 (1980) 716–719.
[3] K.M. Koh and B.P. Tan, The minimum diameter of orientations of complete multipartite graphs, Graphs and Combin. 12 (1996) 333–339.
[4] K.M. Koh and B.P. Tan, The diameter of an orientation of a complete multipartite graph, Discrete Math. 149 (1996) 131–139.
[5] S.B. Maurer, The king chicken theorems, Math. Mag. 53 (1980) 67–80.
[6] G. Rajasekaran and R. Sampathkumar, The orientation number of the complete tripartite graph $K(2, 4, p)$, Submitted.
[7] H. Robbins, A theorem on graphs with an application to a problem of traffic control, Amer. Math. Monthly 46 (1939) 281–283.
[8] L. Soltés, Orientations of graphs minimizing the radius or the diameter, Math. Slovaca 36 (1986) 289–296.