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Coincidence and Common Fixed Point Results under Generalized $(\mathcal{A}, \mathcal{S})_f$-Contractions

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Abstract. Very recently, Shahzad et al. [RACSAM 111 (2017) 307–324] introduced the notion of $(\mathcal{A}, \mathcal{S})$-contractions which unifies several well known nonlinear type contractions (e.g. $\mathcal{R}$-contractions, $\mathcal{Z}$-contractions, $\mathcal{L}$-contractions etc.) in one go. In this paper, we introduce the notion of generalized $(\mathcal{A}, \mathcal{S})_f$-contractions and utilize the same to present some coincidence and common fixed point results for a pair of self-mappings (g, f) defined on a metric space endowed with a binary relation $\mathcal{S}$. In this course, we ought to introduce some new notions namely: $(I, \mathcal{S})$-continuity, $(I, \mathcal{S})$-compatibility and local (g, f)-transitivity. Consequently, several results involving $\mathcal{R}$-contractions and $\mathcal{Z}$-contractions are deduced. Finally, we furnish illustrative examples to demonstrate the utility of our results.

1. Introduction and preliminaries

The tremendous applications of fixed point theory had always inspired the growth of this domain. In 1922, Banach formulated his most simple but very natural result which is now popularly referred as Banach contraction principle. In the course of last several decades, this principle has been extended and generalized in many directions with several applications in many branches. Employing simulation functions, Khojasteh et al. [10] initiated the idea of $\mathcal{Z}$-contractions and utilized the same to cover varied type of nonlinear contractions of the existing literature. Later, Argoubia et al. [5] and Hierro et al. [14] independently sharpened the notion of simulation functions and also proved some coincidence and common fixed point results.

Very recently, Hierro and Shahzad [7] introduced the notion of $\mathcal{R}$-contractions in order to extend several nonlinear contractions such as: $\mathcal{Z}$-contractions, manageable contractions, Meir-Keeler contractions etc. Indeed, $\mathcal{R}$-contractions are associated to $\mathcal{R}$-functions which satisfy two independent conditions involving two sequences of nonnegative real numbers. Soon, inspired by $\mathcal{R}$-contractions, Shahzad et al. [17] introduced the notion of $(\mathcal{A}, \mathcal{S})$-contractions which remains an extension of $(\mathcal{R}, \mathcal{S})$-contractions given in [15] by Hierro and Shahzad wherein the authors proved very interesting results.

In resent years, various results involving fixed point, coincidence point and common fixed point are proved in metric spaces endowed with different types of binary relations (see [1–4, 6, 8, 11, 12, 15–18] and references therein). In this context, we employ a binary relation wherein the involved contractive condition is required to hold merely to those pair of points which are comparable.

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In this paper, we generalize $(\mathcal{A},\mathcal{S})$-contractions to generalized $(\mathcal{A},\mathcal{S})_f$-contractions and utilize the same to prove some existence and uniqueness results for a pair of self-mappings (g, f) defined on a metric space (X, d) endowed with a binary relation $\mathcal{S}$. We also introduce the notions of $(I,\mathcal{S})$-continuity, $(I,\mathcal{S})$-compatibility and locally (g, f)-transitivity. In our main results, the binary relation $\mathcal{S}$ need not to be reflexive, antisymmetric, transitive, f-transitive or even (g, f)-transitive. Furthermore, we replace the completeness assumption by a relatively weaker one namely increasingly precompleteness of an appropriate subspace. Also, we use the new types of continuity and compatibility conditions namely: $(I,\mathcal{S})$-continuity and $(I,\mathcal{S})$-compatibility which are relatively weaker than $\mathcal{S}$-continuity and $(O,\mathcal{S})$-compatibility. We also introduce the notions of generalized $(\mathcal{R},\mathcal{S})_f$-contractions and generalized $(\mathcal{Z},\mathcal{S})_f$-contractions and derive some results involving such contractions as consequences of our results. Finally, we adopt some illustrative examples to exhibit the utility of our results.

For the sake of completeness, we collect here some basic definitions and fundamental results needed in our subsequent discussions.

From now on, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ while $\mathbb{R}$ stands for the set of real numbers. In the sequel, X stands for a nonempty set, I_X refers to the identity mapping on X and $g, f : X \to X$. For brevity, we write gx instead of $g(x)$. A point $x \in X$ is said to be:

(i) *a fixed point of* g if $gx = x$ ($Fix(g)$ denotes the set of all such points);

(ii) *a coincidence point of* (g, f) if $gx = fx$ ($Coin(g, f)$ stands for the set of all such points);

(iii) *a point of coincidence of* (g, f) if there exists $y \in Coin(g, f)$ such that $x = gy = fy$;

(iv) *a common fixed point of* (g, f) if $x = gx = fx$.

Recall that a pair (g, f) is *commuting on* X if $gfx = fgx$, for all $x \in X$ and *weakly compatible* if $gfx = fgx$, for all $x \in Coin(g, f)$. For $x_0 \in X$, the sequence $\{x_n\} \subseteq X$ defined by $x_{n+1} = g^n x_0 = gx_n$, for all $n \in \mathbb{N}_0$, is called *a Picard sequence* based at x_0. Following [15], *a Picard-Jungck sequence* of the pair (g, f) based at a point $x_0 \in X$ is a sequence $\{x_n\} \subseteq X$ such that $fx_{n+1} = gx_n$, for all $n \in \mathbb{N}_0$. This sequence is known as (g, f)-Picard-Jungck sequence.

If (X, d) is a metric space, then $ran(d)$ stands for the range of d, i.e, $ran(d) = \{d(x, y) : x, y \in X\}$. We write $\{x_n\} \to x$ whenever $\{x_n\}$ converges to x. The sequence $\{x_n\} \subseteq X$ is said to be *asymptotically regular* on (X, d) if $\{d(x_{n+1}, x_n)\} \to 0$.

2. Relation-theoretic notions and auxiliary results

A nonempty subset $\mathcal{S}$ of $X \times X$ is said to be *a binary relation on* X. Trivially, $X \times X$ is always a binary relation on X known as *the universal relation*. For simplicity, we write $x\mathcal{S}y$ whenever $(x, y) \in \mathcal{S}$ and write $x\mathcal{S}^{\neq}y$ whenever $x\mathcal{S}y$ and $x \neq y$. Observe that $\mathcal{S}^{\neq} \subseteq \mathcal{S}$. The elements x and y of X are said to be *$\mathcal{S}$-comparable* if either $x\mathcal{S}y$ or $y\mathcal{S}x$. If $x, y \in X$ are $\mathcal{S}$-comparable, then we write $[x, y] \in \mathcal{S}$. Throughout this presentation, $\mathcal{S}$ stands for a binary relation defined on X and $\mathcal{S}_X$ stands for the universal relation on X.

Definition 2.1. *A binary relation $\mathcal{S}$ on X is said to be:*

(i) amorphous *if it has no specific property at all;*

(ii) reflexive *if $x\mathcal{S}x$, for all $x \in X$;*

(iii) transitive *if $x\mathcal{S}y$ and $y\mathcal{S}z$ imply $x\mathcal{S}z$, for any $x, y, z \in X$;*

(iv) antisymmetric *if $x\mathcal{S}y$ and $y\mathcal{S}x$ imply $x = y$, for any $x, y \in X$;*

(v) preorder *if it is reflexive and transitive;*

(vi) partial order *if it is reflexive, transitive and antisymmetric.*

Definition 2.2. *[15] Let $\mathcal{S}$ be a binary relation on a nonempty set X and $g, f : X \to X$. If $gx\mathcal{S}gy$, for all $x, y \in X$ with $fx\mathcal{S}fy$, then g is called* $(f, \mathcal{S})$-nondecreasing. *If $f = I_X$, then g is said to be* $\mathcal{S}$-nondecreasing.

Here it can be pointed out that calling "g is $\mathcal{S}$-nondecreasing" is equivalent to saying that "$\mathcal{S}$ is g-closed" as used by Alam and Imdad [2, 4].

Definition 2.3. *[15] Let $\mathcal{S}$ be a binary relation on a nonempty set X and $g, f : X \to X$. If for every $x \in X$, there exists $y \in X$ such that $gx = fy$ and $fx\mathcal{S}fy$, then we write $gX \subseteq_{\mathcal{S}} fX$.*

Clearly, if $gX \subseteq_{\mathcal{S}} fX$, then $gX \subseteq fX$.

Definition 2.4. *(see [15]) Let $\mathcal{S}$ be a binary relation on a nonempty set X and $g, f : X \to X$. A sequence $\{x_n\}$ is called* a $(g, f, \mathcal{S})$-Picard-Jungck sequence *if it is a (g, f)-Picard-Jungck sequence and $fx_n\mathcal{S}fx_m$, for all $n, m \in \mathbb{N}_0$ with $n < m$. For $f = I_X$, $\{x_n\}$ is called* $(g, \mathcal{S})$-Picard sequence.

Definition 2.5. *Let $\mathcal{S}$ be a binary relation on a nonempty set X and $f : X \to X$. A sequence $\{x_n\} \subseteq X$ is said to be:*

(i) $(f, \mathcal{S})$-nondecreasing *if $fx_n\mathcal{S}fx_{n+1}$, for all $n \in \mathbb{N}_0$;*

(ii) $(f, \mathcal{S})$-increasing *if $fx_n\mathcal{S}^{\ast}fx_{n+1}$, for all $n \in \mathbb{N}_0$.*

Remark 2.6. *On setting $f = I_X$, Definition 2.5 reduces to Definition 5 due to Shahzad et al. [17].*

Here it can be pointed out that Alam and Imdad [2, 4] used the term $\mathcal{S}$-preserving instead of $\mathcal{S}$-nondecreasing in case $f = I_X$.

Definition 2.7. *[3, 15] Let $\mathcal{S}$ be a binary relation on a nonempty set X and $g, f : X \to X$. Then $\mathcal{S}$ is said to be:*

(i) f-transitive *if it is transitive on fX;*

(ii) (g, f)-transitive *if $fx\mathcal{S}fy$ and $fy\mathcal{S}gy$ imply $fx\mathcal{S}gy$, for any $x, y \in X$;*

(iii) (g, f)-compatible *if $fx = fy$ and $fx\mathcal{S}fy$ imply $gx = gy$, for any $x, y \in X$.*

Remark 2.8. *Every transitive binary relation is f-transitive and (g, f)-transitive, whatever g and f. The notions of f-transitivity and (g, f)-transitivity extend the notion of transitivity properly throughout independent notions (for more details one may see [15]).*

Definition 2.9. *[3] Let $\mathcal{S}$ be a binary relation on a nonempty set X and $f : X \to X$. Then $\mathcal{S}$ is said to be:*

(i) locally transitive *if for each (effective) $\mathcal{S}$-nondecreasing sequence $\{x_n\} \subseteq X$ (with range $E := \{x_n : n \in \mathbb{N}_0\}$), the binary relation $\mathcal{S}|_E$ is transitive.*

(ii) locally f-transitive *if for each (effective) $(f, \mathcal{S})$-nondecreasing sequence $\{x_n\}$ (with range $E := \{x_n : n \in \mathbb{N}_0\}$), the binary relation $\mathcal{S}|_E$ is f-transitive.*

Remark 2.10. (i) *Every transitive binary relation is locally transitive.*
(ii) *Every f-transitive binary relation is locally f-transitive, but the converse is not true in general.*

Now, we introduce the notion of locally (g, f)-transitive as follows:

Definition 2.11. *A binary relation $\mathcal{S}$ on a nonempty set X is said to be* locally (g, f)-transitive *if for each (effective) (g, f)-Picard-Jungck iterates $(f, \mathcal{S})$-nondecreasing sequence $\{x_n\}$ (with range $E := \{x_n : n \in \mathbb{N}_0\}$), the binary relation $\mathcal{S}|_E$ is (g, f)-transitive.*

Remark 2.12. *Every (g, f)-transitive binary relation is locally (g, f)-transitive, but the converse is not true in general.*

Proposition 2.13. *If $\mathcal{S}$ is locally f-transitive, then $\mathcal{S}$ is locally (g, f)-transitive.*

Proof. Let $\{x_n\}$ be a (g, f)-Picard-Jungck $(f, \mathcal{S})$-nondecreasing sequence. If $n, m \in \mathbb{N}_0$ are such that $fx_n\mathcal{S}fx_m$ and $fx_m\mathcal{S}gx_m$, then we have $fx_n\mathcal{S}fx_m$ and $fx_m\mathcal{S}fx_{m+1}$(as $fx_{m+1} = gx_m$) so that $fx_n\mathcal{S}fx_{m+1}$ (as $\mathcal{S}$ is locally f-transitive). Hence, $fx_n\mathcal{S}gx_m$. Thus, $\mathcal{S}$ is locally (g, f)-transitive. □

Proposition 2.14. *Let $\mathcal{S}$ be a locally f-transitive (or locally (g, f)-transitive) binary relation on a nonempty set X and $g, f : X \to X$.*

- (a) *If $gX \subseteq fX$, g is $(f, \mathcal{S})$-nondecreasing and there is a point $x_0 \in X$ such that $fx_0\mathcal{S}gx_0$, then there exists a $(g, f, \mathcal{S})$-Picard-Jungck sequence in X based at x_0.*
- (b) *If $gX \subseteq_{\mathcal{S}} fX$, then there exists a $(g, f, \mathcal{S})$-Picard-Jungck sequence in X based at each $x_0 \in X$.*

Proof. First of all, we prove that there exists a (g, f)-Picard-Jungck $(f, \mathcal{S})$-nondecreasing sequence $\{x_n\} \subseteq X$.

- (a) Since $gx_0 \in gX \subseteq fX$, there exists $x_1 \in X$ such that $fx_1 = gx_0$. Similarly, as $gx_1 \in gX \subseteq fX$, there exists $x_2 \in X$ such that $fx_2 = gx_1$. Moreover, $fx_0\mathcal{S}gx_0 = fx_1$ and g is $(f, \mathcal{S})$-nondecreasing so that $gx_0\mathcal{S}gx_1$ which means that $fx_1\mathcal{S}fx_2$. Continuing these arguments, we can construct a sequence $\{x_n\} \subseteq X$ such that $fx_{n+1} = gx_n$ and $fx_n\mathcal{S}fx_{n+1}$, for all $n \in \mathbb{N}_0$, i.e., $\{x_n\}$ is a (g, f)-Picard-Jungck $(f, \mathcal{S})$-nondecreasing sequence.
- (b) Let $x_0 \in X$ be an arbitrary point. Since $gx_0 \in gX \subseteq_{\mathcal{S}} fX$, there exists $x_1 \in X$ such that $fx_1 = gx_0$ and $fx_0\mathcal{S}fx_1$. Again, $gx_1 \in gX \subseteq_{\mathcal{S}} fX$ implies that there exists $x_2 \in X$ such that $fx_2 = gx_1$ and $fx_1\mathcal{S}fx_2$. Thus inductively, one can construct a sequence $\{x_n\} \subseteq X$ such that $fx_{n+1} = gx_n$ and $fx_n\mathcal{S}fx_{n+1}$, for all $n \in \mathbb{N}_0$.

Now, we prove that $fx_n\mathcal{S}fx_m$, for all $n, m \in \mathbb{N}_0$ with $n < m$. As earlier, we have $\{x_n\}$ remains a (g, f)-Picard-Jungck $(f, \mathcal{S})$-nondecreasing sequence. Firstly, assume that $\mathcal{S}$ is locally f-transitive. Let $n, m \in \mathbb{N}_0$ with $n < m$. We have

$$fx_n\mathcal{S}fx_{n+1},\ \ fx_{n+1}\mathcal{S}fx_{n+2},\ \ ...,\ \ fx_{m-1}\mathcal{S}fx_m \Longrightarrow fx_n\mathcal{S}fx_m.$$

Next, assume that $\mathcal{S}$ is locally (g, f)-transitive. For $n \in \mathbb{N}_0$, we have

$$fx_n\mathcal{S}fx_{n+1},\ \ fx_{n+1}\mathcal{S}fx_{n+2} = gx_{n+1} \Longrightarrow fx_n\mathcal{S}gx_{n+1} = fx_{n+2}.$$

Similarly,

$$fx_n\mathcal{S}fx_{n+2},\ \ fx_{n+2}\mathcal{S}fx_{n+3} = gx_{n+2} \Longrightarrow fx_n\mathcal{S}gx_{n+2} = fx_{n+3}.$$

Thus, by induction, we have $fx_n\mathcal{S}fx_m$, for all $n, m \in \mathbb{N}_0$ with $n < m$. □

Observe that the converse of Proposition 2.14 is not true in general as substantiated in the following example:

Example 2.15. *Consider $X = \{0, \dfrac{1}{2}, \dfrac{1}{2^2}, ...\}$. Define a binary relation $\mathcal{S}$ on X as follows:*

$$x\mathcal{S}y \iff \tfrac{1}{2^2} \geq x > y \text{ or } (x, y) \in \{(0,0), (0, \tfrac{1}{2}), (\tfrac{1}{2}, \tfrac{1}{2^2}), (0, \tfrac{1}{2^n}) : n \geq 3\}.$$

Also, define $g, f : X \to X$ by:

$$gx = \tfrac{1}{4}x \text{ and } fx = x, \text{ for all } x \in X.$$

Then $\mathcal{S}$ is locally (g, f)-transitive. However, it is not locally f-transitive. To see this, consider the $\mathcal{S}$-nondecreasing sequence: $x_0 = 0$, $x_n = \frac{1}{2^n} : n \geq 1$, (with range $E := \{x_n : n \in \mathbb{N}_0\}$). Observe that $\mathcal{S}|_E$ is not f-transitive.

Definition 2.16. *[15] Let X be a nonempty set. A self-mapping $g : X \to X$ is said to be* an $\mathcal{S}$-continuous *if $\{gx_n\} \to gy$ whenever $\{x_n\} \subseteq X$ is such that $\{x_n\} \to y \in X$ and $x_n \mathcal{S} x_m$, for all $n, m \in \mathbb{N}_0$ with $n < m$.*

Definition 2.17. *Let X be a nonempty set. A self-mapping $g : X \to X$ is said to be* an $(I, \mathcal{S})$-continuous *if $\{gx_n\} \to gy$ whenever $\{x_n\} \subseteq X$ is an $\mathcal{S}$-increasing sequence such that $\{x_n\} \to y \in X$. For $\mathcal{S} = \mathcal{S}_X$, g is called* I-continuous.

Remark 2.18. *Continuity $\Longrightarrow$ $\mathcal{S}$-continuity $\Longrightarrow$ $(I, \mathcal{S})$-continuity.*

Definition 2.19. *[9, 15] Let (X, d) be a metric space endowed with a binary relation $\mathcal{S}$ and $g, f : X \to X$. Then the pair (g, f) is said to be* an $(O, \mathcal{S})$-compatible *if*

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0,$$

whenever $\{x_n\} \subseteq X$ is such that $fx_n \mathcal{S} fx_m$, for all $n < m$ and $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = z \in X$.

Definition 2.20. *Let (X, d) be a metric space endowed with a binary relation $\mathcal{S}$ and $g, f : X \to X$. Then the pair (g, f) is said to be* an $(I, \mathcal{S})$-compatible *if*

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0,$$

whenever $\{x_n\} \subseteq X$ is $(f, \mathcal{S})$-increasing and $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = z \in X$. For $\mathcal{S} = \mathcal{S}_X$, the pair (g, f) is said to be I-compatible.

Remark 2.21. *Commutativity $\Longrightarrow$ weakly compatible $\Longrightarrow$ $(O, \mathcal{S})$-compatibility $\Longrightarrow$ $(I, \mathcal{S})$-compatibility.*

Definition 2.22. *Let (X, d) be a metric space. A subset $B \subseteq X$ is said to be* an $(\mathcal{S}, d)$-increasingly complete *if every $\mathcal{S}$-increasing Cauchy sequence $\{x_n\} \subseteq B$ converges to a point $y \in B$.*

Definition 2.23. *(see[17]) Let (X, d) be a metric space. A subset $B \subseteq X$ is said to be* precomplete *if each Cauchy sequence $\{x_n\} \subseteq B$ converges to some $x \in X$.*

Remark 2.24. *Every complete subset of X is precomplete.*

Definition 2.25. *(see [15]) Let (X, d) be a metric space endowed with a binary relation $\mathcal{S}$. A subset $B \subseteq X$ is said to be* an $(\mathcal{S}, d)$-increasingly precomplete *if each $\mathcal{S}$-increasing Cauchy sequence $\{x_n\} \subseteq B$ converges to some $x \in X$.*

Remark 2.26. *Every precomplete subset of X is $(\mathcal{S}, d)$-increasingly precomplete whatever the binary relation $\mathcal{S}$.*

Definition 2.27. *(see [15]) Let (X, d) be a metric space equipped with a binary relation $\mathcal{S}$. A subset $B \subseteq X$ is said to be* an $(\mathcal{S}, d)$-increasingly regular *if for every $\mathcal{S}$-increasing sequence $\{x_n\} \subseteq X$ with $\{x_n\} \to y \in X$, we have $x_n \mathcal{S} y$, for all $n \in \mathbb{N}_0$.*

The following lemma is needed in the sequel.

Lemma 2.28. *[13] Let (X, d) be a metric space and $\{x_n\}$ a sequence in X. If $\{x_n\}$ is not a Cauchy sequence in X, then there exist $\epsilon_0 > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that $k \le n(k) \le m(k)$, $d(x_{n(k)}, x_{m(k)-1}) \le \epsilon_0 < d(x_{n(k)}, x_{m(k)})$, $\forall k \in \mathbb{N}_0$. Moreover, if $\{x_n\}$ is asymptotically regular, then*

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)-1}) = \epsilon_0.$$

3. Generalized contractivity notions and auxiliary results

In what follows: $N(g,f,x,y) = \max\{d(fx,fy), d(fx,gx), d(fy,gx)\}$.

Definition 3.1. *[5, 10, 14] A mapping $\zeta : [0,\infty)\times[0,\infty) \to \mathbb{R}$ is called* a simulation function *if it satisfies the following conditions:*

(ζ_1) $\zeta(u,v) < v-u$, for all $u,v>0$;

(ζ_2) if $\{a_n\}$ and $\{b_n\}$ are two sequences in $(0,\infty)$ such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n > 0$ and $a_n < b_n$, for all $n\in\mathbb{N}$, then $\limsup_{n\to\infty}\zeta(a_n,b_n) < 0$.

The set of all simulation functions is denoted by $\mathcal{Z}$.

Inspired by [10], we will use simulation functions to present some generalized contractions on metric spaces in the following sense:

Definition 3.2. *Let (X,d) be a metric space endowed with a binary relation $\mathcal{S}$ and $g,f : X\to X$. If there exists a simulation function ζ such that*

$$\zeta(d(gx,gy), N(g,f,x,y)) \geq 0, \text{ for all } x,y\in X \text{ such that } fx\mathcal{S}^{\varkappa}fy \text{ and } gx\mathcal{S}^{\varkappa}gy,$$

then g is called a generalized $(\mathcal{Z},\mathcal{S})_f$-contraction w.r.t. ζ.
By choosing $f = I_X$, $\mathcal{S} = \mathcal{S}_X$ and $f = I_X$ with $\mathcal{S} = \mathcal{S}_X$, we respectively deduce generalized $(\mathcal{Z},\mathcal{S})$-contractions, generalized $\mathcal{Z}_f$-contractions *and* generalized $\mathcal{Z}$-contractions *w.r.t. ζ.*

Definition 3.3. *[7] Let B be a nonempty subset of $\mathbb{R}$ and $\varrho : B\times B\to\mathbb{R}$ a function. Then ϱ is said to be* an $\mathcal{R}$-function *if it satisfies the following conditions:*

(ϱ_1) If $\{a_n\}\subseteq(0,\infty)\cap B$ is a sequence such that $\varrho(a_{n+1},a_n)>0$, for all $n\in\mathbb{N}$, then $\{a_n\}\to 0$.

(ϱ_2) If $\{a_n\},\{b_n\}\subseteq(0,\infty)\cap B$ are two sequences converging to the same limit $L\geq 0$ such that $L<a_n$ and $\varrho(a_n,b_n)>0$, for all $n\in\mathbb{N}$, then $L=0$.

The family of all $\mathcal{R}$-functions with domain $B\times B$ is denoted by $\mathcal{R}_B$. In some cases, the following condition is also considered.

(ϱ_3) If $\{a_n\},\{b_n\}\subseteq(0,\infty)\cap B$ are two sequences such that $\{b_n\}\to 0$ and $\varrho(a_n,b_n)>0$, for all $n\in\mathbb{N}$, then $\{a_n\}\to 0$.

Proposition 3.4. *[7] Every simulation function is an $\mathcal{R}$-function satisfying ϱ_3.*

Proposition 3.5. *[7] If $\varrho\in\mathcal{R}_B$, then $\varrho(u,u)\leq 0$, for all $u\in(0,\infty)\cap B$.*

Inspired by [7], we will use $\mathcal{R}$-functions to present some generalized contractions on metric spaces in the following sense:

Definition 3.6. *Let (X,d) be a metric space endowed with a binary relation $\mathcal{S}$ and $g,f : X\to X$. If there exists an $\mathcal{R}$-function $\varrho\in\mathcal{R}_B$ such that $ran(d)\subseteq B$ and*

$$\varrho(d(gx,gy), N(g,f,x,y)) > 0, \ \forall\, x,y\in X \text{ such that } fx\mathcal{S}^{\varkappa}fy \text{ and } gx\mathcal{S}^{\varkappa}gy, \tag{1}$$

then g is called a generalized $(\mathcal{R},\mathcal{S})_f$-contraction w.r.t. ϱ.
By choosing $f = I_X$, $\mathcal{S} = \mathcal{S}_X$ and $f = I_X$ with $\mathcal{S} = \mathcal{S}_X$, we respectively deduce generalized $(\mathcal{R},\mathcal{S})$-contractions, generalized $\mathcal{R}_f$-contractions *and* generalized $\mathcal{R}$-contractions *w.r.t. ϱ.*

The following proposition is immediate.

Proposition 3.7. *If $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is a simulation function, then every generalized $(\mathcal{Z},\mathcal{S})_f$-contraction w.r.t. ζ is a generalized $(\mathcal{R},\mathcal{S})_f$-contraction w.r.t. ζ.*

Definition 3.8. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. We say that $\{(a_n,b_n)\}$ is* a $(g,f,\mathcal{S})_N$-sequence *if there exist two sequences $\{x_n\},\{y_n\} \subseteq X$ such that $fx_n\mathcal{S}fy_n$, $a_n = d(gx_n,gy_n) > 0$ and $b_n = N(g,f,x_n,y_n) > 0$, where $N(g,f,x_n,y_n) = \max\{d(fx_n,fy_n), d(fx_n,gx_n), d(fy_n,gx_n)\}$, for all $n \in \mathbb{N}_0$. If $\mathcal{S}$ is the universal relation $\mathcal{S}_X$, then $\{(a_n,b_n)\}$ is called* a $(g,f)_N$-sequence.

Inspired by [17], we introduce the notion of generalized $(\mathcal{A},\mathcal{S})_f$-contractions as follows:

Definition 3.9. *Let (X,d) be a metric space, $B \subseteq \mathbb{R}$ such that $ran(d) \subseteq B$ and $g,f : X \to X$. If there exists a function $\varrho : B \times B \to \mathbb{R}$ such that g and ϱ satisfy the following conditions:*

($\mathcal{A}_1$) *If $\{x_n\} \subseteq X$ is a (g,f)-Picard-jungck $(f,\mathcal{S})$-increasing sequence of g such that $\varrho(d(fx_{n+1},fx_{n+2}),N(g,f,x_n,x_{n+1})) > 0$, for all $n \in \mathbb{N}_0$, then $\{fx_n\}$ is asymptotically regular on (X,d) (i.e., $\{d(fx_n,fx_{n+1})\} \to 0$);*

($\mathcal{A}_2$) *If $\{(a_n,b_n)\} \subseteq B \times B$ is a $(g,f,\mathcal{S})_N$-sequence such that $\{a_n\}$ and $\{b_n\}$ converge to the same limit $L \geq 0$ such that $L < a_n$ and $\varrho(a_n,b_n) > 0$, for all $n \in \mathbb{N}_0$, then $L = 0$;*

($\mathcal{A}_3$) *$\varrho\Big(d(gx,gy), N(g,f,x,y)\Big) > 0$, for all $x,y \in X$ such that $fx\mathcal{S}^{\varkappa}fy$ and $gx\mathcal{S}^{\varkappa}gy$.*

Then g is said to be a generalized $(\mathcal{A},\mathcal{S})_f$-contraction w.r.t. ϱ.
The set of all generalized $(\mathcal{A},\mathcal{S})_f$-contractions with respect to $\varrho : B \times B \to \mathbb{R}$ is denoted by $\mathcal{A}_{\mathcal{S},f,B}$.
By choosing $f = I_X$, $\mathcal{S} = \mathcal{S}_X$ and $f = I_X$ with $\mathcal{S} = \mathcal{S}_X$, we respectively deduce generalized $(\mathcal{A},\mathcal{S})$-contractions, generalized $\mathcal{A}_f$-contractions *and* generalized $\mathcal{A}$-contractions *w.r.t. ϱ.*

Some times, we also consider the following condition:

($\mathcal{A}_4$) *If $\{(a_n,b_n)\}$ is a $(g,f,\mathcal{S})_N$-sequence such that $\{b_n\} \to 0$ and $\varrho(a_n,b_n) > 0$, for all $n \in \mathbb{N}_0$, then $\{a_n\} \to 0$.*

Remark 3.10. *Observe that conditions ($\mathcal{A}_1$), ($\mathcal{A}_2$) and ($\mathcal{A}_4$) are established under the existence of a sequence with suitable conditions. Conventionally, in the case of nonexistence of such sequences the conditions ($\mathcal{A}_1$), ($\mathcal{A}_2$) and ($\mathcal{A}_4$) are assumed to be satisfied vacuously.*

Proposition 3.11. *If g is generalized $(\mathcal{R},\mathcal{S})_f$-contraction with respect to $\varrho : B \times B \to \mathbb{R}$, then it is generalized $(\mathcal{A},\mathcal{S})_f$-contraction (w.r.t. ϱ).*

Proof. The proof follows from the fact that (ϱ_1) $\Rightarrow$ ($\mathcal{A}_1$), (ϱ_2) $\Rightarrow$ ($\mathcal{A}_2$), (ϱ_3) $\Rightarrow$ ($\mathcal{A}_4$) and (1) is equivalent to ($\mathcal{A}_3$). □

Lemma 3.12. *If $\varrho(u,v) \leq v - u$, for all $u,v > 0$ and ($\mathcal{A}_2$) holds, then ($\mathcal{A}_1$) holds.*

Proof. Let $\{x_n\} \subseteq X$ be a (g,f)-Picard-jungck $(f,\mathcal{S})$-increasing sequence of g such that

$$\varrho(d(fx_{n+1},fx_{n+2}),N(g,f,x_n,x_{n+1})) > 0 \quad \text{, for all} \quad n \in \mathbb{N}_0.$$

Observe that

$$N(g,f,x_n,x_{n+1}) = \max\{d(fx_n,fx_{n+1}), d(fx_n,gx_n), d(fx_{n+1},gx_n)\} = d(fx_n,fx_{n+1}).$$

Now, as $\varrho(d(fx_{n+1},fx_{n+2}),N(g,f,x_n,x_{n+1})) > 0$, $fx_n \neq fx_{n+1}$, for all $n \in \mathbb{N}_0$, $\varrho(u,v) \leq v-u$, for all $u,v > 0$, we have

$$d(fx_n,fx_{n+1}) - d(fx_{n+1},fx_{n+2}) \geq \varrho(d(fx_{n+1},fx_{n+2}), d(fx_n,fx_{n+1})) > 0,$$

which implies that $d(fx_{n+1},fx_{n+2}) < d(fx_n,fx_{n+1})$ so that $\{d(fx_n,fx_{n+1})\}$ is a strictly decreasing sequence of positive real numbers. Hence, $\{d(fx_n,fx_{n+1})\} \to L \geq 0$. Let $\{a_n\}$ and $\{b_n\}$ be the two sequences of positive real numbers defined by $a_n = d(fx_{n+1},fx_{n+2})$ and $b_n = d(fx_n,fx_{n+1})$, for all $n \in \mathbb{N}_0$. Then $a_n > 0$ and $b_n > 0$, for all $n \in \mathbb{N}_0$ and $\{(a_n,b_n)\} \subseteq B \times B$ is a $(g,f,\mathcal{S})_N$-sequence such that $\{a_n\}$ and $\{b_n\}$ converge to the same limit $L \geq 0$ and satisfying that $L < a_n$ (as $\{a_n\}$ is strictly decreasing sequence of positive real numbers converging to L) and $\varrho(a_n,b_n) > 0$, for all $n \in \mathbb{N}_0$. Hence, $L = 0$ (due to ($\mathcal{A}_2$)). Therefore, we have $\lim_{n\to 0} d(fx_n,fx_{n+1}) = 0$, i.e., $\{fx_n\}$ is asymptotically regular. Hence, ($\mathcal{A}_1$) holds. □

Lemma 3.13. *If $\varrho(u,v) \leq v-u$, for all $u,v>0$, then $(\mathcal{A}_4)$ holds.*

Proof. Let $\{(a_n,b_n)\}$ be a $(g,f,\mathcal{S})_N$-sequence such that $\{b_n\}\to 0$ and $\varrho(a_n,b_n)>0$, for all $n\in\mathbb{N}_0$, then there exist two sequences $\{x_n\},\{y_n\}\subseteq X$ such that $fx_n\mathcal{S}fy_n$, $a_n=d(gx_n,gy_n)>0$ and $b_n=N(g,f,x_n,y_n)>0$, for all $n\in\mathbb{N}_0$. Since $\varrho(u,v)\leq v-u$, for all $u,v>0$ and $\varrho(a_n,b_n)>0$, for all $n\in\mathbb{N}_0$, we have $b_n-a_n\geq\varrho(a_n,b_n)>0$, for all $n\in\mathbb{N}_0$. Therefore, we have $b_n>a_n>0$, for all $n\in\mathbb{N}_0$, which on letting $n\to\infty$ implies that $\lim\limits_{n\to\infty}a_n=0$ (as $\{b_n\}\to 0$). Hence, $(\mathcal{A}_4)$ holds. □

Remark 3.14. *Observe that (in view of Lemmas 3.12 and 3.13) if $\varrho(u,v)\leq v-u$, for all $u,v>0$ and $(\mathcal{A}_2)$ holds, then $(\mathcal{A}_1)$ and $(\mathcal{A}_4)$ both hold.*

4. Main results

Firstly, we present a result on the existence of a coincidence point under $(I,\mathcal{S})$-continuity, which runs as follows:

Theorem 4.1. *Let (X,d) be a metric space equipped with a binary relation $\mathcal{S}$ and $g,f:X\to X$ two $(I,\mathcal{S})$-continuous mappings such that g is generalized $(\mathcal{A},\mathcal{S})_f$-contraction w.r.t. $\varrho: B\times B\to\mathbb{R}$. Assume that*

(*a*) *there exists a $(g,f,\mathcal{S})$-Picard-Jungck sequence in X;*

(*b*) *gX is $(\mathcal{S},d)$-increasingly precomplete;*

(*c*) *the pair (g,f) is $(I,\mathcal{S})$-compatible.*

Then the pair (g,f) has a coincidence point. Indeed, if $\{x_n\}$ is any $(g,f,\mathcal{S})$-Picard-Jungck sequence, then either $\{fx_n\}$ contains a coincidence point of the pair (g,f) or $\{fx_n\}$ converges to a coincidence point of the pair (g,f).

Before presenting the proof, let us point out the advantages of the hypotheses utilized in this theorem over earlier ones.

- Due to Proposition 2.14, hypothesis (*a*) is guaranteed if one of the following conditions holds.
  - (*i*) If $\mathcal{S}$ is locally f-transitive (or locally (g,f)-transitive), $gX\subseteq fX$, g is $(f,\mathcal{S})$-nondecreasing and there exists a point $x_0\in X$ such that $fx_0\mathcal{S}gx_0$;
  - (*ii*) If $\mathcal{S}$ is locally f-transitive (or locally (g,f)-transitive) and $gX\subseteq_{\mathcal{S}} fX$.

  Herein, we observe that the binary relation $\mathcal{S}$ is locally f-transitive (or locally (g,f)-transitive), and indeed, transitivity $\Rightarrow$ f-transitivity $\Rightarrow$ locally f-transitivity $\Rightarrow$ locally (g,f)-transitivity, (g,f)-transitivity $\Rightarrow$ locally (g,f)-transitivity. Moreover, $\mathcal{S}$ need not to be reflexive or antisymmetric.

- g and f are $(I,\mathcal{S})$-continuous, and indeed, continuity $\Rightarrow$ $\mathcal{S}$-continuity $\Rightarrow$ $(I,\mathcal{S})$-continuity;

- gX is $(\mathcal{S},d)$-increasingly precomplete which is relatively weaker than the following conditions:
  - (*i*) gX is precomplete;
  - (*ii*) X or gX is complete;
  - (*iii*) there exists a complete subset $Y\subseteq X$ such that $gX\subseteq Y\subseteq X$;
  - (*iv*) X is complete and gX is closed.

  Moreover, if any one of these four preceeding conditions holds, then gX is $(\mathcal{S},d)$-increasingly precomplete;

- (g,f) is $(I,\mathcal{S})$-compatible, and indeed, commutativity $\Rightarrow$ $(O,\mathcal{S})$-compatibility $\Rightarrow$ $(I,\mathcal{S})$-compatibility.

Proof. Observe that, hypothesis (a) ensures the existence of a $(g, f, \mathcal{S})$-Picard-Jungck sequence $\{x_n\} \subseteq X$, such that $fx_{n+1} = gx_n$ and $fx_n \mathcal{S} fx_m$, for all $n, m \in \mathbb{N}_0$ with $n < m$. If $fx_{n_0+1} = fx_{n_0}$, for some $n_0 \in \mathbb{N}_0$, then x_{n_0} is a coincidence point of the pair (g, f). Assume that $fx_n \neq fx_{n+1}$, for all $n \in \mathbb{N}_0$. Then, $fx_n \mathcal{S}^{\varkappa} fx_{n+1}$ (i.e., $\{x_n\}$ is $(f, \mathcal{S})$-increasing) and $gx_n \mathcal{S}^{\varkappa} gx_{n+1}$, for all $n \in \mathbb{N}_0$. Using $(\mathcal{A}_3)$, we have

$$0 < \varrho(d(gx_n, gx_{n+1}), N(g, f, x_n, x_{n+1})) = \varrho(d(fx_{n+1}, fx_{n+2}), N(g, f, x_n, x_{n+1})),$$

where

$$N(g, f, x_n, x_{n+1}) = \max\{d(fx_n, fx_{n+1}), d(fx_n, gx_n), d(fx_{n+1}, gx_n)\} = d(fx_n, fx_{n+1}).$$

Therefore, we have

$$\varrho(d(fx_{n+1}, fx_{n+2}), d(fx_n, fx_{n+1})) > 0, \quad \text{for all } n \in \mathbb{N}_0.$$

Applying $(\mathcal{A}_1)$, we deduce that $\{fx_n\}$ is asymptotically regular, i.e.,

$$\lim_{n\to 0} d(fx_n, fx_{n+1}) = 0. \tag{2}$$

Now, we prove that $\{fx_n\}$ is a Cauchy sequence. To accomplish this, let on contrary that $\{fx_n\}$ is not Cauchy, then Lemma 2.28 and (2) guarantee the existence of an $\epsilon_0 > 0$ and two subsequences $\{fx_{n(k)}\}$ and $\{fx_{m(k)}\}$ of $\{fx_n\}$ such that $k \leq n(k) \leq m(k)$, $d(fx_{n(k)}, fx_{m(k)-1}) \leq \epsilon_0 < d(fx_{n(k)}, fx_{m(k)})$, $\forall k \in \mathbb{N}_0$ and

$$\lim_{k\to\infty} d(fx_{n(k)}, fx_{m(k)}) = \lim_{k\to\infty} d(fx_{n(k)-1}, fx_{m(k)-1}) = \epsilon_0. \tag{3}$$

Since $\{d(fx_{n(k)}, fx_{m(k)})\}$ and $\{d(fx_{n(k)-1}, fx_{m(k)-1})\}$ converge to ϵ_0, there exists $k_0 \in \mathbb{N}_0$ such that $d(fx_{n(k)}, fx_{m(k)}) > 0$ and $d(fx_{n(k)-1}, fx_{m(k)-1}) > 0$, $\forall k \geq k_0$. In particular, $d(gx_{n(k)-1}, gx_{m(k)-1}) = d(fx_{n(k)}, fx_{m(k)}) > 0$ and $d(fx_{n(k)-1}, fx_{m(k)-1}) > 0$, for all $k \geq k_0$ so that $gx_{n(k)-1} \mathcal{S}^{\varkappa} gx_{m(k)-1}$ and $fx_{n(k)-1} \mathcal{S}^{\varkappa} fx_{m(k)-1}$, for all $k \geq k_0$ (in view of (a) and Proposition 2.14). On using $(\mathcal{A}_3)$, we have

$$\begin{aligned} 0 &< \varrho(d(gx_{n(k)-1}, gx_{m(k)-1}), N(g, f, x_{n(k)-1}, x_{m(k)-1})) \\ &= \varrho(d(fx_{n(k)}, fx_{m(k)}), N(g, f, x_{n(k)-1}, x_{m(k)-1})), \quad \text{for all } k \geq k_0, \end{aligned} \tag{4}$$

where

$$\begin{aligned} N(g, f, x_{n(k)-1}, x_{m(k)-1}) &= \max\{d(fx_{n(k)-1}, fx_{m(k)-1}), d(fx_{n(k)-1}, gx_{n(k)-1}), d(fx_{m(k)-1}, gx_{n(k)-1})\} \\ &= \max\{d(fx_{n(k)-1}, fx_{m(k)-1}), d(fx_{n(k)-1}, fx_{n(k)}), d(fx_{m(k)-1}, fx_{n(k)})\}. \end{aligned}$$

Using (2), (3) and triangle inequality one can prove that $\{N(g, f, x_{n(k)-1}, x_{m(k)-1})\} \to \epsilon_0$. Let $L = \epsilon_0$, $a_k = d(fx_{n(k)}, fx_{m(k)}) > 0$ and $b_k = N(g, f, x_{n(k)-1}, x_{m(k)-1}) > 0$, for all $k \geq k_0$. Then, $\{(a_k, b_k)\} \subseteq B \times B$ is a $(g, f, \mathcal{S})_N$-sequence such that $\{a_k\}$ and $\{b_k\}$ converge to the same limit $L \geq 0$. Since $L = \epsilon_0 < d(fx_{n(k)}, fx_{m(k)}) = a_k$ and $\varrho(a_k, b_k) > 0$, for all $k \geq k_0$ (in view of (4)), therefore the condition $(\mathcal{A}_2)$ guarantees that $\epsilon_0 = L = 0$, a contradiction. Thus, $\{fx_{n+1} = gx_n\} \subseteq gX$ is a Cauchy sequence which is $\mathcal{S}$-increasing. Since gX is $(\mathcal{S}, d)$-increasingly precomplete (in view of the condition (b)), there exists $y \in X$ such that $\{fx_n\} \to y$. Moreover, as g and f are $(I, \mathcal{S})$-continuous, we have $\{gfx_n\} \to gy$ and $\{ffx_n\} \to fy$.

As $\{x_n\}$ is $(f, \mathcal{S})$-increasing and $\{gx_n = fx_{n+1}\} \to y$, therefore $(I, \mathcal{S})$- compatibility of the pair (g, f) yields that

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0.$$

Observe that $d(fy, gy) = d\Big(\lim_{n\to\infty} ffx_{n+1}, \lim_{n\to\infty} gfx_n\Big) = \lim_{n\to\infty} d(fgx_n, gfx_n) = 0$. Hence, $gy = fy$ so that y is a coincidence point of the pair (g, f). This concludes the proof. □

Theorem 4.2. *Conclusions of Theorem 4.1 remain true if $(\mathcal{A}_1)$ is replaced by the following condition:*

- $\varrho(u, v) \leq v - u$, *for all $u, v > 0$.*

Proof. This theorem is immediate in view of Lemma 3.12 and Theorem 4.1. □

Now, we prove a coincidence point result under $\mathcal{S}$-increasing regularity.

Theorem 4.3. *Let (X, d) be a metric space equipped with a binary relation $\mathcal{S}$ and $g, f : X \to X$ such that g is generalized $(\mathcal{A}, \mathcal{S})_f$-contraction w.r.t. $\varrho : B \times B \to \mathbb{R}$. Assume that*

- (a) *there exists a $(g, f, \mathcal{S})$-Picard-Jungck sequence in X;*
- (b) *(X, d) is $(\mathcal{S}, d)$-increasingly regular;*
- (c) *$(\mathcal{A}_4)$ holds and g is $(f, \mathcal{S})$-nondecreasing.*

If in addition, at lest, one of the following conditions holds:

- (d) *fX is $(\mathcal{S}, d)$-increasingly complete;*
- (e) *fX is $(\mathcal{S}, d)$-increasingly precomplete, the pair (g, f) is $(I, \mathcal{S})$-compatible, f is $\mathcal{S}$-nondecreasing and f is $(I, \mathcal{S})$-continuous and injective on fX,*

then the pair (g, f) has a coincidence point. Indeed, if $\{x_n\}$ is any $(g, f, \mathcal{S})$-Picard-Jungck sequence, then either $\{fx_n\}$ contains a coincidence point of the pair (g, f) or $\{fx_n\}$ converges to a coincidence point of the pair (g, f).

Proof. Following the proof of Theorem 4.1, we can prove that the sequence $\{fx_n\}$ is an asymptotically regular $\mathcal{S}$-increasing Cauchy sequence. Now, we distinguish two cases as follows:
Case 1. Assume that the condition (d) holds. Since fX is $(\mathcal{S}, d)$-increasingly complete, there exists $z \in fX$ such that $\{fx_n\} \to z$. Now, we prove that any point $y \in f^{-1}z$ is a coincidence point of the pair (g, f). Let $y \in X$ be an arbitrary point such that $y = fz$. Since $\{fx_n\} \to fy$ and (X, d) is $(\mathcal{S}, d)$-increasingly regular (in view of (b)), we have $fx_n\mathcal{S}fy$, for all $n \in \mathbb{N}_0$. Now, if there exists some $n_0 \in \mathbb{N}_0$ such that $fx_{n_0} = fz$, then $fx_{n_0+1} \neq fz$ and hence the set $\{n \in \mathbb{N}_0 : fx_n \neq fz\}$ is infinite. Thus, there exists a subsequence $\{fx_{n(k)}\}$ of $\{fx_n\}$ such that $d(fx_{n(k)}, fz) > 0$, for all $k \in \mathbb{N}_0$. So, $fx_{n(k)}\mathcal{S}^{\varkappa}fz$, for all $k \in \mathbb{N}_0$. Let $P = \{k \in \mathbb{N}_0 : gx_{n(k)} = gz\}$. Here, arise two sub-cases. Firstly, if P is finite, then there exists $k_0 \in \mathbb{N}_0$ such that $gx_{n(k)} \neq gz$, for all $k \geq k_0$. Since $fx_{n(k)}\mathcal{S}fz$, for all $k \in \mathbb{N}_0$ and g is $(f, \mathcal{S})$-nondecreasing, therefore $gx_{n(k)}\mathcal{S}gz$, for all $k \geq k_0$ and henceforth, $gx_{n(k)}\mathcal{S}^{\varkappa}gz$, for all $k \geq k_0$. Therefore, on using the contractivity condition $(\mathcal{A}_3)$, we have

$$\varrho(d(gx_{n(k)}, gz), N(g, f, x_{n(k)}, z)) > 0, \quad \text{for all } k \geq k_0, \tag{5}$$

where

$$\begin{aligned} N(g, f, x_{n(k)}, z) &= \max\{d(fx_{n(k)}, fz), d(fx_{n(k)}, gx_{n(k)}), d(fz, gx_{n(k)})\} \\ &= \max\{d(fx_{n(k)}, fz), d(fx_{n(k)}, fx_{n(k)+1}), d(fz, fx_{n(k)+1})\}. \end{aligned}$$

Let $a_k = d(gx_{n(k)}, gz)$ and $b_k = N(g, f, x_{n(k)}, z)$, for all $k \geq k_0$. Then $a_k > 0$ and $b_k > 0$, for all $k \geq k_0$. Furthermore, $\{b_k\} \to 0$ and (5) guarantees that $\varrho(a_k, b_k) > 0$, for all $k \geq k_0$. Hence, on applying $(\mathcal{A}_4)$, we get that $\{a_k\} \to 0$ so that $\{fx_{n(k)+1} = gx_{n(k)}\} \to gz$. As $\{fx_{n(k)}\} \subseteq \{fx_n\}$ and $\{fx_n\} \to fz$, we conclude that $gz = fz$. Therefore, z is a coincidence point of the pair (g, f). Secondly, if P is infinite, then there exists a subsequence $\{gx_{n'(k)}\}$ of $\{gx_{n(k)}\}$ such that $gx_{n'(k)} = gz$, for all $k \in \mathbb{N}_0$. As $fx_{n'(k)+1} = gx_{n'(k)} = gz$, for all $k \in \mathbb{N}_0$, we have $\{fx_{n'(k)}\} \to gz$. Since $\{fx_{n'(k)}\} \subseteq \{fx_n\}$ and $\{fx_n\} \to fz$, we conclude that $fz = gz$. Therefore, z is a coincidence point of the pair (g, f) in both the sub-cases.
Case 2. Assume that the condition (e) holds. Since fX is $(\mathcal{S}, d)$-increasingly precomplete and $\{fx_n\} \subseteq fX$ is an $\mathcal{S}$-increasing Cauchy sequence, then there exists $y \in X$ such that $\{fx_n\} \to y$. As f is $(I, \mathcal{S})$-continuous, we have $\{ffx_n\} \to fy$. Moreover, the $(I, \mathcal{S})$-compatibility of the pair (g, f) leads to

$$\lim_{n\to\infty} d(fy, gfx_n) = \lim_{n\to\infty} d(ffx_{n+1}, gfx_n) = \lim_{n\to\infty} d(fgx_n, gfx_n) = 0.$$

Thus, we have

$$\{gfx_n\} \to fy. \tag{6}$$

Let $Q = \{n \in \mathbb{N}_0 : gfx_n = gy\}$. Again, we have two sub-cases depending on Q. Firstly, assume that Q is infinite, then there exists a subsequence $\{fx_{n(k)}\}$ of $\{fx_n\}$ such that $gfx_{n(k)} = gy$, for all $k \in \mathbb{N}_0$. Hence, $\{gfx_{n(k)}\} \to gy$. Since $\{gfx_n\} \to fy$, then $gy = fy$, proving y is a coincidence point of the pair (g, f) and hence the proof is finished in this sub-case. Secondly, if Q is finite, then there exists an $n_0 \in \mathbb{N}_0$ such that $gfx_n \neq gy$, for all $n \geq n_0$. Let us assume that

$$gfx_n \neq gy, \quad \text{for all } n \in \mathbb{N}_0. \tag{7}$$

Since X is $(\mathcal{S}, d)$-increasingly regular, $\{fx_n\}$ is $\mathcal{S}$-increasing sequence, $\{fx_n\} \to y$ and $fx_n\mathcal{S}fx_m$, for all $n < m$, therefore $fx_n\mathcal{S}y$, for all $n \in \mathbb{N}_0$. Moreover, as f is $\mathcal{S}$-nondecreasing, we have

$$ffx_n\mathcal{S}fy, \quad \text{for all } n \in \mathbb{N}_0. \tag{8}$$

As earlier, we distinguish two sub-sub-cases depending on $Q^* = \{n \in \mathbb{N}_0 : fx_n = y\}$. Firstly, assume that Q^* is finite, then there exists $n_0 \in \mathbb{N}_0$ such that $fx_n \neq y$, for all $n \geq n_0$. Then $ffx_n \neq fy$, for all $n \geq n_0$ (as f is injective). Hence, $ffx_n\mathcal{S}^{\neq}fy$, for all $n \geq n_0$. As g is $(f, \mathcal{S})$-nondecreasing, therefore due to (7) and (8), we have $gfx_n\mathcal{S}^{\neq}gy$, for all $n \geq n_0$. Let $a_n = d(gfx_n, gy) > 0$ and $b_n = N(g, f, fx_n, y) > 0$, for all $n \geq n_0$, where $N(g, f, fx_n, y) = \max\{d(ffx_n, fy), d(ffx_n, gfx_n), d(fy, gfx_n)\}$. Applying the contractivity condition $(\mathcal{A}_3)$, we have

$$\varrho(a_n, b_n) = \varrho(d(gfx_n, gy), N(g, f, fx_n, y)) > 0, \quad \text{for all } n \geq n_0. \tag{9}$$

Since $\{ffx_n\} \to fy$ and $\{gfx_n\} \to fy$, therefore $\{b_n\} \to 0$. Thus, on using the condition $(\mathcal{A}_4)$, we have $\{a_n\} \to 0$ so that $\{gfx_n\} \to gy$. Hence, $gy = fy$ (due to(6)) so that y is a coincidence point of the pair (g, f). Secondly, assume that Q^* is infinite. Then there exists a subsequence $\{fx_{n(k)}\} \subseteq \{fx_n\}$ such that $fx_{n(k)} = y$, for all $k \in \mathbb{N}_0$ which implies that $gfx_{n(k)} = gy$, for all $k \in \mathbb{N}_0$. Hence, $\{gfx_{n(k)}\} \to gy$. As $\{gfx_n\} \to fy$ (due to (6)), we have $gy = fy$ so that y is a coincidence point of the pair (g, f) (in all cases). This completes the proof. □

Theorem 4.4. *Conclusions of Theorem 4.3 remain true if conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_4)$ are replaced by the following condition:*

- $\varrho(u, v) \leq v - u$, *for all* $u, v > 0$.

Proof. This theorem is immediate in view of Lemmas 3.12, 3.13 and Theorem 4.3. □

Lemma 4.5. *Under the hypotheses of Theorem 4.1 (or Theorem 4.3), let $x, y \in Coin(g, f)$ and assume that fx and fy are $\mathcal{S}$-comparable and $\varrho(u, u) \leq 0$, for all $u > 0$. Then $fx = fy$.*

Proof. Suppose, on contrary, that $fx \neq fy$. Since fx and fy are $\mathcal{S}$-comparable, therefore without loss of generality, we can assume that $fx\mathcal{S}fy$. As $gx = fx$ and $gy = fy$, we have $fx\mathcal{S}^{\neq}fy$ and $gx\mathcal{S}^{\neq}gy$. As $\varrho(u, u) \leq 0$, for all $u > 0$, on applying $(\mathcal{A}_3)$, we have

$$0 \geq \varrho(d(fx, fy), d(fx, fy)) = \varrho(d(gx, gy), N(g, f, x, y)) > 0,$$

a contradiction. Therefore, $fx = fy$. □

Next, we prove a corresponding uniqueness result as follows:

Theorem 4.6. *If in addition to the hypotheses of Theorem 4.1 (or Theorem 4.3), we assume that, for all distinct coincidence points $x, y \in Coin(g, f)$, fx and fy are $\mathcal{S}$-comparable and $\varrho(u, u) \leq 0$, for all $u > 0$. Then the pair (g, f) has a unique point of coincidence. Moreover, If g or f is injective on $Coin(g, f)$, then the pair (g, f) has a unique coincidence point.*

Proof. In view of Theorem 4.1 (or Theorem 4.3) the set $Coin(g, f)$ is nonempty so that the pair (g, f) has, at least, one coincidence point. Let z and z^* be two points of coincidence of the pair (g, f), then there exist $x, y \in Coin(g, f)$ such that $z = gx = fx$ and $z^* = gy = fy$. Lemma 4.5 implies that $z = fx = fy = z^*$. Hence, (g, f) has a unique point of coincidence.

Now, assume that f (or g) is injective on $Coin(g, f)$ and let $x, y \in Coin(g, f)$. On contrary, suppose that $x \neq y$. By Lemma 4.5, we have $gx = fx = fy = gy$. Since f (or g) is injective on $Coin(g, f)$, we obtain $x = y$, a contradiction. Thus, (g, f) has a unique coincidence point. □

Now, we present a common fixed point result, which runs as follows:

Theorem 4.7. *If in addition to the hypotheses of Theorem 4.6, we assume that g and f are weakly compatible, then the pair (g, f) has a unique common fixed point.*

Proof. Theorem 4.6 guarantees the existence of a unique coincidence point of the pair (g, f), let x be such point and let $z \in X$ be such that $z = gx = fx$. As g and f are weakly compatible, we have $gz = gfx = fgx = fz$. Thus, z is a coincidence point of g and f. As x is unique, we must have $x = z = gx = fx$. Therefore, x is a common fixed point of (g, f) which is indeed unique (in view of the uniqueness of the coincidence point of (g, f)). □

5. Some consequences

In this section, as consequences of our results, we derive several results involving coincidence point, common fixed point and fixed point results.

Firstly, we derive the following coincidence point results by setting $\mathcal{S} = \mathcal{S}_X$ in Theorems 4.1 and 4.3 respectively.

Corollary 5.1. *Let (X, d) be a metric space and $g, f : X \to X$ two I-continuous mappings such that g is generalized $\mathcal{A}_f$-contraction w.r.t. $\varrho : B \times B \to \mathbb{R}$. Assume that*

(*a*) *there exists a (g, f)-Picard-Jungck sequence in X;*

(*b*) *gX is increasingly precomplete;*

(*c*) *the pair (g, f) is I-compatible.*

Then the pair (g, f) has a coincidence point. Indeed, if $\{x_n\}$ is any (g, f)-Picard-Jungck sequence, then either $\{fx_n\}$ contains a coincidence point of the pair (g, f) or $\{fx_n\}$ converges to a coincidence point of the pair (g, f).

Corollary 5.2. *Let (X, d) be a metric space and $g, f : X \to X$ such that g is generalized $\mathcal{A}_f$-contraction w.r.t. $\varrho : B \times B \to \mathbb{R}$. Assume that there exists a (g, f)-Picard-Jungck sequence in X and $(\mathcal{A}_4)$ holds. If in addition, at least, one of the following conditions holds:*

(*a*) *fX is increasingly complete;*

(*b*) *fX is increasingly precomplete, the pair (g, f) is I-compatible, f is I-continuous and injective on fX,*

then the pair (g, f) has a coincidence point. Indeed, if $\{x_n\}$ is any (g, f)-Picard-Jungck sequence, then either $\{fx_n\}$ contains a coincidence point of the pair (g, f) or $\{fx_n\}$ converges to a coincidence point of the pair (g, f).

The following common fixed point result is a sharpened version of Theorem 33 due to Hierro and Shahzad [15].

Corollary 5.3. *Let (X, d) be a metric space equipped with a binary relation $\mathcal{S}$ and $g, f : X \to X$ two $(I, \mathcal{S})$-continuous mappings such that g is generalized $(\mathcal{R}, \mathcal{S})_f$-contraction w.r.t. an $\mathcal{R}$-function $\varrho \in \mathcal{R}_B$. Assume that*

(*a*) *there exists a $(g, f, \mathcal{S})$-Picard-Jungck sequence in X;*

(*b*) *gX is $(\mathcal{S}, d)$-increasingly precomplete;*

(*c*) *the pair (g, f) is $(I, \mathcal{S})$-comparable.*

Then the pair (g, f) has a coincidence point. Moreover, if g and f are weakly compatible and for all distinct points $x, y \in Coin(g, f)$, fx and fy are $\mathcal{S}$-comparable, then (g, f) has a unique common fixed point.

Proof. As every generalized $(\mathcal{R}, \mathcal{S})_f$-contraction is generalized $(\mathcal{A}, \mathcal{S})_f$-contraction, Theorem 4.1 ensures the existence of a coincidence point of the pair (g, f). Also, since every $\varrho \in \mathcal{R}_B$ satisfies $\varrho(u, u) \leq 0$, for all $u > 0$, therefore Theorem 4.7 proves that (g, f) has a unique common fixed point. □

The following common fixed point result is a sharpened version of Theorem 37 due to Hierro and Shahzad [15].

Corollary 5.4. *Let (X, d) be a metric space equipped with a binary relation $\mathcal{S}$ and $g, f : X \to X$ such that g is generalized $(\mathcal{R}, \mathcal{S})_f$-contraction w.r.t. an $\mathcal{R}$-function $\varrho \in \mathcal{R}_B$. Assume that*

(*a*) *there exists a $(g, f, \mathcal{S})$-Picard-Jungck sequence in X;*

(*b*) *(X, d) is $(\mathcal{S}, d)$-increasingly regular;*

(*c*) *$(\mathcal{A}_4)$ holds (or $\varrho(u, v) \leq v - u$, $\forall\, u, v > 0$) and g is $(f, \mathcal{S})$-nondecreasing.*

If in addition, at least, one of the following conditions holds:

(*d*) *fX is $(\mathcal{S}, d)$-increasingly complete;*

(*e*) *fX is $(\mathcal{S}, d)$-increasingly precomplete, the pair (g, f) is $(I, \mathcal{S})$-compatible, f is $\mathcal{S}$-nondecreasing and f is $(I, \mathcal{S})$-continuous and injective on fX,*

then the pair (g, f) has a coincidence point. Moreover, if g and f are weakly compatible and for all distinct points $x, y \in Coin(g, f)$, fx and fy are $\mathcal{S}$-comparable, then (g, f) has a unique common fixed point.

Proof. As every generalized $(\mathcal{R}, \mathcal{S})_f$-contraction is generalized $(\mathcal{A}, \mathcal{S})_f$-contraction, Theorem 4.3 (if $\mathcal{A}_4$ holds) (or Theorem 4.4 if $\varrho(u, v) \leq v - u$, $\forall\, u, v > 0$) ensures the existence of a coincidence point of the pair (g, f). Also, since every $\varrho \in \mathcal{R}_B$ satisfies $\varrho(u, u) \leq 0$, for all $u > 0$, therefore Theorem 4.7 proves that (g, f) has a unique common fixed point. □

The following common fixed point results are obtained by assuming g to be generalized $(\mathcal{Z}, \mathcal{S})_f$-contraction. These results present sharpened versions of the main result of [14].

Corollary 5.5. *Let (X, d) be a metric space equipped with a binary relation $\mathcal{S}$ and $g, f : X \to X$ two $(I, \mathcal{S})$-continuous mappings such that g is generalized $(\mathcal{Z}, \mathcal{S})_f$-contraction w.r.t. a simulation function ζ. Assume that*

(*a*) *there exists a $(g, f, \mathcal{S})$-Picard-Jungck sequence in X;*

(*b*) *gX is $(\mathcal{S}, d)$-increasingly precomplete;*

(*c*) *the pair (g, f) is $(I, \mathcal{S})$-comparable.*

Then the pair (g, f) has a coincidence point. Moreover, if g and f are weakly compatible and for all distinct points $x, y \in Coin(g, f)$, fx and fy are $\mathcal{S}$-comparable, then (g, f) has a unique common fixed point.

Proof. This corollary is immediate in view of Proposition 2.14 and Corollary 5.3. □

Corollary 5.6. *Let (X, d) be a metric space equipped with a binary relation $\mathcal{S}$ and $g, f : X \to X$ such that g is generalized $(\mathcal{Z}, \mathcal{S})_f$-contraction w.r.t. a simulation function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$. Assume that*

(*a*) *there exists a $(g, f, \mathcal{S})$-Picard-Jungck sequence in X;*

(*b*) (X, d) *is* $(\mathcal{S}, d)$*-increasingly regular;*

(*c*) $(\mathcal{A}_4)$ *holds (or* $\varrho(u, v) \leq v - u,\ \forall\ u, v > 0$ *) and* g *is* $(f, \mathcal{S})$*-nondecreasing.*

If in addition, at least, one of the following conditions holds:

(*d*) fX *is* $(\mathcal{S}, d)$*-increasingly complete;*

(*e*) fX *is* $(\mathcal{S}, d)$*-increasingly precomplete, the pair* (g, f) *is* $(I, \mathcal{S})$*-compatible,* f *is* $\mathcal{S}$*-nondecreasing and* f *is* $(I, \mathcal{S})$*-continuous and injective on* fX*,*

then the pair (g, f) *has a coincidence point. Moreover, if* T *and* f *are weakly compatible and for all distinct points* $x, y \in Coin(g, f)$*,* fx *and* fy *are* $\mathcal{S}$*-comparable, then* (g, f) *has a unique common fixed point.*

Proof. This corollary is immediate in view of Proposition 2.14 and Corollary 5.4. □

The following two fixed point results remain sharpened versions of the main result of Shahzad et al. [17].

Corollary 5.7. *Let* (X, d) *be a metric space equipped with a binary relation* $\mathcal{S}$ *and* $g : X \to X$ *be an* $(I, \mathcal{S})$*-continuous mapping such that* g *is generalized* $(\mathcal{A}, \mathcal{S})$*-contraction w.r.t.* $\varrho : B \times B \to \mathbb{R}$*. Assume that* gX *is* $(\mathcal{S}, d)$*-increasingly precomplete and there exists a* $(g, \mathcal{S})$*-Picard sequence in* X*. Then* g *has a fixed point.*

Proof. Taking $f = I_X$, in Theorem 4.1, one can derive the required. □

Corollary 5.8. *Let* (X, d) *be a metric space equipped with a binary relation* $\mathcal{S}$ *and* $g : X \to X$ *such that* g *is generalized* $(\mathcal{A}, \mathcal{S})$*-contraction w.r.t.* $\varrho : B \times B \to \mathbb{R}$*. Assume that*

(*a*) *there exists a* $(g, \mathcal{S})$*-Picard sequence in* X*;*

(*b*) X *is* $(\mathcal{S}, d)$*-increasingly precomplete;*

(*c*) (X, d) *is* $(\mathcal{S}, d)$*-increasingly regular.*

If in addition, at least, one of the following conditions holds:

(*d*) $(\mathcal{A}_4)$ *holds and* g *is* $\mathcal{S}$*-nondecreasing;*

(*e*) $\varrho(u, v) \leq v - u$*, for all* $u, v > 0$ *and* g *is* $\mathcal{S}$*-nondecreasing,*

then g *has a fixed point.*

Proof. If the condition (*d*) holds, then taking $f = I_X$ in Theorem 4.3, we deduce this corollary. On the other case, assume that the condition (*e*) holds, then setting $f = I_X$ in Theorem 4.4, one can derive the required. □

The reader may particularize the previous fixed point results to the cases:

- g is generalized $(\mathcal{R}, \mathcal{S})$-contraction w.r.t. an $\mathcal{R}$-function $\varrho \in \mathcal{R}_B$ (in which the main result and Corollaries 28-33 of Hierro et al. [14] are generalized).

- g is generalized $(\mathcal{Z}, \mathcal{S})$-contraction w.r.t a simulation function ζ (in which the main result of Khojasteh et al. [10] is generalized).

6. Illustrative examples

In this section, we illustrate some examples to show the utility of our results.

Example 6.1. *Let X and $\mathcal{S}$ be defined as in Example 2.15 and let d be the usual metric on X. Denote $B = ran(d) = X$ and define $\varrho : B \times B \to \mathbb{R}$ by $\varrho(u, v) = v - u$, for all $u, v \in B$. We assert that g is generalized $(\mathcal{A}, \mathcal{S})_f$-contraction w.r.t. ϱ.*
$(\mathcal{A}_1)$*: Assume that $\{x_n\} \subseteq X$ is a (g, f)-Picard-Jungck $(f, \mathcal{S})$-increasing sequence of g based on some nonzero $x_0 \in X$, otherwise $fx_n = 0$, for all n, and hence $\{x_n\}$ is not $(f, \mathcal{S})$-increasing. Now, assume that $x_0 = \frac{1}{2^{n_0}}$, for some $n_0 \geq 1$, then $x_n = \frac{1}{2^{n_0+2n}}$, for all n, and hence we have $\{d(fx_n, fx_{n+1}) = |\frac{1}{2^{n_0+2n}} - \frac{1}{2^{n_0+2(n+1)}}|\} \to 0$.*
$(\mathcal{A}_2)$*: Assume that $\{(a_n, b_n)\} \subseteq B \times B$ is a $(g, f, \mathcal{S})_N$-sequence satisfying $\{a_n\}$ and $\{b_n\}$ converge to the same limit $L \geq 0$ such that $L < a_n$ and $\varrho(a_n, b_n) > 0$, for all $n \in \mathbb{N}_0$, then there exist two sequences $\{x_n\}, \{y_n\} \subseteq X$ such that $fx_n\mathcal{S}fy_n$, $a_n = d(gx_n, gy_n) > 0$ and $b_n = N(g, f, x_n, y_n) > 0$, for all $n \in \mathbb{N}$, wherein we distinguish three cases:*
Case 1. Both $\{x_n\}$ and $\{y_n\}$ are constant sequences in X. Then there exist $r, s \in \mathbb{N}$ such that $x_n = \frac{1}{2^r}$ and $y_n = \frac{1}{2^s}$, for all $n \in \mathbb{N}$. Hence, $a_n = d(gx_n, gy_n) = |gx_n - gy_n| = \left|\frac{1}{2^r} - \frac{1}{2^s}\right|$, for all $n \in \mathbb{N}$ so that $\{a_n\} \to \left|\frac{1}{2^r} - \frac{1}{2^s}\right| = L$ which contradicts the fact that $L < a_n$, for all n. So, this case is impossible.
Case 2. One of $\{x_n\}$ and $\{y_n\}$ is constant sequence in X. Assume that $\{x_n\}$ is constant sequence while $\{y_n\}$ is not constant sequence. Then $\{y_n\}$, $\{gy_n\}$ and $\{fy_n\}$ converge to zero and there exists $r \in \mathbb{N}$ such that $x_n = \frac{1}{2^r}$, for all $n \in \mathbb{N}$. Thus, $\{gx_n = \frac{1}{2^{r+2}}\} \to \frac{1}{2^{r+2}}$ and $\{fx_n = \frac{1}{2^r}\} \to \frac{1}{2^r}$. Now,

$$a_n = d(gx_n, gy_n) = |gx_n - gy_n| \to \frac{1}{2^{r+2}} \text{ as } n \to \infty,$$

where as

$$b_n = \max\{d(fx_n, fy_n), d(fx_n, gx_n), d(fy_n, gx_n)\} \geq d(fx_n, fy_n) = |fx_n - fy_n| \to \frac{1}{2^r} \text{ as } n \to \infty,$$

which is a contradiction to the fact that $\{a_n\}$ and $\{b_n\}$ converging to the same limit. Hence, this case is also impossible.
Case 3. Both $\{x_n\}$ and $\{y_n\}$ are not constant sequences in X. Then $\{x_n\}$, $\{gx_n\}$, $\{fx_n\}$, $\{y_n\}$, $\{gy_n\}$ and $\{fy_n\}$ are all strictly decreasing sequences in X and converge to zero. Hence, $\{a_n\}$ and $\{b_n\}$ are also strictly decreasing sequences and converge to zero, i.e., $L = 0$.
$(\mathcal{A}_3)$*: Let $x, y \in X$ be such that $fx\mathcal{S}^{\varkappa}fy$ and $gx\mathcal{S}^{\varkappa}gy$. Observe that $fx \neq fy$ and $gx \neq gy$ implies $x \neq y$. Now, we have*

$$N(g, f, x, y) = \max\{d(fx, fy), d(fx, gx), d(fy, gx)\} = \max\{|x - y|, \frac{3}{4}x, \frac{1}{4}|4y - x|\} > \frac{1}{4}|x - y| = d(gx, gy).$$

Hence, $\varrho(d(gx, gy), N(g, f, x, y)) = N(g, f, x, y) - d(gx, gy) > 0$. Therefor, g is generalized $(\mathcal{A}, \mathcal{S})_f$-contraction w.r.t. ϱ. By a routine calculation, one can easily show that all the remaining hypotheses of Theorem 4.7 are satisfied. Therefor, (g, f) has a unique common fixed point (namely $x = 0$).

Observe that the binary relation $\mathcal{S}$ in Example 6.1 is not reflexive, not antisymmetric, not transitive, not f-transitive, not (g, f)-transitive, not locally transitive and not locally f-transitive . Henceforth, our results are genuine extension of several corresponding results proved under binary relations which are earlier required to be at least reflexive, antisymmetric, transitive, f-transitive or (g, f)-transitive.

Example 6.2. *Let $X = \{-1, 0, 1\}$ equipped with the usual metric and the binary relation $\mathcal{S}$ defined by:*

$$x\mathcal{S}y \iff (x, y) \in \{(-1, 0), (0, -1), (0, 0), (0, 1), (1, 1)\}.$$

Define $g, f : X \to X$ by

$$gx = 0 \text{ and } fx = x, \text{ for all } x \in X.$$

Observe that $\mathcal{S}$ is (g, f)-transitive. Put $B = ran(d) = [0, \infty)$ and define $\varrho : B \times B \to \mathbb{R}$ by $\varrho(u, v) = v$, for all $u, v \in B$. We assert that g is generalized $(\mathcal{A}, \mathcal{S})_f$-contraction w.r.t. ϱ.
$(\mathcal{A}_1)$*: It is impossible to have such kind of sequence, as it is impossible to find a (g, f)-Picard-Jungck sequence $\{x_n\} \subseteq X$ such that $fx_n\mathcal{S}^{\varkappa}fx_{n+1}$, for all $n \in \mathbb{N}_0$.*

($\mathcal{A}_2$): We claim that it is again impossible to have such kind of sequences because if $\{(a_n, b_n)\} \subseteq B \times B$ is a $(g, f, \mathcal{S})_N$-sequence such that $\{a_n\}$ and $\{b_n\}$ converge to the same limit $L \geq 0$ and $L < a_n$, for all $n \in \mathbb{N}_0$, then there exist two sequences $\{x_n\}, \{y_n\} \subseteq X$ such that $fx_n\mathcal{S}fy_n$, $a_n = d(gx_n, gy_n) > 0$ and $b_n = N(g, f, x_n, y_n) > 0$, for all $n \in \mathbb{N}_0$, which is impossible as $gx = 0$, for all $x \in X$.
($\mathcal{A}_3$): obvious.
Hence, g is generalized $(\mathcal{A}, \mathcal{S})_f$-contraction w.r.t. ϱ. By a routine calculation, one can verify the remaining conditions of Theorem 4.3. Now in view of Theorem 4.3, (g, f) has a coincidence point (namely $x = 0$).

Observe that the binary relation $\mathcal{S}$ in Example 6.2 is not reflexive, not antisymmetric, not transitive and not f-transitive. Henceforth, our results are a proper extension of several corresponding results proved under binary relations which are required to be at least reflexive, antisymmetric, transitive or f-transitive.

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