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Extragradient Algorithms for Variational Inequality, Fixed Point and Generalized Mixed Equilibrium Problems

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Abstract. The purpose of this paper is to investigate variational inequalities, fixed point problems and generalized mixed equilibrium problems. An extragradient iterative algorithm is investigated in the framework of Hilbert spaces. Weak convergence theorems for common solutions are established.

1. Introduction

In real world, there are many problems are reduced to finding solutions of equilibrium problems, which cover variational inequalities, fixed point problems, saddle point problems, complementarity problems as special cases. Equilibrium problem, which was first introduced by Fan [1] and further studied by Blum and Oettli [2], has been extensively studied as an effective and powerful tool for a wide class of real world problems which arise in economics, finance, image reconstruction, ecology, transportation, network and related optimization problems; see [2-17] and the references therein. For solving solutions of variational inequalities, projection algorithms are efficient. However, they request the involving monotone mappings are inverse-strongly monotone; [18]. To relax the restriction on inverse-strongly monotone, extragradient algorithms, which have been extensively studied [19-23], are considered for a variational inequality involving a continuous and monotone mapping in this paper.

The organization of this paper is as follows. In Section 2, we provide some necessary preliminaries which play an important role. In Section 3, an extragradient projection algorithm is introduced and the convergence analysis is also given. A weak convergence theorem is established in the framework of Hilbert spaces. Some subresults are also provided as corollaries of the main results in this section.

2. Preliminaries

From now on, we always assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and C is a nonempty, closed, and convex subset of H. $\mathbb{R}$ is denoted by the set of real numbers. Let F be a bifunction of $C \times C$ into $\mathbb{R}$. Consider the problem: find a p such that

$$F(p, y) \geq 0, \quad \forall y \in C. \tag{2.1}$$

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In this paper, the solution set of the problem is denoted by $EP(F)$, i.e.,

$$EP(F) = \{p \in C : F(p, y) \geq 0, \quad \forall y \in C\}.$$

The above problem is first introduced by Ky Fan [1]. In the sense of Blum and Oettli [2], the Ky Fan inequality is also called an equilibrium problem.

Recently, the "so-called" generalized mixed equilibrium problem has been investigated by many authors: The generalized mixed equilibrium problem is to find $p \in C$ such that

$$F(p, y) + \langle Ap, y - p\rangle + \varphi(y) - \varphi(p) \geq 0, \quad \forall y \in C, \tag{2.2}$$

where $\varphi : C \to \mathbb{R}$ is a real valued function and $A : C \to E^*$ is mapping. We use $GMEP(F, A, \varphi)$ to denote the solution set of the equilibrium problem. That is,

$$GMEP(F, A, \varphi) := \{p \in C : F(p, y) + \langle Ap, y - p\rangle + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in C\}.$$

Next, we give some special cases:

If $A = 0$, then the problem (2.2) is equivalent to find $p \in C$ such that

$$F(p, y) + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in C, \tag{2.3}$$

which is called the mixed equilibrium problem.

If $F = 0$, then the problem (2.2) is equivalent to find $p \in C$ such that

$$\langle Ap, y - p\rangle + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in C, \tag{2.4}$$

which is called the mixed variational inequality of Browder type.

If $\varphi = 0$, then the problem (2.2) is equivalent to find $p \in C$ such that

$$F(p, y) + \langle Ap, y - p\rangle \geq 0, \quad \forall y \in C, \tag{2.5}$$

which is called the generalized equilibrium problem.

If $A = 0$ and $\varphi = 0$, then the problem (2.2) is equivalent to (2.1).

Let $F(x, y) = \langle Ax, y - x\rangle$, $\forall x, y \in C$. we see that the problem (2.1) is reduced to the following classical variational inequality. Find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{2.6}$$

It is known that $x \in C$ is a solution to (2.6) if and only if x is a fixed point of the mapping $P_C(I - \rho A)$, where $\rho > 0$ is a constant, and I is the identity mapping.

For solving the above equilibrium problems, let us assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x, x) = 0, \forall x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0, \forall x, y \in C$;

(A3)
$$\limsup_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y), \forall x, y, z \in C;$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and weakly lower semi-continuous.

Let $T : C \to C$ be a mapping. In this paper, we use $F(T)$ to stand for the set of fixed points. Recall that the mapping T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

T is said to be κ-strictly pseudocontractive if there exits a constant $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

It is clear the class of κ-strictly pseudocontractive include the class of nonexpansive mappings as a special case.

Let $A : C \to H$ be a mapping. Recall that A is said to be monotone if

$$\langle Ax - Ay, x - y\rangle \ge 0, \quad \forall x, y \in C.$$

A is said to be κ-inverse strongly monotone if there exits a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \ge \kappa\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is clear that κ-inverse strongly monotone is monotone and Lipschitiz continuous.

A set-valued mapping $T : H \to 2^H$ is said to be monotone if, for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \ge 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if, for any $(x, f) \in H\times H$, $\langle x-y, f-g\rangle \ge 0$ for all $(y, g) \in G(T)$ implies $f \in Tx$. The class of monotone operators is one of the most important classes of operators. Within the past several decades, many authors have been devoting to the studies on the existence and convergence of zero points for maximal monotone operators.

In order to prove our main results, we need the following lemmas.

Lemma 2.1 [24] *Let C be a nonempty, closed, and convex subset of H, and $S : C \to C$ a strictly pseudocontractive mapping. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$, and $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$, then $x = Sx$.*

Lemma 2.2. [24] *Let $S : C \to C$ be a κ-strictly pseudocontractive mapping. Define $S_t : C \to C$ by $S_t x = tx+(1-t)Sx$ for each $x \in C$. Then, as $t \in [\kappa, 1)$, S_t is nonexpansive such that $F(S_t) = F(S)$.*

Lemma 2.3 [25] *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnegative sequences satisfying the following condition:*

$$a_{n+1} \le (1 + b_n)a_n + c_n, \quad \forall n \ge n_0,$$

where n_0 is some nonnegative integer, $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then the limit $\lim_{n\to\infty} a_n$ exists.

Lemma 2.4. [2] *Let C be a nonempty closed convex subset of H, and $F : C\times C \to \mathbb{R}$ a bifunction satisfying $(A1)$-$(A4)$. Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C.$$

Further, define

$$T_r x = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C\}$$

for all $r > 0$ and $x \in H$. Then, the following hold:

(a) *T_r is single-valued;*

(b) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y\rangle;$$

(c) *$F(T_r) = EP(F)$;*

(d) *$EP(F)$ is closed and convex.*

Lemma 2.5. [26] *Let* $\{a_n\}_{n=1}^{\infty}$ *be real numbers in* $[0,1]$ *such that* $\sum_{n=1}^{\infty} a_n = 1$. *Then we have the following.*

$$\|\sum_{i=1}^{\infty} a_i x_i\|^2 \leq \sum_{i=1}^{\infty} a_i \|x_i\|^2,$$

for any given bounded sequence $\{x_n\}_{n=1}^{\infty}$ *in* H.

Lemma 2.6. [27] *Let* A *be a monotone mapping of* C *into* H *and* $N_C v$ *the normal cone to* C *at* $v \in C$, *i.e.,*

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \quad \forall u \in C\}$$

and define a mapping T *on* C *by*

$$Tv = \begin{cases} Av + N_C v, & v \in C \\ \emptyset, & v \notin C. \end{cases}$$

Then T *is maximal monotone and* $0 \in Tv$ *if and only if* $\langle Av, u - v\rangle \geq 0$ *for all* $u \in C$.

Lemma 2.7. [28] *Let* $0 < p \leq t_n \leq q < 1$ *for all* $n \geq 1$. *Suppose that* $\{x_n\}$, *and* $\{y_n\}$ *are sequences in* H *such that*

$$\limsup_{n\to\infty} \|x_n\| \leq d, \quad \limsup_{n\to\infty} \|y_n\| \leq d$$

and

$$\lim_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = d$$

hold for some $r \geq 0$. *Then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

3. Main Results

Theorem 3.1. *Let* C *be a nonempty closed convex subset of a Hilbert space* H. *Let* $A : C \to H$ *be a* L*-Lipschitz continuous and monotone mapping and let* $T : C \to C$ *be a* κ*-strictly psuedocontractive mapping. Let* $N \geq 1$ *be some positive integer. Let* F_m *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *which satisfies* $(A1)$*-*$(A4)$. *Let* $\varphi_m : C \to \mathbb{R}$ *be a lower semicontinuous and convex function and let* $B_m : C \to H$ *be a continuous and monotone mapping for each* $1 \leq m \leq N$. *Assume that* $\mathcal{F} := \cap_{m=1}^{N} GMEP(F_m, B_m, \varphi_m) \cap F(T) \cap VI(C, A) \neq \emptyset$. *Let* $\{\lambda_n\}$, $\{r_{n,m}\}$ *be positive real number sequences. Let* $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha''_n\}$, $\{\beta_n\}$ *and* $\{\delta_{n,m}\}$ *be real number sequences in* $(0,1)$. *Let* $\{x_n\}$ *be a sequence generated in the following process:*

$$\begin{cases} x_1 \in H, \\ F_m(z_{n,m}, z) + \langle B_m z_{n,m}, z - z_{n,m}\rangle + \varphi_m(z) - \varphi_m(z_{n,m}) + \frac{1}{r_{n,m}}\langle z - z_{n,m}, z_{n,m} - x_n\rangle \geq 0, \quad \forall z \in C, \\ y_n = \mathrm{Proj}_C(\sum_{m=1}^{N} \delta_{n,m} z_{n,m} - \lambda_n A \sum_{m=1}^{N} \delta_{n,m} z_{n,m}), \\ x_{n+1} = \alpha_n x_n + \alpha'_n(\beta_n \mathrm{Proj}_C(\sum_{m=1}^{N} \delta_{n,m} z_{n,m} - \lambda_n A y_n) + (1-\beta_n) T \mathrm{Proj}_C(\sum_{m=1}^{N} \delta_{n,m} z_{n,m} - \lambda_n A y_n)) + \alpha''_n e_n, \end{cases}$$

where $\{e_n\}$ *is a bounded sequence in* C. *Assume that* $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha''_n\}$, $\{\beta_n\}$, $\{\delta_{n,m}\}$, $\{\lambda_n\}$, $\{r_{n,m}\}$ *satisfy the following restrictions:*

(1) $\alpha_n + \alpha'_n + \alpha''_n = 1$, $0 < a \leq \alpha_n \leq b < 1$;

(2) $\kappa \leq \beta_n \leq c < 1$

(3) $\sum_{m=1}^{\infty} \delta_{n,m} = 1$, *and* $0 < d \leq \delta_{n,m} \leq 1$;

(4) $\liminf_{n\to\infty} r_{n,m} > 0$, $\sum_{n=1}^{\infty} |\alpha''_n| < \infty$ *and* $m_1 \leq \lambda_n \leq m_2$, *where* $m_1, m_2 \in (0, 1/L)$.

Then $\{x_n\}$ converges weakly to some point $\bar{x} \in \mathcal{F}$.

Proof. First, we prove that the sequence $\{x_n\}$ is bounded. Define $G_m(p, y) = F_m(p, y)+\langle B_m p, y-p\rangle+\varphi(y)-\varphi(p)$, $\forall p, y \in C$. It is easy to see that the bifunction G satisfies the conditions (A1)-(A4). Therefore, generalized mixed equilibrium problem is equivalent to the following equilibrium problem: find $p \in C$ such that $G_m(p, y) \geq 0$, $\forall y \in C$. Fix $p \in \mathcal{F}$ and set $u_n = \mathrm{Proj}_C(\sum_{m=1}^N \delta_{n,m} z_{n,m} - \lambda_n A y_n)$ and $v_n = \sum_{m=1}^N \delta_{n,m} z_{n,m}$. It follows that $y_n = \mathrm{Proj}_C(v_n - \lambda_n A v_n)$. Hence, we have

$$\begin{aligned}
\|u_n - p\|^2 &\leq \|v_n - \lambda_n A y_n - p\|^2 - \|v_n - \lambda_n A y_n - u_n\|^2 \\
&= \|v_n - p\|^2 - \|v_n - u_n\|^2 + 2\lambda_n(\langle A y_n - Ap, p - y_n\rangle + \langle Ap, p - y_n\rangle \\
&\quad + \langle A y_n, y_n - u_n\rangle) \\
&\leq \|v_n - p\|^2 - \|v_n - y_n\|^2 - \|y_n - u_n\|^2 + 2\langle v_n - \lambda_n A y_n - y_n, u_n - y_n\rangle.
\end{aligned}$$

Since A is Lipschitz continuous, we have

$$\begin{aligned}
\langle v_n - \lambda_n A y_n - y_n, u_n - y_n\rangle &\leq \|v_n - \lambda_n A y_n - y_n\|\|u_n - y_n\| \\
&\leq \lambda_n L\|v_n - y_n\|\|u_n - y_n\|.
\end{aligned}$$

Hence, we have

$$\|u_n - p\|^2 \leq \|v_n - p\|^2 + (\lambda_n^2 L^2 - 1)\|v_n - y_n\|^2. \tag{3.1}$$

Since $T_{r_{n,m}} = \{z \in C : G_m(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$ is firmly nonexpansive, we have

$$\begin{aligned}
\|v_n - p\|^2 &\leq \|\sum_{m=1}^N \delta_{n,m} z_{n,m} - p\|^2 \\
&\leq \sum_{m=1}^N \delta_{n,m}\|T_{r_{n,m}} x_n - p\|^2 \\
&\leq \|x_n - p\|^2.
\end{aligned} \tag{3.2}$$

Substituting (3.2) into (3.1), we find

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 + (\lambda_n^2 L^2 - 1)\|v_n - y_n\|^2.$$

Set $T_n = \beta_n I + (1 - \beta_n)T$. Using Lemma 2.2, we find that T_n is nonexpansive and $F(T_n) = F(T)$. Hence, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + \alpha_n'\|T_n u_n - p\|^2 + \alpha_n''\|e_n - p\| \\
&\leq \alpha_n\|x_n - p\|^2 + \alpha_n'\|u_n - p\|^2 + \alpha_n''\|e_n - p\| \\
&\leq \alpha_n\|x_n - p\|^2 + \alpha_n'(\|x_n - p\|^2 + (\lambda_n^2 L^2 - 1)\|v_n - y_n\|^2) + \alpha_n''\|e_n - p\| \\
&\leq \|x_n - p\|^2 + \alpha_n'(\lambda_n^2 L^2 - 1)\|v_n - y_n\|^2 + \alpha_n''\|e_n - p\| \\
&\leq \|x_n - p\|^2 + \alpha_n''\|e_n - p\|.
\end{aligned} \tag{2.3}$$

Using Lemma 2.3., we see that the $\lim_{n\to\infty} \|x_n - p\|$ exists. This obtains that $\{x_n\}$ is bounded. Since $\{x_n\}$ is bounded, we may assume that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to $\bar{x}$. Using (3.3), we find that

$$\beta_n(1 - \lambda_n^2 L^2)\|v_n - y_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n''\|e_n - p\|.$$

It follows from the restrictions (2) and (4), we see that $\lim_{n\to\infty} \|v_n - y_n\| = 0$. Note that

$$\|y_n - u_n\| \leq \lambda L\|v_n - y_n\|.$$

It follows that $\lim_{n\to\infty} \|y_n - u_n\| = 0$. This implies that

$$\lim_{n\to\infty} \|v_n - u_n\| = 0. \tag{3.4}$$

Using Lemma 2.4, we see that

$$\begin{aligned}\|z_{n,m}-p\|^2 &= \|T_{r_{n,m}}x_n - T_{r_{n,m}}p\|^2\\ &\le \langle T_{r_{n,m}}x_n - T_{r_{n,m}}p, x_n - p\rangle\\ &= \frac{1}{2}(\|z_{n,m}-p\|^2 + \|x_n-p\|^2 - \|z_{n,m}-x_n\|^2).\end{aligned}$$

It follows that

$$\|z_{n,m}-p\|^2 \le \|x_n-p\|^2 - \|z_{n,m}-x_n\|^2.$$

Since $v_n = \sum_{m=1}^N \delta_{n,m} z_{n,m}$, where $\sum_{m=1}^N \delta_{n,m} = 1$, we find that

$$\begin{aligned}\|v_n-p\|^2 &\le \sum_{m=1}^N \delta_{n,m}\|z_{n,m}-p\|^2\\ &\le \|x_n-p\|^2 - \sum_{m=1}^N \delta_{n,m}\|z_{n,m}-x_n\|^2.\end{aligned}$$

Since $\|\cdot\|^2$ is convex, we see that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n\|x_n-p\|^2 + \alpha_n'\|T_nu_n-p\|^2 + \alpha_n''\|e_n-p\|\\ &\le \alpha_n\|x_n-p\|^2 + \alpha_n'\|u_n-p\|^2 + \alpha_n''\|e_n-p\|\\ &\le \alpha_n\|x_n-p\|^2 + \alpha_n'\|v_n-p\|^2 + \alpha_n''\|e_n-p\|\\ &\le \|x_n-p\|^2 - \alpha_n'\sum_{m=1}^N \delta_{n,m}\|z_{n,m}-x_n\|^2 + \alpha_n''\|e_n-p\|.\end{aligned}$$

This implies that

$$(1-\alpha_n)\delta_{n,m}\|z_{n,m}-x_n\|^2 \le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \alpha_n''\|e_n-p\|.$$

It follows that

$$\lim_{n\to\infty}\|z_{n,m}-x_n\| = 0. \tag{2.5}$$

Based on the mapping A, define a maximal monotone mapping S by:

$$Sx = \begin{cases} Ax + N_Cx, & x \in C,\\ \emptyset, & x \notin C.\end{cases}$$

For any given $(x,y) \in G(S)$, we have $y - Ax \in N_Cx$. It follows that

$$\langle y - Ax, x - z\rangle \ge 0, \quad \forall z \in C.$$

Using the definition of u_n, we have

$$\langle x - u_n, \frac{u_n - v_n}{\lambda_n} + Ay_n\rangle \ge 0.$$

Since A is monotone, we have

$$\begin{aligned}\langle x-u_{n_i}, y\rangle &\ge \langle x-u_{n_i}, Ax\rangle\\ &\ge \langle x-u_{n_i}, Ax\rangle - \langle x-u_{n_i}, Ay_{n_i} + \frac{u_{n_i}-v_{n_i}}{\lambda_{n_i}}\rangle\\ &= \langle x-u_{n_i}, Ax - Au_{n_i}\rangle + \langle x-u_{n_i}, Au_{n_i} - Ay_{n_i}\rangle - \langle x-u_{n_i}, \frac{u_{n_i}-v_{n_i}}{\lambda_{n_i}}\rangle\\ &\ge \langle x-u_{n_i}, Au_{n_i} - Ay_{n_i}\rangle - \langle x-u_{n_i}, \frac{u_{n_i}-v_{n_i}}{\lambda_{n_i}}\rangle.\end{aligned}$$

Since $\|v_n - x_n\| \le \sum_{m=1}^{N} \delta_{n,m}\|z_{n,m} - x_n\|$, We find from (3.5) that $\lim_{n\to\infty} \|v_n - x_n\| = 0$. Note the fact that

$$\|u_n - x_n\| \le \|u_n - v_n\| + \|v_n - x_n\|.$$

It follows that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.6}$$

Since $\{x_{n_i}\}$ converges weakly to $\bar{x}$, we find that $u_{n_i} \rightharpoonup \bar{x}$. It follows that $\langle x - \xi, y\rangle \ge 0$. Since S is maximal monotone, we find that $0 \in S\bar{x}$. Using Lemma 2.6, we find that $\xi \in VI(C, A)$.

Next, we show that $\bar{x} \in \cap_{m=1}^{N} GMEP(F_m, B_m, \varphi_m)$. In view of (3.5), we see that $\{z_{n_i,m}\}$ converges weakly to $\bar{x}$ for each $m \ge 1$. Using the fact that $z_{n,m} = T_{r_{n,m}} x_n$, we have

$$F_m(z_{n,m}, z) + \langle B_m z_{n,m}, z - z_{n,m}\rangle + \varphi(z) - \varphi(z_{n,m}) + \frac{1}{r_{n,m}}\langle z - z_{n,m}, z_{n,m} - x_n\rangle \ge 0, \quad \forall z \in C.$$

Using the assumption (A2), we see that

$$\langle z - z_{n_i,m}, \frac{z_{n_i,m} - x_{n_i}}{r_{n_i,m}}\rangle \ge G_m(z, z_{n_i,m}), \quad \forall z \in C.$$

Using the assumption (A4), we see from (3.5) that $G_m(z, \bar{x}) \le 0$, $\forall z \in C$. For t_m with $0 < t_m \le 1$, and $z \in C$, set

$$z_{t_m} = (1 - t_m)\bar{x} + t_m z, \quad 1 \le m \le N.$$

Since $z_{t_m} \in C$, we find that $G_m(z_{t_m}, \bar{x}) \le 0$. Since

$$0 = G_m(z_{t_m}, z_{t_m}) \le t_m G_m(z_{t_m}, z) + (1 - t_m)G_m(z_{t_m}, \bar{x}) \le t_m G_m(z_{t_m}, z),$$

we see that $G_m(z_{t_m}, z) \ge 0$, $\forall z \in C$. Letting $t_m \downarrow 0$, one sees that $G_m(\bar{x}, z) \ge 0$, $\forall z \in C$. This implies that $\bar{x} \in GMEP(F_m, B_m, \varphi_m)$ for each $m \ge 1$. This proves that $\bar{x} \in \cap_{m=1}^{N} GMEP(F_m, B_m, \varphi_m)$.

Now, we are in a position to show that $\bar{x}$ is a fixed point of T. Since $\lim_{n\to\infty} \|x_n - p\|$ exists, we put $\lim_{n\to\infty} \|x_n - p\| = d > 0$. It follows that

$$\lim_{n\to\infty} \|x_{n+1} - p\| = \lim_{n\to\infty} \|\alpha_n\Big(x_n - p + \alpha_n''(e_n - T_n u_n)\Big) + (1 - \alpha_n)\Big(T_n u_n - p + \alpha_n''(e_n - T_n u_n)\Big)\| = d.$$

Note that

$$\begin{aligned}\limsup_{n\to\infty} \|x_n - p + \alpha_n''(e_n - T_n u_n)\| &\le \limsup_{n\to\infty} \|x_n - p\| + \limsup_{n\to\infty} \alpha_n''\|e_n - T_n u_n\| \\ &\le d\end{aligned}$$

and

$$\begin{aligned}\limsup_{n\to\infty} \|T_n u_n - p + \alpha_n''(e_n - T_n u_n)\| &\le \limsup_{n\to\infty} \|T_n u_n - p\| + \limsup_{n\to\infty} \alpha_n''\|e_n - T_n u_n\| \\ &\le \limsup_{n\to\infty} \|u_n - p\| + \limsup_{n\to\infty} \alpha_n''\|e_n - T_n u_n\| \\ &\le d.\end{aligned}$$

Using Lemma 2.7, we find

$$\lim_{n\to\infty} \|x_n - T_n u_n\| = 0. \tag{3.7}$$

On the other hand, we have

$$\begin{aligned}\|T_n x_n - x_n\| &\le \|T_n x_n - T_n u_n\| + \|T_n u_n - x_n\| \\ &\le \|x_n - u_n\| + \|T_n u_n - x_n\|.\end{aligned}$$

It follows from (3.6) and (3.7) that

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{3.8}$$

This implies from (3.8) that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Using Lemma 2.1, we find that $\bar{x} \in F(T)$. Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ converging weakly to ξ, where $\xi \neq \bar{x}$. Similarly, we find that $\xi \in \mathcal{F}$. Using Opial's condition, we find that

$$\begin{aligned} d &= \liminf_{i\to\infty}\|x_{n_i} - \bar{x}\| < \liminf_{i\to\infty}\|x_{n_i} - \xi\| \\ &= \liminf_{j\to\infty}\|x_j - \xi\| < \liminf_{j\to\infty}\|x_j - \bar{x}\| = d. \end{aligned}$$

This is a contradiction. Hence $\bar{x} = \xi$. This completes the proof.

If T is nonexpansive, we have the following.

Corollary 3.2. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $A : C \to H$ be a L-Lipschitz continuous and monotone mapping and let $T : C \to C$ be a nonexpansive mapping. Let $N \geq 1$ be some positive integer. Let F_m be a bifunction from $C\times C$ to $\mathbb{R}$ which satisfies (A1)-(A4). Let $\varphi_m : C \to \mathbb{R}$ be a lower semicontinuous and convex function and let $B_m : C \to H$ be a continuous and monotone mapping for each $1 \leq m \leq N$. Assume that $\mathcal{F} := \cap_{m=1}^{N} GMEP(F_m, B_m, \varphi_m) \cap F(T) \cap VI(C, A) \neq \emptyset$. Let $\{\lambda_n\}$, $\{r_{n,m}\}$ be positive real number sequences. Let $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha''_n\}$ and $\{\delta_{n,m}\}$ be real number sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence generated in the following process:*

$$\begin{cases} x_1 \in H, \\ F_m(z_{n,m}, z) + \langle B_m z_{n,m}, z - z_{n,m}\rangle + \varphi_m(z) - \varphi_m(z_{n,m}) + \frac{1}{r_{n,m}}\langle z - z_{n,m}, z_{n,m} - x_n\rangle \geq 0, \quad \forall z \in C, \\ y_n = \mathrm{Proj}_C(\sum_{m=1}^{N}\delta_{n,m}z_{n,m} - \lambda_n A\sum_{m=1}^{N}\delta_{n,m}z_{n,m}), \\ x_{n+1} = \alpha_n x_n + \alpha'_n T\mathrm{Proj}_C(\sum_{m=1}^{N}\delta_{n,m}z_{n,m} - \lambda_n A y_n) + \alpha''_n e_n, \end{cases}$$

where $\{e_n\}$ is a bounded sequence in C. Assume that $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha''_n\}$, $\{\delta_{n,m}\}$, $\{\lambda_n\}$, $\{r_{n,m}\}$ satisfy the following restrictions:

(1) $\alpha_n + \alpha'_n + \alpha''_n = 1$, $0 < a \leq \alpha_n \leq b < 1$;

(2) $\sum_{m=1}^{\infty}\delta_{n,m} = 1$, *and* $0 < d \leq \delta_{n,m} \leq 1$;

(3) $\liminf_{n\to\infty} r_{n,m} > 0$, $\sum_{n=1}^{\infty}|\alpha''_n| < \infty$ *and* $m_1 \leq \lambda_n \leq m_2$, *where* $m_1, m_2 \in (0, 1/L)$.

Then $\{x_n\}$ converges weakly to some point $\bar{x} \in \mathcal{F}$.

If $B_m = 0$, we find the following result on mixed equilibrium problem.

Corollary 3.3. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $A : C \to H$ be a L-Lipschitz continuous and monotone mapping and let $T : C \to C$ be a κ-strictly psuedocontractive mapping. Let $N \geq 1$ be some positive integer. Let F_m be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)-(A4). Let $\varphi_m : C \to \mathbb{R}$ be a lower semicontinuous and convex function for each $1 \leq m \leq N$. Assume that $\mathcal{F} := \cap_{m=1}^{N} GMEP(F_m, \varphi_m) \cap F(T) \cap VI(C, A) \neq \emptyset$. Let $\{\lambda_n\}$, $\{r_{n,m}\}$ be positive real number sequences. Let $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha''_n\}$, $\{\beta_n\}$ and $\{\delta_{n,m}\}$ be real number sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence generated in the following process:*

$$\begin{cases} x_1 \in H, \\ y_n = \mathrm{Proj}_C(\sum_{m=1}^{N}\delta_{n,m}z_{n,m} - \lambda_n A\sum_{m=1}^{N}\delta_{n,m}z_{n,m}), \\ x_{n+1} = \alpha_n x_n + \alpha'_n(\beta_n\mathrm{Proj}_C(\sum_{m=1}^{N}\delta_{n,m}z_{n,m} - \lambda_n A y_n) + (1-\beta_n)T\mathrm{Proj}_C(\sum_{m=1}^{N}\delta_{n,m}z_{n,m} - \lambda_n A y_n)) + \alpha''_n e_n, \end{cases}$$

where $\{e_n\}$ is a bounded sequence in C and $z_{n,m}$ is such that

$$F_m(z_{n,m}, z) + \varphi_m(z) - \varphi_m(z_{n,m}) + \frac{1}{r_{n,m}}\langle z - z_{n,m}, z_{n,m} - x_n\rangle \geq 0, \quad \forall z \in C.$$

Assume that $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha''_n\}$, $\{\beta_n\}$, $\{\delta_{n,m}\}$, $\{\lambda_n\}$, $\{r_{n,m}\}$ satisfy the following restrictions:

(1) $\alpha_n + \alpha'_n + \alpha''_n = 1$, $0 < a \leq \alpha_n \leq b < 1$;

(2) $\kappa \leq \beta_n \leq c < 1$

(3) $\sum_{m=1}^{\infty} \delta_{n,m} = 1$, *and* $0 < d \leq \delta_{n,m} \leq 1$;

(4) $\liminf_{n\to\infty} r_{n,m} > 0$, $\sum_{n=1}^{\infty} |\alpha''_n| < \infty$ *and* $m_1 \leq \lambda_n \leq m_2$, *where* $m_1, m_2 \in (0, 1/L)$.

Then $\{x_n\}$ *converges weakly to some point* $\bar{x} \in \mathcal{F}$.

If $A = 0$, we find from Theorem 3.1 the following result.

Corollary 3.4. *Let* C *be a nonempty closed convex subset of a Hilbert space* H. *Let* $A : C \to H$ *be a L-Lipschitz continuous and monotone mapping and let* $T : C \to C$ *be a* κ*-strictly psuedocontractive mapping. Let* $N \geq 1$ *be some positive integer. Let* F_m *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *which satisfies* $(A1)$-$(A4)$. *Let* $\varphi_m : C \to \mathbb{R}$ *be a lower semicontinuous and convex function and let* $B_m : C \to H$ *be a continuous and monotone mapping for each* $1 \leq m \leq N$. *Assume that* $\mathcal{F} := \cap_{m=1}^{N} GMEP(F_m, B_m, \varphi_m) \cap F(T) \cap VI(C, A) \neq \emptyset$. *Let* $\{\lambda_n\}$, $\{r_{n,m}\}$ *be positive real number sequences. Let* $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha''_n\}$, $\{\beta_n\}$ *and* $\{\delta_{n,m}\}$ *be real number sequences in* $(0, 1)$. *Let* $\{x_n\}$ *be a sequence generated in the following process:*

$$\begin{cases} x_1 \in H, \\ F_m(z_{n,m}, z) + \langle B_m z_{n,m}, z - z_{n,m}\rangle + \varphi_m(z) - \varphi_m(z_{n,m}) + \frac{1}{r_{n,m}}\langle z - z_{n,m}, z_{n,m} - x_n\rangle \geq 0, \quad \forall z \in C, \\ y_n = \sum_{m=1}^{N} \delta_{n,m} z_{n,m}, \\ x_{n+1} = \alpha_n x_n + \alpha'_n(\beta_n \mathrm{Proj}_C(\sum_{m=1}^{N} \delta_{n,m} z_{n,m} - \lambda_n A y_n) + (1-\beta_n) T \mathrm{Proj}_C(\sum_{m=1}^{N} \delta_{n,m} z_{n,m} - \lambda_n A y_n)) + \alpha''_n e_n, \end{cases}$$

where $\{e_n\}$ *is a bounded sequence in* C. *Assume that* $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha''_n\}$, $\{\beta_n\}$, $\{\delta_{n,m}\}$, $\{\lambda_n\}$, $\{r_{n,m}\}$ *satisfy the following restrictions:*

(1) $\alpha_n + \alpha'_n + \alpha''_n = 1$, $0 < a \leq \alpha_n \leq b < 1$;

(2) $\kappa \leq \beta_n \leq c < 1$

(3) $\sum_{m=1}^{\infty} \delta_{n,m} = 1$, *and* $0 < d \leq \delta_{n,m} \leq 1$;

(4) $\liminf_{n\to\infty} r_{n,m} > 0$, $\sum_{n=1}^{\infty} |\alpha''_n| < \infty$.

Then $\{x_n\}$ *converges weakly to some point* $\bar{x} \in \mathcal{F}$.

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References

[1] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Student 63 (1994) 123-145.

[2] F.E. Browder, W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert spaces, J. Math. Anal. Appl. 20 (1967) 197–228.

[3] S.Y. Cho, X. Qin, On the strong convergence of an iterative process for asymptotically strict pseudocontractions and equilibrium problems, Appl. Math. Comput. 235 (2014) 430-438.

[4] S.Y. Cho, X. Qin, S.M. Kang, Iterative processes for common fixed points of two different families of mappings with applications, J. Global Optim. 57 (2013) 1429-1446.

[5] J.H. Chen, Iterations for equilibrium and fixed point problems, J. Nonlinear Funct. Anal. 2013 (2013) Article ID 4.

[6] S.Y. Cho, S.M. Kang, Approximation of common solutions of variational inequalities via strict pseudocontractions, Acta Math. Sci. 32 (2012), 1607-1618.

[7] S.Y. Cho, W. Li, S.M. Kang, Convergence analysis of an iterative algorithm for monotone operators, J. Inequal. Appl. 2013 (2013) Article ID 199.

[8] K. Fan, A minimax inequality and its application, in: O. Shisha (Ed.), Inequalities, vol. 3, Academic Press, New York, 1972, 103-113.

[9] R.H. He, Coincidence theorem and existence theorems of solutions for a system of Ky Fan type minimax inequalities in FC-spaces, Adv. Fixed Point Theory, 2 (2012) 47-57.

[10] S. Lv, C. Wu, Convergence of iterative algorithms for a generalized variational inequality and a nonexpansive mapping, Eng. Math. Lett. 1 (2012) 44-57.

[11] S. Park, A review of the KKM theory on ϕ_A-space or GFC-spaces, Adv. Fixed Point Theory 3 (2013) 355-382.

[12] S. Lv, Generalized systems of variational inclusions involving (A, η)-monotone mappings, Adv. Fixed Point Theory 1 (2011) 15-26.

[13] H. Iiduka, Strong convergence for an iterative method for the triple-hierarchical constrained optimization problem, Nonlinear Anal. 71 (2009) e1292-e1297.

[14] P.Q. Khanh, L.M. Luu, On the existence of solutions to vector quasivariational inequalities and quasicomplementarity problems with applications to traffic network equilibria, J. Optim. Theory Appl. 123 (2004) 533-548.

[15] J.K. Kim, Strong convergence theorems by hybrid projection methods for equilibrium problems and fixed point problems of the asymptotically quasi-ϕ-nonexpansive mappings, Fixed Point Theory Appl. 2011 (2011) Article ID 10.

[16] D.Y. Lin, P.W. Leong, An N-path user equilibrium for transportation networks, Appl. Math. Modelling 38 (2014) 667-682.

[17] N. Nadezhkina, W. Takahashi, Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings, J. Optim. Theory Appl. 128 (2006) 191-201.

[18] S. Plubtieng, W. Sriprad, An extragradient method and proximal point algorithm for inverse strongly monotone operators and maximal monotone operators in Banach spaces, Fixed Point Theory Appl. 2009 (2009) Article ID 591874.

[19] X. Qin, S.Y. Cho, S.M. Kang, An extragradient-type method for generalized equilibrium problems involving strictly pseudocontractive mappings, J. Global Optim. 49 (2011) 679-693.

[20] X. Qin, M. Shang, Y. Su, Strong convergence of a general iterative algorithm for equilibrium problems and variational inequality problems, Math. Comput. Modelling 48 (2008) 1033-1046.

[21] R.T. Rockafellar, On the maximality of sums of nonlinear monotone operators, Trans. Amer. Math. Soc. 149 (1970) 75-88.

[22] J. Schu, Weak and strong convergence of fixed points of asymptotically nonexpansive mappings, Bull. Austral. Math. Soc. 43 (1991) 153-159.

[23] W. Takahashi, M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone Mappings, J. Optim. Theory Appl.118 (2003) 417-428.

[24] K.K. Tan, H.K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iterative process, J. Math. Anal. Appl. 178 (1993) 301-308.

[25] R. Wangkeeree, An extragradient approximation method for equilibrium problems and fixed point problems of a countable family of nonexpansive mappings, Fixed Point Theory Appl. 2008 (2008) Article ID 134148.

[26] Z.M. Wang, W. Lou, A new iterative algorithm of common solutions to quasi-variational inclusion and fixed point problems, J. Math. Comput. Sci. 3 (2013) 57-72.

[27] L. Yang, F. Zhao, J.K. Kim, Hybrid projection method for generalized mixed equilibrium problem and fixed point problem of infinite family of asymptotically quasi-ϕ-nonexpansive mappings in Banach spaces, Appl. Math. Comput. 218 (2012) 6072-6082.

[28] H. Zegeye, N. Shahzad, Strong convergence theorem for a common point of solution of variational inequality and fixed point problem, Adv. Fixed Point Theory, 2 (2012), 374-397.