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Approximation by Complex Modified Szász-Mirakjan-Stancu Operators in Compact Disks

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Abstract. In this paper, we establish some theorems on approximation and Voronovskaja type results for complex modified Szász-Mirakjan-Stancu operators attached to analytic functions having exponential growth on compact disks. Also, we estimate the rate of convergence and the exact order of approximation.

1. Introduction

For a real function of real variable $f:[0,\infty)\to\mathbb{R}$, Szász-Mirakjan operators are defined as

$$S_n(f;x)=e^{-nx}\sum_{j=0}^{\infty}\frac{(nx)^j}{j!}f\left(\frac{j}{n}\right)\ ,\ x\in[0,\infty)\,,$$

where the convergence of $S_n(f;x)$ to $f(x)$ under the exponential growth condition on f that is $\left|f(x)\right|\leq Ce^{Bx}$, for all $x\in[0,\infty)$, with $C,B>0$ was proved in [2].

Concerning the convergence of complex Szász-Mirakjan operators in the complex plane, the first result was established by J.J. Gergen, F.G. Dressel and W.H. Purcell [9]. Note that in the above mentioned paper, no quantitative estimate of this convergence result was obtained. Then, S. G. Gal [4] obtained quantitative estimates for the convergence and Voronovskaja's theorem of complex Szász-Mirakjan operators attached to analytic functions satisfying a suitable exponential-type growth condition. In [5], for the analytic functions without exponential-type growth conditions, S. G. Gal gave Voronovskaja type result with quantitative estimate and the exact order in approximation for these operators. We may also mention that similar results for the well-known complex approximating operators were obtained by S. G. Gal in his book [3].

Very recently, N. Çetin and N. İspir [1] introduced the complex modified Szász-Mirakjan operators, which are defined by

$$S_n(f;a_n,b_n;z)=e^{-\frac{a_n}{b_n}z}\sum_{j=0}^{\infty}\frac{(a_nz)^j}{j!b_n^j}f(\frac{jb_n}{a_n})\quad z\in\mathbb{C}\,;n\in\mathbb{N} \tag{1}$$

where $\{a_n\},\{b_n\}$ are given sequences of strictly positive numbers such that $\lim\limits_{n\to\infty}\frac{b_n}{a_n}=0$ and $\frac{b_n}{a_n}\leq 1$. In [1], the authors obtained Voronovskaja type results and estimated the exact orders of approximation and also

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proved that the complex modified Szász-Mirakjan operators preserve the geometric properties in unit disk. Recently, many researchers have studied intensively Stancu-type generalization of several complex operators (see [6–8, 10, 11]). Inspired by such type operators we would like to study the Stancu-type generalization of the operators (1).

In the present paper, we introduce the complex modified Szász-Mirakjan-Stancu operators as follows:

$$S_n^{(\alpha,\beta)}(f;a_n,b_n;z)=e^{-\frac{a_n}{b_n}z}\sum_{j=0}^{\infty}\frac{(a_nz)^j}{j!b_n^j}f\left(\frac{(j+\alpha)\,b_n}{a_n+\beta b_n}\right)\quad z\in\mathbb{C}\,;n\in\mathbb{N} \tag{2}$$

where $\{a_n\},\{b_n\}$ are given sequences of strictly positive numbers such that $\lim_{n\to\infty}\frac{b_n}{a_n}=0$ and $\frac{b_n}{a_n}\leq 1$ and α,β are two given real parameters satisfying the condition $0\leq\alpha\leq\beta$. Also, $D_R=\{z\in\mathbb{C}:|z|<R,\ 1<R<\infty\}$, the function $f:[R,\infty)\cup\overline{D_R}\to\mathbb{C}$ is continuous in $[R,\infty)\cup\overline{D_R}$, analytic in D_R and f has a suitable exponential growth condition in defined domain. We note that for $\alpha=\beta=0$, these operators become the complex modified Szász-Mirakjan operators defined by (1).

In this study, we investigate approximation properties of the complex modified Szász-Mirakjan-Stancu operators attached to analytic functions having suitable exponential growth on compact disks. Then, we obtain Voronovskaja type results and estimate the exact orders in approximation by complex modified Szász-Mirakjan-Stancu operators and their derivatives.

2. Auxiliary Results

Now, we will give the following auxiliary results which include some properties of the operators defined by (1) and (2).

Lemma 2.1. *For all $n\in\mathbb{N},k\in\mathbb{N}\cup\{0\}$, $0\leq\alpha\leq\beta$ and $z\in\mathbb{C}$, we have*

$$S_n^{(\alpha,\beta)}(e_{k+1};a_n,b_n;z)=\frac{b_nz}{a_n+\beta b_n}\left(S_n^{(\alpha,\beta)}(e_k;a_n,b_n;z)\right)'+\frac{a_nz+\alpha b_n}{a_n+\beta b_n}S_n^{(\alpha,\beta)}(e_k;a_n,b_n;z) \tag{3}$$

where $e_k(z)=z^k$.

Proof. From the formula (2), we can write

$$S_n^{(\alpha,\beta)}(e_k;a_n,b_n;z)=e^{-\frac{a_n}{b_n}z}\sum_{j=0}^{\infty}\frac{(a_nz)^j}{j!b_n^j}\left(\frac{(j+\alpha)\,b_n}{a_n+\beta b_n}\right)^k.$$

Differentiating with respect to $z\neq 0$, by direct computation, we get

$$\begin{aligned}
&\left(S_n^{(\alpha,\beta)}(e_k;a_n,b_n;z)\right)'\\
&=-\frac{a_n}{b_n}e^{-\frac{a_n}{b_n}z}\sum_{j=0}^{\infty}\frac{(a_nz)^j}{j!b_n^j}\left(\frac{(j+\alpha)\,b_n}{a_n+\beta b_n}\right)^k+e^{-\frac{a_n}{b_n}z}\sum_{j=0}^{\infty}\frac{j\,(a_nz)^{j-1}a_n}{j!b_n^j}\left(\frac{(j+\alpha)\,b_n}{a_n+\beta b_n}\right)^k\\
&=-\frac{a_n}{b_n}S_n^{(\alpha,\beta)}(e_k;a_n,b_n;z)+\frac{a_n+\beta b_n}{b_nz}e^{-\frac{a_n}{b_n}z}\sum_{j=0}^{\infty}\frac{(a_nz)^j}{j!b_n^j}\left(\frac{(j+\alpha)\,b_n}{a_n+\beta b_n}\right)^{k+1}-\frac{\alpha}{z}e^{-\frac{a_n}{b_n}z}\sum_{j=0}^{\infty}\frac{(a_nz)^j}{j!b_n^j}\left(\frac{(j+\alpha)\,b_n}{a_n+\beta b_n}\right)^k\\
&=-\frac{a_n}{b_n}S_n^{(\alpha,\beta)}(e_k;a_n,b_n;z)+\frac{a_n+\beta b_n}{b_nz}S_n^{(\alpha,\beta)}(e_{k+1};a_n,b_n;z)-\frac{\alpha}{z}S_n^{(\alpha,\beta)}(e_k;a_n,b_n;z)\\
&=-\frac{a_nz+\alpha b_n}{b_nz}S_n^{(\alpha,\beta)}(e_k;a_n,b_n;z)+\frac{a_n+\beta b_n}{b_nz}S_n^{(\alpha,\beta)}(e_{k+1};a_n,b_n;z),
\end{aligned}$$

which implies the recurrence relation in the statement. □

Lemma 2.2. *Let α, β be satisfying $0 \leq \alpha \leq \beta$. Denoting $e_v(z) = z^v$ and $S_n^{(0,0)}(e_v; a_n, b_n)$ by $S_n(e_v; a_n, b_n)$, for all $n, k \in \mathbb{N} \cup \{0\}$, we have the following recursive relation for the images of the monomials e_k under $S_n^{(\alpha,\beta)}$ in terms of $S_n(e_v; a_n, b_n)$, $v = 0, 1, ...k$,*

$$S_n^{(\alpha,\beta)}(e_k; a_n, b_n; z) = \sum_{v=0}^{k} \binom{k}{v} \frac{a_n^v (\alpha b_n)^{k-v}}{(a_n + \beta b_n)^k} S_n(e_v; a_n, b_n; z).$$

Proof. This formula can be easily proved by mathematical induction. It is clear that this formula is true for $k = 0$. Now supposing that it is true for $k = r$, it implies

$$S_n^{(\alpha,\beta)}(e_r; a_n, b_n; z) = \sum_{v=0}^{r} \binom{r}{v} \frac{a_n^v (\alpha b_n)^{r-v}}{(a_n + \beta b_n)^r} S_n(e_v; a_n, b_n; z).$$

Using (3), we obtain

$$\begin{aligned} S_n^{(\alpha,\beta)}(e_{r+1}; a_n, b_n; z) &= \frac{b_n z}{a_n + \beta b_n} \sum_{v=0}^{r} \binom{r}{v} \frac{a_n^v (\alpha b_n)^{r-v}}{(a_n + \beta b_n)^r} (S_n(e_v; a_n, b_n; z))' \\ &\quad + \frac{a_n z + \alpha b_n}{a_n + \beta b_n} \sum_{v=0}^{r} \binom{r}{v} \frac{a_n^v (\alpha b_n)^{r-v}}{(a_n + \beta b_n)^r} S_n(e_v; a_n, b_n; z) \\ &= \sum_{v=0}^{r} \binom{r}{v} \frac{a_n^{v+1} (\alpha b_n)^{r-v}}{(a_n + \beta b_n)^{r+1}} \left\{ \frac{z b_n}{a_n} (S_n(e_v; a_n, b_n; z))' + \left(z + \frac{\alpha b_n}{a_n} \right) S_n(e_v; a_n, b_n; z) \right\}. \end{aligned}$$

By applying the recurrence formula for the complex modified Szász-Mirakjan operators obtained in [1], proof of Theorem 3, namely

$$S_n(e_{v+1}; a_n, b_n; z) = \frac{z b_n}{a_n} (S_n(e_v; a_n, b_n; z))' + z S_n(e_v; a_n, b_n; z),$$

it follows that

$$\begin{aligned} S_n^{(\alpha,\beta)}(e_{r+1}; a_n, b_n; z) &= \sum_{v=0}^{r} \binom{r}{v} \frac{a_n^{v+1} (\alpha b_n)^{r-v}}{(a_n + \beta b_n)^{r+1}} \left\{ S_n(e_{v+1}; a_n, b_n; z) + \frac{\alpha b_n}{a_n} S_n(e_v; a_n, b_n; z) \right\} \\ &= \sum_{v=1}^{r+1} \binom{r}{v-1} \frac{a_n^v (\alpha b_n)^{r-v+1}}{(a_n + \beta b_n)^{r+1}} S_n(e_v; a_n, b_n; z) + \sum_{v=0}^{r} \binom{r}{v} \frac{a_n^v (\alpha b_n)^{r-v+1}}{(a_n + \beta b_n)^{r+1}} S_n(e_v; a_n, b_n; z) \\ &= \sum_{v=0}^{r+1} \binom{r+1}{v} \frac{a_n^v (\alpha b_n)^{r-v+1}}{(a_n + \beta b_n)^{r+1}} S_n(e_v; a_n, b_n; z). \end{aligned}$$

This completes the proof of lemma. □

Lemma 2.3. *If we denote $S_n(e_k; a_n, b_n; z) = S_n^{(0,0)}(e_k; a_n, b_n; z)$, where $e_k(z) = z^k$, then for all $|z| \leq r$ with $r \geq 1$, $n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$, we have*

$$|S_n(e_k; a_n, b_n; z)| \leq k! r^k.$$

Proof. We use the following recurrence formula obtained in the proof of Theorem 3 (i) in [1]

$$S_n(e_{k+1};a_n,b_n;z)=\frac{zb_n}{a_n}\left(S_n(e_k;a_n,b_n;z)\right)'+zS_n(e_k;a_n,b_n;z)$$

for all $z\in\mathbb{C},k\in\mathbb{N}\cup\{0\},n\in\mathbb{N}$. Clearly, since $S_n(e_0;a_n,b_n;z)=1$, we get

$$|S_n(e_1;a_n,b_n;z)|\leq r$$

for all $|z|\leq r$. Then, for $k=1$ we obtain

$$|S_n(e_2;a_n,b_n;z)|\leq\frac{rb_n}{a_n}\left|\left(S_n(e_1;a_n,b_n;z)\right)'\right|+r\left|S_n(e_1;a_n,b_n;z)\right|.$$

Taking into account that from Lemma 2 in [1], $S_n(e_k;a_n,b_n;z)$ is a polynomial of degree k, by the well-known Bernstein's inequality we get

$$\left|\left(S_n(e_k;a_n,b_n;z)\right)'\right|\leq\frac{k}{r}\max\left\{|S_n(e_k;a_n,b_n;z)|:|z|\leq r\right\}.$$

Therefore, by the last inequality, we have

$$\begin{aligned}|S_n(e_2;a_n,b_n;z)| &\leq \frac{b_n}{a_n}\|S_n(e_1;a_n,b_n;z)\|_r+r\left|S_n(e_1;a_n,b_n;z)\right|\\ &\leq r\left(r+\frac{b_n}{a_n}\right).\end{aligned}$$

By writing for $k=2,3,...$, step by step we easily obtain

$$\begin{aligned}|S_n(e_k;a_n,b_n;z)| &\leq \prod_{j=1}^{k}\left[r+(j-1)\frac{b_n}{a_n}\right]\\ &\leq r^k\prod_{j=1}^{k}\left[1+(j-1)\frac{b_n}{a_n}\right]\leq r^k k!\end{aligned}$$

for all $|z|\leq r,k\in\mathbb{N}\cup\{0\},n\in\mathbb{N}$. □

3. Approximation by Complex Modified Szász-Mirakjan-Stancu Operators

Upper estimates for $S_n^{(\alpha,\beta)}(f;a_n,b_n;z)$ can be expressed by the following theorem.

Theorem 3.1. *Let* $D_R=\{z\in\mathbb{C}:|z|<R\}$ *be with* $1<R<+\infty$ *and suppose that* $f:[R,+\infty)\cup\overline{D_R}\to\mathbb{C}$ *is continuous in* $[R,+\infty)\cup\overline{D_R}$ *and analytic in* D_R*, i.e.* $f(z)=\sum_{k=0}^{\infty}c_kz^k$*, for all* $z\in D_R$*, and that there exist* $M,C,B>0$ *and* $A\in\left(\frac{1}{R},1\right)$*, with the property* $|c_k|\leq M\frac{A^k}{k!}$*, for all* $k=0,1,2,...$*, (which implies* $\left|f(z)\right|\leq Me^{A|z|}$ *for all* $z\in D_R$ *) and* $\left|f(x)\right|\leq Ce^{Bx}$*, for all* $x\in[R,+\infty)$.

i) Let $0\leq\alpha\leq\beta$ *and* $1\leq r<\frac{1}{A}$ *be arbitrary fixed. For all* $|z|\leq r$ *and* $n\in\mathbb{N}$*, we have*

$$\left|S_n^{(\alpha,\beta)}(f;a_n,b_n;z)-f(z)\right|\leq\frac{b_n\left[a_n(1+\beta)+\beta b_n\right]}{a_n\left(a_n+\beta b_n\right)}C_{r,A},$$

where

$$C_{r,A}=M\sum_{k=1}^{\infty}(k+1)(rA)^k<\infty.$$

ii) Let $0 \le \alpha \le \beta$ *and* $1 \le r < r_1 < \frac{1}{A}$. *Then, for all* $|z| \le r$ *and* $n, p \in \mathbb{N}$, *we have*

$$\left| \left(S_n^{(\alpha,\beta)}(f; a_n, b_n; z) \right)^{(p)} - f^{(p)}(z) \right| \le \frac{b_n \left[a_n (1+\beta) + \beta b_n \right]}{a_n (a_n + \beta b_n)} \frac{C_{r_1,A} p! r_1}{(r_1 - r)^{p+1}},$$

where $C_{r_1,A}$ *is given as at the above point (i).*

Proof. *i*) Reasoning exactly as in the case of complex modified Szász-Mirakjan operators in the proof of Theorem 3 (*i*) in [1], we can write

$$S_n^{(\alpha,\beta)}(f; a_n, b_n; z) = \sum_{k=0}^{\infty} c_k S_n^{(\alpha,\beta)}(e_k; a_n, b_n; z)$$

for all $z \in D_R$, which immediately implies

$$\left| S_n^{(\alpha,\beta)}(f; a_n, b_n; z) - f(z) \right| \le \sum_{k=1}^{\infty} |c_k| \left| S_n^{(\alpha,\beta)}(e_k; a_n, b_n; z) - e_k(z) \right|$$

since $S_n^{(\alpha,\beta)}(e_0; a_n, b_n; z) = 1$ for all $z \in \mathbb{C}$. By using Lemma 2.2, we obtain

$$\begin{aligned} S_n^{(\alpha,\beta)}(e_k; a_n, b_n; z) - e_k(z) = &\sum_{v=0}^{k-1} \binom{k}{v} \frac{a_n^v (\alpha b_n)^{k-v}}{(a_n + \beta b_n)^k} \left[S_n(e_v; a_n, b_n; z) - e_v(z) \right] + \sum_{v=0}^{k-1} \binom{k}{v} \frac{a_n^v (\alpha b_n)^{k-v}}{(a_n + \beta b_n)^k} e_v(z) \\ &+ \frac{a_n^k}{(a_n + \beta b_n)^k} \left[S_n(e_k; a_n, b_n; z) - e_k(z) \right] + \left(\frac{a_n^k}{(a_n + \beta b_n)^k} - 1 \right) e_k(z), \end{aligned}$$

which by passing to the norm $\|.\|_r$ implies

$$\begin{aligned} \left\| S_n^{(\alpha,\beta)}(e_k; a_n, b_n) - e_k \right\|_r \le &\sum_{v=0}^{k-1} \binom{k}{v} \frac{a_n^v (\alpha b_n)^{k-v}}{(a_n + \beta b_n)^k} \| S_n(e_v; a_n, b_n) - e_v \|_r + \sum_{v=0}^{k-1} \binom{k}{v} \frac{a_n^v (\alpha b_n)^{k-v}}{(a_n + \beta b_n)^k} r^v \\ &+ \frac{a_n^k}{(a_n + \beta b_n)^k} \| S_n(e_k; a_n, b_n) - e_k \|_r + \left(1 - \frac{a_n^k}{(a_n + \beta b_n)^k} \right) r^k \end{aligned}$$

for all $|z| \le r$. Using the inequalities

$$\| S_n(e_k; a_n, b_n) - e_k \|_r \le \frac{(k+1)!}{2} \frac{b_n}{a_n} r^{k-1}$$

obtained in the proof of Theorem 3 (*i*) in [1] and

$$1 - \prod_{j=1}^{k} x_j \le \sum_{j=1}^{k} \left(1 - x_j \right), \quad 0 \le x_j \le 1, \quad j = 1, 2, ..., k,$$

we get

$$\left\|S_n^{(\alpha,\beta)}(e_k;a_n,b_n)-e_k\right\|_r \leq \left(\frac{a_n+\alpha b_n}{a_n+\beta b_n}\right)^k \frac{(k+1)!}{2}\frac{b_n}{a_n}r^{k-1}+\left[\left(\frac{a_n+\alpha b_n}{a_n+\beta b_n}\right)^k-\frac{a_n^k}{(a_n+\beta b_n)^k}\right]r^k$$
$$+\left(1-\frac{a_n^k}{(a_n+\beta b_n)^k}\right)r^k$$
$$\leq (k+1)!\frac{b_n}{a_n}r^{k-1}+2r^k\left(1-\frac{a_n^k}{(a_n+\beta b_n)^k}\right)$$
$$\leq (k+1)!\frac{b_n}{a_n}r^{k-1}+2r^k\frac{k\beta b_n}{a_n+\beta b_n}$$
$$\leq \frac{b_n\left[a_n(1+\beta)+\beta b_n\right]}{a_n(a_n+\beta b_n)}(k+1)!r^k.$$

This immediately implies

$$\left|S_n^{(\alpha,\beta)}(f;a_n,b_n;z)-f(z)\right| \leq \sum_{k=1}^{\infty}|c_k|\left|S_n^{(\alpha,\beta)}(e_k;a_n,b_n;z)-e_k(z)\right|$$
$$\leq \sum_{k=1}^{\infty}M\frac{A^k}{k!}\frac{b_n\left[a_n(1+\beta)+\beta b_n\right]}{a_n(a_n+\beta b_n)}(k+1)!r^k$$
$$=\frac{b_n\left[a_n(1+\beta)+\beta b_n\right]}{a_n(a_n+\beta b_n)}M\sum_{k=1}^{\infty}(k+1)(rA)^k$$
$$=\frac{b_n\left[a_n(1+\beta)+\beta b_n\right]}{a_n(a_n+\beta b_n)}C_{r,A},$$

where

$$C_{r,A}=M\sum_{k=1}^{\infty}(k+1)(rA)^k<\infty$$

for all $1\leq r<\frac{1}{A}$. We note that $f(z)=\sum\limits_{k=1}^{\infty}z^{k+1}$ and its derivative $f'(z)=\sum\limits_{k=1}^{\infty}(k+1)z^k$ are absolutely and uniformly convergent in any compact disk included in the open unit disk.

ii) Denoting by γ the circle of radius $r_1>r$ and center 0, for any $|z|\leq r$ and $v\in\gamma$, we have $|v-z|\geq r_1-r$. By the Cauchy's formula, for all $|z|\leq r$ and $n\in\mathbb{N}$, it follows

$$\left|\left(S_n^{(\alpha,\beta)}(f;a_n,b_n;z)\right)^{(p)}-f^{(p)}(z)\right| = \frac{p!}{2\pi}\left|\int_\gamma \frac{S_n^{(\alpha,\beta)}(f;a_n,b_n;v)-f(v)}{(v-z)^{p+1}}dv\right|$$
$$\leq \frac{b_n\left[a_n(1+\beta)+\beta b_n\right]}{a_n(a_n+\beta b_n)}C_{r_1,A}\frac{p!r_1}{(r_1-r)^{p+1}}$$

which proves (*ii*) and the theorem. □

Now, we give Voronovskaja-type result in compact disks for $S_n^{(\alpha,\beta)}(f;a_n,b_n;z)$.

Theorem 3.2. *Suppose that the hypotheses on the function f and on the constants R,M,C,B,A in the statement of Theorem 3.1 hold. Also, let $0\leq\alpha\leq\beta$ and $1\leq r<\frac{1}{A}$. Then, for all $n\in\mathbb{N}$ and $|z|\leq r$, we have the following*

Voronovskaja-type result

$$\left| S_n^{(\alpha,\beta)}(f;a_n,b_n;z) - f(z) - \frac{(\alpha-\beta z)\,b_n}{a_n+\beta b_n} f'(z) - \frac{b_n}{2a_n} z f''(z) \right|$$
$$\leq \left(\frac{b_n}{a_n}\right)^2 M_r(f) + \frac{b_n^2}{(a_n+\beta b_n)^2} M_{r,1}^{(\alpha,\beta)}(f) + \frac{b_n^2}{2a_n(a_n+\beta b_n)} M_{r,2}^{(\alpha,\beta)}(f),$$

where

$$M_r(f) = \frac{3MA\,|z|}{r^2} \sum_{k=2}^{\infty} (k+1)(rA)^{k-1} < \infty,$$
$$M_{r,1}^{(\alpha,\beta)}(f) = M\left(\alpha^2+\alpha\beta+2\beta^2\right) \sum_{k=0}^{\infty} k(k-1)(Ar)^k < \infty,$$
$$M_{r,2}^{(\alpha,\beta)}(f) = MA(\alpha+\beta) \sum_{k=0}^{\infty} k(k+1)(Ar)^{k-1} < \infty.$$

Proof. For all $z \in D_R$, we consider

$$S_n^{(\alpha,\beta)}(f;a_n,b_n;z) - f(z) - \frac{(\alpha-\beta z)\,b_n}{a_n+\beta b_n} f'(z) - \frac{b_n}{2a_n} z f''(z)$$
$$= S_n(f;a_n,b_n;z) - f(z) - \frac{b_n}{2a_n} z f''(z) + S_n^{(\alpha,\beta)}(f;a_n,b_n;z) - S_n(f;a_n,b_n;z) - \frac{(\alpha-\beta z)\,b_n}{a_n+\beta b_n} f'(z).$$

Taking $f(z) = \sum\limits_{k=0}^{\infty} c_k z^k$, we immediately obtain

$$S_n^{(\alpha,\beta)}(f;a_n,b_n;z) - f(z) - \frac{(\alpha-\beta z)\,b_n}{a_n+\beta b_n} f'(z) - \frac{b_n}{2a_n} z f''(z)$$
$$= \sum_{k=0}^{\infty} c_k \left(S_n(e_k;a_n,b_n;z) - z^k - \frac{b_n}{2a_n} k(k-1) z^{k-1} \right)$$
$$+ \sum_{k=0}^{\infty} c_k \left(S_n^{(\alpha,\beta)}(e_k;a_n,b_n;z) - S_n(e_k;a_n,b_n;z) - \frac{(\alpha-\beta z)\,b_n}{a_n+\beta b_n} k z^{k-1} \right).$$

By Theorem 4 in [1], for all $|z| \leq r$ we have

$$\left| S_n(f;a_n,b_n;z) - f(z) - \frac{b_n}{2a_n} z f''(z) \right| \leq \left(\frac{b_n}{a_n}\right)^2 M_r(f),$$

where

$$M_r(f) = \frac{3MA\,|z|}{r^2} \sum_{k=2}^{\infty} (k+1)(rA)^{k-1} < \infty.$$

Next, to estimate the second sum, using Lemma 2.2, we rewrite as follows.

$$S_n^{(\alpha,\beta)}(e_k;a_n,b_n;z)-S_n(e_k;a_n,b_n;z)-\frac{(\alpha-\beta z)\,b_n}{a_n+\beta b_n}kz^{k-1}$$
$$=\sum_{v=0}^{k-1}\binom{k}{v}\frac{a_n^v(\alpha b_n)^{k-v}}{(a_n+\beta b_n)^k}S_n(e_v;a_n,b_n;z)-\left(1-\frac{a_n^k}{(a_n+\beta b_n)^k}\right)S_n(e_k;a_n,b_n;z)-\frac{(\alpha-\beta z)\,b_n}{a_n+\beta b_n}kz^{k-1}$$
$$=\sum_{v=0}^{k-2}\binom{k}{v}\frac{a_n^v(\alpha b_n)^{k-v}}{(a_n+\beta b_n)^k}S_n(e_v;a_n,b_n;z)+\frac{k\alpha a_n^{k-1}b_n}{(a_n+\beta b_n)^k}S_n(e_{k-1};a_n,b_n;z)$$
$$-\sum_{v=0}^{k-1}\binom{k}{v}\frac{a_n^v(\beta b_n)^{k-v}}{(a_n+\beta b_n)^k}S_n(e_k;a_n,b_n;z)-\frac{(\alpha-\beta z)\,b_n}{a_n+\beta b_n}kz^{k-1}$$
$$=\sum_{v=0}^{k-2}\binom{k}{v}\frac{a_n^v(\alpha b_n)^{k-v}}{(a_n+\beta b_n)^k}S_n(e_v;a_n,b_n;z)+\frac{k\alpha a_n^{k-1}b_n}{(a_n+\beta b_n)^k}\left[S_n(e_{k-1};a_n,b_n;z)-z^{k-1}\right]$$
$$-\sum_{v=0}^{k-2}\binom{k}{v}\frac{a_n^v(\beta b_n)^{k-v}}{(a_n+\beta b_n)^k}S_n(e_k;a_n,b_n;z)-\frac{k\beta a_n^{k-1}b_n}{(a_n+\beta b_n)^k}\left[S_n(e_k;a_n,b_n;z)-z^k\right]$$
$$+\frac{k\alpha b_n}{a_n+\beta b_n}z^{k-1}\left(\frac{a_n^{k-1}}{(a_n+\beta b_n)^{k-1}}-1\right)+\frac{k\beta b_n}{a_n+\beta b_n}z^k\left(1-\frac{a_n^{k-1}}{(a_n+\beta b_n)^{k-1}}\right).$$

Also, using Lemma 2.3 and the following inequalities

$$1-\frac{a_n^k}{(a_n+\beta b_n)^k}\le\sum_{j=1}^{k}\left(1-\frac{a_n}{a_n+\beta b_n}\right)=\frac{k\beta b_n}{a_n+\beta b_n},$$

$$|S_n(e_k;a_n,b_n;z)-e_k(z)|\le\frac{(k+1)!}{2}\frac{b_n}{a_n}r^{k-1}\text{ (see in the proof of Theorem 3 in [1]),}$$

$$\left|\sum_{v=0}^{k-2}\binom{k}{v}\frac{a_n^v(\alpha b_n)^{k-v}}{(a_n+\beta b_n)^k}S_n(e_v;a_n,b_n;z)\right|\le\sum_{v=0}^{k-2}\binom{k}{v}\frac{a_n^v(\alpha b_n)^{k-v}}{(a_n+\beta b_n)^k}|S_n(e_v;a_n,b_n;z)|$$
$$=\sum_{v=0}^{k-2}\frac{k(k-1)}{(k-v)(k-v-1)}\binom{k-2}{v}\frac{a_n^v(\alpha b_n)^{k-v}}{(a_n+\beta b_n)^k}|S_n(e_v;a_n,b_n;z)|$$
$$\le\frac{k(k-1)}{2}\frac{(\alpha b_n)^2}{(a_n+\beta b_n)^2}(k-2)!r^{k-2}\sum_{v=0}^{k-2}\binom{k-2}{v}\frac{a_n^v(\alpha b_n)^{k-v-2}}{(a_n+\beta b_n)^{k-2}}$$
$$\le\frac{k(k-1)}{2}\frac{(\alpha b_n)^2}{(a_n+\beta b_n)^2}(k-2)!r^{k-2},$$

we obtain

$$\begin{aligned}
&\left|S_n^{(\alpha,\beta)}(e_k;a_n,b_n;z)-S_n(e_k;a_n,b_n;z)-\frac{(\alpha-\beta z)\,b_n}{a_n+\beta b_n}kz^{k-1}\right|\\
&\le \sum_{v=0}^{k-2}\binom{k}{v}\frac{a_n^v\,(\alpha b_n)^{k-v}}{(a_n+\beta b_n)^k}\left|S_n(e_v;a_n,b_n;z)\right|+\frac{k\alpha a_n^{k-1}b_n}{(a_n+\beta b_n)^k}\left|S_n(e_{k-1};a_n,b_n;z)-z^{k-1}\right|\\
&+\sum_{v=0}^{k-2}\binom{k}{v}\frac{a_n^v\,(\beta b_n)^{k-v}}{(a_n+\beta b_n)^k}\left|S_n(e_k;a_n,b_n;z)\right|+\frac{k\beta a_n^{k-1}b_n}{(a_n+\beta b_n)^k}\left|S_n(e_k;a_n,b_n;z)-z^k\right|\\
&+\frac{k\alpha b_n}{a_n+\beta b_n}|z|^{k-1}\left|\frac{a_n^{k-1}}{(a_n+\beta b_n)^{k-1}}-1\right|+\frac{k\beta b_n}{a_n+\beta b_n}|z|^k\left|1-\frac{a_n^{k-1}}{(a_n+\beta b_n)^{k-1}}\right|\\
&\le \frac{k\,(k-1)}{2}\frac{(\alpha b_n)^2}{(a_n+\beta b_n)^2}(k-2)!r^{k-2}+\frac{k\alpha b_n^2}{2a_n\,(a_n+\beta b_n)}k!r^{k-2}\\
&+\frac{k\,(k-1)}{2}\frac{(\beta b_n)^2}{(a_n+\beta b_n)^2}k!r^k+\frac{k\beta b_n^2}{2a_n\,(a_n+\beta b_n)}(k+1)!r^{k-1}+\frac{k\,(k-1)\,\alpha\beta b_n^2}{(a_n+\beta b_n)^2}r^{k-1}+\frac{k\,(k-1)\,(\beta b_n)^2}{(a_n+\beta b_n)^2}r^k\\
&\le \frac{b_n^2}{(a_n+\beta b_n)^2}r^k\left[\frac{k\,(k-1)}{2}\alpha^2\,(k-2)!+\frac{k\,(k-1)}{2}\beta^2k!+k\,(k-1)\,\alpha\beta+k\,(k-1)\,\beta^2\right]\\
&+\frac{b_n^2}{2a_n\,(a_n+\beta b_n)}r^{k-1}\left[k\alpha k!+k\beta\,(k+1)!\right]\\
&\le \frac{b_n^2}{(a_n+\beta b_n)^2}\left[\alpha^2+\alpha\beta+2\beta^2\right]k\,(k-1)\,k!r^k+\frac{b_n^2}{2a_n\,(a_n+\beta b_n)}\left[\alpha+\beta\right]k\,(k+1)!r^{k-1}.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\left|\sum_{k=0}^{\infty}c_k\left(S_n^{(\alpha,\beta)}(e_k;a_n,b_n;z)-S_n(e_k;a_n,b_n;z)-\frac{(\alpha-\beta z)\,b_n}{a_n+\beta b_n}kz^{k-1}\right)\right|\\
&\le \sum_{k=0}^{\infty}|c_k|\left|S_n^{(\alpha,\beta)}(e_k;a_n,b_n;z)-S_n(e_k;a_n,b_n;z)-\frac{(\alpha-\beta z)\,b_n}{a_n+\beta b_n}kz^{k-1}\right|\\
&\le \frac{M\left(\alpha^2+\alpha\beta+2\beta^2\right)b_n^2}{(a_n+\beta b_n)^2}\sum_{k=0}^{\infty}k\,(k-1)\,(rA)^k+\frac{MA\,(\alpha+\beta)\,b_n^2}{2a_n\,(a_n+\beta b_n)}\sum_{k=0}^{\infty}k\,(k+1)\,(rA)^{k-1},
\end{aligned}$$

where for $rA<1$ the series are convergent. This completes the proof. □

Now, we will obtain the exact orders in approximation by complex modified Szász-Mirakjan-Stancu operators and their derivatives.

Theorem 3.3. *Suppose that the hypotheses on the function f and on the constants R, M, C, B, A in the statement of Theorem 3.1 hold and let $1\le r<\frac{1}{A}$ be fixed. Then, for all $n\in\mathbb{N}$ and $|z|\le r$, we have*

$$\left\|S_n^{(\alpha,\beta)}(f;a_n,b_n)-f\right\|_r\sim\frac{b_n}{a_n},\quad n\in\mathbb{N}$$

where the constants in the equivalence depend only on f, α, β and r, if f is not a polynomial of degree ≤ 0 for $0<\alpha\le\beta$, if f is not a polynomial of degree ≤ 1 for $\alpha=\beta=0$ and if f is not of the form $f(z)=Ce^{2\beta z}$ with $A\neq 2\beta$ for $0=\alpha<\beta$.

Proof. For all $|z| \le r$ and $n \in \mathbb{N}$, we have

$$S_n^{(\alpha,\beta)}(f;a_n,b_n;z) - f(z) = \frac{b_n}{a_n}\left\{\frac{a_n}{b_n}\frac{(\alpha-\beta z)\,b_n}{a_n+\beta b_n}f'(z) + \frac{z}{2}f''(z)\right.$$
$$+\frac{b_n}{a_n}\left(\frac{a_n}{b_n}\right)^2\left[S_n^{(\alpha,\beta)}(f;a_n,b_n;z) - f(z) - \frac{(\alpha-\beta z)\,b_n}{a_n+\beta b_n}f'(z) - \frac{b_n}{2a_n}zf''(z)\right]\Bigg\}$$
$$= \frac{b_n}{a_n}\Bigg\{(\alpha-\beta z)\,f'(z) + \frac{z}{2}f''(z) + \frac{b_n}{a_n}\left(\frac{a_n}{b_n}\right)^2\Bigg[S_n^{(\alpha,\beta)}(f;a_n,b_n;z) - f(z)$$
$$\left.\left. - \frac{(\alpha-\beta z)\,b_n}{a_n+\beta b_n}f'(z) - \frac{b_n}{2a_n}zf''(z) - \frac{\beta b_n^2}{a_n(a_n+\beta b_n)}(\alpha-\beta z)\,f'(z)\right]\right\}.$$

Using the following inequality

$$\|F+G\| \ge |\|F\| - \|G\|| \ge \|F\| - \|G\|$$

and denoting $e_1(z) = z$, we obtain

$$\left\|S_n^{(\alpha,\beta)}(f;a_n,b_n) - f\right\|_r \ge \frac{b_n}{a_n}\Bigg[\left\|(\alpha-\beta e_1)\,f' + \frac{e_1}{2}f''\right\|_r - \frac{b_n}{a_n}\left(\frac{a_n}{b_n}\right)^2\Bigg\|S_n^{(\alpha,\beta)}(f;a_n,b_n) - f$$
$$\left. - \frac{(\alpha-\beta e_1)\,b_n}{a_n+\beta b_n}f' - \frac{b_n}{2a_n}e_1f'' - \frac{\beta b_n^2}{a_n(a_n+\beta b_n)}(\alpha-\beta e_1)\,f'\Bigg\|_r\right].$$

Taking into account the hypotheses on f, we can write $\left\|(\alpha-\beta e_1)\,f' + \frac{e_1}{2}f''\right\|_r > 0$. Indeed, assuming the contrary, it follows that

$$(\alpha-\beta z)\,f'(z) + \frac{z}{2}f''(z) = 0$$

for all $z \in \overline{D_r}$. Here, we have three different cases. If $0 < \alpha \le \beta$, denoting $y(z) = f'(z)$, searching $y(z)$ in the form $y(z) = \sum_{k=0}^{\infty}\delta_k z^k$ and replacing in the above differential equation, we easily obtain $\delta_k = 0$ for all $k = 0, 1, ...,$ which implies that $f(z)$ is a polynomial of degree ≤ 0, a contradiction. If $\alpha = \beta = 0$, then we immediately get $f''(z) = 0$ for all $|z| \le r$, i.e. f is a polynomial of degree ≤ 1, a contradiction. If $0 = \alpha < \beta$, the differential equation easily gives the solution $f(z) = Ce^{2\beta z}$, $C \in \mathbb{C}$ arbitrary complex constant, which is a contradiction.

Now, by Theorem 3.2, we immediately obtain

$$\left(\frac{a_n}{b_n}\right)^2\left\|S_n^{(\alpha,\beta)}(f;a_n,b_n) - f - \frac{(\alpha-\beta e_1)\,b_n}{a_n+\beta b_n}f' - \frac{b_n}{2a_n}e_1f'' - \frac{\beta b_n^2}{a_n(a_n+\beta b_n)}(\alpha-\beta e_1)\,f'\right\|_r$$
$$\le \left(\frac{a_n}{b_n}\right)^2\left\|S_n^{(\alpha,\beta)}(f;a_n,b_n) - f - \frac{(\alpha-\beta e_1)\,b_n}{a_n+\beta b_n}f' - \frac{b_n}{2a_n}e_1f''\right\|_r + \frac{a_n}{a_n+\beta b_n}\left\|\beta(\alpha-\beta e_1)\,f'\right\|_r$$
$$\le M_r(f) + M_{r,1}^{(\alpha,\beta)}(f) + M_{r,2}^{(\alpha,\beta)}(f) + \beta(\alpha+\beta r)\left\|f'\right\|_r,$$

there exists an index n_1 (depending on f, α, β and r only) such that for all $n \ge n_1$, we have

$$\left\|(\alpha-\beta e_1)\,f' + \frac{e_1}{2}f''\right\|_r - \frac{b_n}{a_n}\left(\frac{a_n}{b_n}\right)^2\Bigg\|S_n^{(\alpha,\beta)}(f;a_n,b_n) - f$$
$$- \frac{(\alpha-\beta e_1)\,b_n}{a_n+\beta b_n}f' - \frac{b_n}{2a_n}e_1f'' - \frac{\beta b_n^2}{a_n(a_n+\beta b_n)}(\alpha-\beta e_1)\,f'\Bigg\|_r$$
$$\ge \frac{1}{2}\left\|(\alpha-\beta e_1)\,f' + \frac{e_1}{2}f''\right\|_r,$$

which implies

$$\left\|S_n^{(\alpha,\beta)}(f;a_n,b_n)-f\right\|_r \geq \frac{b_n}{2a_n}\left\|(\alpha-\beta e_1)f'+\frac{e_1}{2}f''\right\|_r$$

for all $n \geq n_1$. For $1 \leq n \leq n_1-1$, we get

$$\left\|S_n^{(\alpha,\beta)}(f;a_n,b_n)-f\right\|_r \geq \frac{b_n}{a_n}M_r(f)$$

with $M_r(f)=\frac{a_n}{b_n}\left\|S_n^{(\alpha,\beta)}(f;a_n,b_n)-f\right\|_r>0$. Therefore, finally we obtain

$$\left\|S_n^{(\alpha,\beta)}(f;a_n,b_n)-f\right\|_r \geq \frac{b_n}{a_n}C_r(f)$$

for all n, with

$$C_r(f)=\min\left\{M_{r,1}(f),...,M_{r,n_1-1}(f),\frac{1}{2}\left\|(\alpha-\beta e_1)f'+\frac{e_1}{2}f''\right\|_r\right\},$$

which combined with Theorem 3.1 (i), we get the desired conclusion. □

Theorem 3.4. *Suppose that the hypotheses on the function f and on the constants R, M, C, B, A in the statement of Theorem 3.1 hold and let $1 \leq r < r_1 < \frac{1}{A}$ and $p \in \mathbb{N}$ be fixed. Then, for all $n \in \mathbb{N}$ and $|z| \leq r$, we have*

$$\left\|\left(S_n^{(\alpha,\beta)}(f;a_n,b_n)\right)^{(p)}-f^{(p)}\right\|_r \sim \frac{b_n}{a_n},\ n \in \mathbb{N}$$

where the constants in the equivalence depend only on f, α, β, p, r_1 and r, if f is not a polynomial of degree $\leq p-1$ for $0 < \alpha \leq \beta$, if f is not a polynomial of degree $\leq p$ for $\alpha = \beta = 0$ and if f is not of the form $f(z) = Ce^{2\beta z}$ with $A \neq 2\beta$ for $0 = \alpha < \beta$.

Proof. Since the upper estimate is obtained in Theorem 3.1 (ii), it remains to prove the lower estimate. Denoting by γ the circle of radius r_1 and center 0 (where $r_1 > r \geq 1$), for all $|z| \leq r$ and $v \in \gamma$, we have the inequality $|v-z| \geq r_1 - r$.
By the Cauchy's formula, it follows that for all $|z| \leq r$ and $n \in \mathbb{N}$

$$\left(S_n^{(\alpha,\beta)}(f;a_n,b_n;z)\right)^{(p)}-f^{(p)}(z)=\frac{p!}{2\pi i}\int_\gamma \frac{S_n^{(\alpha,\beta)}(f;a_n,b_n;v)-f(v)}{(v-z)^{p+1}}dv.$$

For all $v \in \gamma$ and $n \in \mathbb{N}$, we have

$$S_n^{(\alpha,\beta)}(f;a_n,b_n;v)-f(v)=\frac{b_n}{a_n}\left\{(\alpha-\beta v)f'(v)+\frac{v}{2}f''(v)+\frac{b_n}{a_n}\left[\left(\frac{a_n}{b_n}\right)^2\left(S_n^{(\alpha,\beta)}(f;a_n,b_n;v)-f(v)\right.\right.\right.$$
$$\left.\left.\left.-\frac{(\alpha-\beta v)b_n}{a_n+\beta b_n}f'(v)-\frac{b_n}{2a_n}vf''(v)\right)-\frac{\beta a_n}{(a_n+\beta b_n)}(\alpha-\beta v)f'(v)\right]\right\}.$$

By using Cauchy's formula, we get

$$\left(S_n^{(\alpha,\beta)}(f;a_n,b_n;z)\right)^{(p)}-f^{(p)}(z)=\frac{b_n}{a_n}\left\{\left[(\alpha-\beta z)f'(z)+\frac{z}{2}f''(z)\right]^{(p)}\right.$$

$$+\frac{b_n}{a_n}\left[\frac{p!}{2\pi i}\int_\gamma \frac{\left(\frac{a_n}{b_n}\right)^2\left(S_n^{(\alpha,\beta)}(f;a_n,b_n;v)-f(v)-\frac{(\alpha-\beta v)b_n}{a_n+\beta b_n}f'(v)-\frac{b_n}{2a_n}vf''(v)\right)}{(v-z)^{p+1}}dv\right.$$

$$\left.-\frac{p!}{2\pi i}\int_\gamma \frac{\frac{\beta a_n}{(a_n+\beta b_n)}(\alpha-\beta v)f'(v)}{(v-z)^{p+1}}dv\right]\Bigg\}.$$

Now, passing to the norm $\|.\|_r$, for all $n\in\mathbb{N}$ it follows that

$$\left\|\left(S_n^{(\alpha,\beta)}(f;a_n,b_n)\right)^{(p)}-f^{(p)}\right\|_r \geq \frac{b_n}{a_n}\left\{\left\|\left[(\alpha-\beta e_1)f'+\frac{e_1}{2}f''\right]^{(p)}\right\|_r\right.$$

$$-\frac{b_n}{a_n}\left\|\frac{p!}{2\pi i}\int_\gamma \frac{\left(\frac{a_n}{b_n}\right)^2\left(S_n^{(\alpha,\beta)}(f;a_n,b_n;v)-f(v)-\frac{(\alpha-\beta v)b_n}{a_n+\beta b_n}f'(v)-\frac{b_n}{2a_n}vf''(v)\right)}{(v-z)^{p+1}}dv\right.$$

$$\left.\left.-\frac{p!}{2\pi i}\int_\gamma \frac{\frac{\beta a_n}{(a_n+\beta b_n)}(\alpha-\beta v)f'(v)}{(v-z)^{p+1}}dv\right\|_r\right\}.$$

By Theorem 3.2, for all $n\in\mathbb{N}$ we obtain

$$\left\|\frac{p!}{2\pi i}\int_\gamma \frac{\left(\frac{a_n}{b_n}\right)^2\left(S_n^{(\alpha,\beta)}(f;a_n,b_n;v)-f(v)-\frac{(\alpha-\beta v)b_n}{a_n+\beta b_n}f'(v)-\frac{b_n}{2a_n}vf''(v)\right)}{(v-z)^{p+1}}dv\right.$$

$$\left.-\frac{p!}{2\pi i}\int_\gamma \frac{\frac{\beta a_n}{(a_n+\beta b_n)}(\alpha-\beta v)f'(v)}{(v-z)^{p+1}}dv\right\|_r\Bigg\}$$

$$\leq \frac{p!}{2\pi}\frac{2\pi r_1}{(r_1-r)^{p+1}}\left[M_{r_1}(f)+M_{r_1,1}^{(\alpha,\beta)}(f)+M_{r_1,2}^{(\alpha,\beta)}(f)\right]+\frac{p!}{2\pi}\frac{2\pi r_1}{(r_1-r)^{p+1}}\beta(\alpha+\beta r_1)\left\|f'\right\|_{r_1}.$$

Taking into account the hypotheses on f we have $\left\|\left[(\alpha-\beta e_1)f'+\frac{e_1}{2}f''\right]^{(p)}\right\|_r>0$. The remain of the proof can be easily shown by exactly the lines in [8] (see also [3]). Therefore, we omit the details. □

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