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Scattering Theory of Impulsive Sturm-Liouville Equations

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Abstract. In this paper, we investigate scattering theory of the impulsive Sturm–Liouville boundary value problem (ISBVP). In particular, we find the Jost solution and the scattering function of this problem. We also study the properties of the Jost function and the scattering function of this ISBVP. Furthermore, we present two examples by getting Jost function and scattering function of the impulsive boundary value problem. Besides, we examine the eigenvalues of these boundary value problems given in examples in detail.

1. Introduction

Let us consider the Sturm–Liouville boundary value problem

$$-y'' + q(x)y = \lambda^2 y, \quad x \in [0, \infty), \tag{1}$$

$$y(0) = 0, \tag{2}$$

where λ is a spectral parameter, q is a real-valued function and

$$\int_0^\infty x|q(x)|\mathrm{d}x < \infty. \tag{3}$$

Under the condition (3), the equation (1) has a solution $e(x, \lambda)$ satisfying the condition

$$\lim_{x\to\infty} e(x, \lambda)e^{-i\lambda x} = 1, \quad \lambda \in \overline{\mathbb{C}}_+ := \{\lambda \in \mathbb{C} : \operatorname{Im} \lambda \geq 0\}. \tag{4}$$

$e(x, \lambda)$ is called the Jost solution of (1) [7]. The Jost solution is analytic with respect to λ in $\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \operatorname{Im} \lambda > 0\}$ and continuous up to the real axis. It is well-known that the solution $e(x, \lambda)$ has an integral representation

$$e(x, \lambda) = e^{i\lambda x} + \int_x^\infty K(x, t)e^{i\lambda t}\mathrm{d}t, \quad \lambda \in \overline{\mathbb{C}}_+ \tag{5}$$

where the kernel $K(x, t)$ may be expressed in terms of the potential function q [7]. The function

$$e(\lambda) := e(0, \lambda) = 1 + \int_0^\infty K(0, t)e^{i\lambda t}\mathrm{d}t, \quad \lambda \in \overline{\mathbb{C}}_+$$

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is called the Jost function of (1). It is well known from [7] that under the condition (3), $e(\lambda)$ has a finite number of zeros in the half complex plane $\mathbb{C}_+$. They are all simple and lie on the imaginary axis. Let $i\lambda_k$, $k = 1, 2, ..., n$ be the zeros of the Jost function $e(\lambda)$, numbered in the order of increase of their module $(0 < \lambda_1 < \lambda_2 < ... < \lambda_n)$, and let m_k^{-1} be the norm of the function $e(x, i\lambda_k)$ in $L_2(0, \infty)$, i.e.,

$$m_k^{-2} = \int_0^\infty e^2(x, i\lambda_k)\mathrm{d}x, \quad k = 1, 2, ..., n,$$

and

$$\mathcal{S}(\lambda) := \frac{\overline{e(\lambda)}}{e(\lambda)}, \quad \lambda \in (-\infty, \infty).$$

The function $\mathcal{S}(\lambda)$ is the scattering function of (1)-(2). The collection

$$\{\mathcal{S}(\lambda), \lambda \in (-\infty, \infty); \lambda_k, m_k, k = 1, 2, ..., n\} \tag{6}$$

is the scattering data of boundary value problem (1)-(2). When the potential function q is given, the problem of finding scattering data (6) and learning the properties of scattering data is the direct problem for quantum scattering theory. Conversely, the problem of finding the potential function q according to the scattering data given in (6) is an inverse problem of quantum scattering theory. Quantum scattering theory of (1)-(2) was investigated in detail in [3, 5–7] and corresponding references cited therein. Let us consider the impulsive Sturm–Liouville boundary value problem (ISBVP)

$$-y'' + q(x)y = \lambda^2 y, \quad x \in [0, 1) \cup (1, \infty), \tag{7}$$

$$y(0) = 0, \tag{8}$$

$$y(1^+) = \alpha y(1^-), \quad y'(1^+) = \beta y'(1^-), \tag{9}$$

where λ is a spectral parameter, α, β are real numbers, $\alpha\beta \neq 0$ and q is a real valued function satisfying the condition

$$\int_0^\infty x|q(x)|\mathrm{d}x < \infty.$$

Note that, the condition (9) is an impulsive condition for the equation (7). In literature impulsive conditions are called different kinds of names. Some of this names are jump condition, interface condition, point interaction condition and transmission condition. In particular, regular impulsive boundary value problems have been investigated by Mukhtarov *et al.* [8–13]. Singular impulsive problems have been studied in [2, 16–20]. In [1, 4, 15], the authors have examined the general theory of impulsive differential equations. In this paper, we investigate the scattering theory of ISBVP (7)-(8). In particular, we find Jost solution, Jost function and the scattering function of (7)-(8). We also studied the properties of the scattering function (7)-(8). Furthermore, we obtain the scattering function and Jost function of two different impulsive boundary value problem as an example and examine the properties of these functions. At the end, we investigate the eigenvalues of the impulsive boundary value problem given in these examples.

2. Jost solution and scattering function of the impulsive equations

Let $S(x, \lambda^2)$ and $C(x, \lambda^2)$ are the fundamental solution of (7) in the interval $[0, 1)$ satisfying the initial conditions

$$S(0, \lambda^2) = 0, \quad S'(0, \lambda^2) = 1,$$

and

$$C(0, \lambda^2) = 1, \quad C'(0, \lambda^2) = 0,$$

respectively. It is clear that the solutions $S(x,\lambda^2)$ and $C(x,\lambda^2)$ are entire functions of λ and

$$W[S(x,\lambda^2), C(x,\lambda^2)] = -1, \quad \lambda \in \mathbb{C},$$

where $W[y_1, y_2]$ denotes the wronskian of the solutions y_1 and y_2 of the equation (7). We consider the following function

$$E(x,\lambda) = \begin{cases} a(\lambda)\, C\left(x,\lambda^2\right) + b(\lambda)\, S\left(x,\lambda^2\right) & , \quad x \in [0,1) \\ e(x,\lambda) & , \quad x \in (1,\infty), \end{cases} \tag{10}$$

for $\lambda \in \overline{\mathbb{C}}_+$, where $e(x,\lambda)$ defined by (4) and (5). Using the impulsive condition (9), we find the coefficients $a(\lambda)$ and $b(\lambda)$:

$$\begin{aligned} E(1^-,\lambda) &= \frac{1}{\alpha} E(1^+,\lambda) \\ E'(1^-,\lambda) &= \frac{1}{\beta} E'(1^+,\lambda). \end{aligned}$$

It follows from (10) that

$$a(\lambda)\, C\left(1,\lambda^2\right) + b(\lambda)\, S\left(1,\lambda^2\right) = \frac{1}{\alpha} e(1,\lambda) \tag{11}$$

$$a(\lambda)\, C'\left(1,\lambda^2\right) + b(\lambda)\, S'\left(1,\lambda^2\right) = \frac{1}{\beta} e'(1,\lambda). \tag{12}$$

Using (11) and (13), we obtain

$$a(\lambda) = -\frac{1}{\alpha\beta}\left[\alpha e'(1,\lambda) S\left(1,\lambda^2\right) - \beta e(1,\lambda) S'(1,\lambda^2)\right], \quad \lambda \in \overline{\mathbb{C}}_+, \tag{13}$$

$$b(\lambda) = \frac{1}{\alpha\beta}\left[\alpha e'(1,\lambda) C\left(1,\lambda^2\right) - \beta e(1,\lambda) C'(1,\lambda^2)\right], \quad \lambda \in \overline{\mathbb{C}}_+. \tag{14}$$

The function $E(x,\lambda)$ is the Jost solution of the impulsive Sturm–Liouville boundary value problem (7)-(9), where $a(\lambda)$ and $b(\lambda)$ are defined by (13) and (14), respectively. So we get

$$E(0,\lambda) = a(\lambda)$$

by (13), i.e., the function $a(\lambda)$ is the Jost function of (7)-(9). Note that the function $a(\lambda)$ is analytic in $\mathbb{C}_+$ and continuous up to the real axis.

Theorem 2.1. *For all $\lambda \in \mathbb{R} \setminus \{0\}$, $a(\lambda) \neq 0$.*

Proof. It is clear that ([7])

$$W[e(x,\lambda), e(x,-\lambda)] = -2i\lambda, \quad \lambda \in \mathbb{R} \setminus \{0\}.$$

Now, we will consider the following solution of (7)-(9)

$$F(x,\lambda) = \begin{cases} S\left(x,\lambda^2\right) & , \quad x \in [0,1) \\ c(\lambda) e(x,\lambda) + d(\lambda) e(x,-\lambda) & , \quad x \in (1,\infty). \end{cases}$$

Using the impulsive condition (9), we get that

$$c(\lambda) = -\frac{1}{2i\lambda}\left[\alpha S(1,\lambda^2) e'(1,-\lambda) - \beta S'(1,\lambda^2) e(1,-\lambda)\right], \quad \lambda \in \mathbb{R} \setminus \{0\}, \tag{15}$$

$$d(\lambda) = \frac{1}{2i\lambda}\left[\alpha S(1,\lambda^2) e'(1,\lambda) - \beta S'(1,\lambda^2) e(1,\lambda)\right], \quad \lambda \in \mathbb{R} \setminus \{0\}. \tag{16}$$

It follows from (13), (15) and (16) that

$$d(\lambda) = -\frac{\alpha\beta}{2i\lambda}a(\lambda), \quad c(\lambda) = \overline{d(\lambda)} = \frac{\alpha\beta}{2i\lambda}\overline{a(\lambda)}, \quad \lambda \in \mathbb{R} \setminus \{0\}. \tag{17}$$

Assume that, there exists a $\lambda_0 \in \mathbb{R} \setminus \{0\}$ such that $a(\lambda_0) = 0$. Since $a(\lambda_0) = 0$, we find $c(\lambda_0) = d(\lambda_0) = 0$ by using (17). Then the solution $F(x, \lambda_0)$ is equal to zero identically. So this is a trivial solution of (7)-(9), this gives a contradiction, i.e., $a(\lambda_0) \neq 0$ for all $\lambda \in \mathbb{R} \setminus \{0\}$. □

It can be easily seen from (10) that for all $\lambda \in \mathbb{R} \setminus \{0\}$,

$$\overline{E(x,\lambda)} = E(x, -\lambda).$$

The function

$$\mathcal{S}(\lambda) = \frac{\overline{E(0,\lambda)}}{E(0,\lambda)} = \frac{\overline{a(\lambda)}}{a(\lambda)} \quad \lambda \in \mathbb{R} \setminus \{0\} \tag{18}$$

is the scattering function of the impulsive boundary value problem (7)-(9). It is clear from (13) and (18) that

$$\mathcal{S}(\lambda) = \frac{\alpha e'(1,-\lambda)S(1,\lambda^2) - \beta e(1,-\lambda)S'(1,\lambda^2)}{\alpha e'(1,\lambda)S(1,\lambda^2) - \beta e(1,\lambda)S'(1,\lambda^2)}, \tag{19}$$

for all $\lambda \in \mathbb{R} \setminus \{0\}$. It follows from (19) that

$$\mathcal{S}(0) = \lim_{\lambda \to 0} \mathcal{S}(\lambda) = 1.$$

Theorem 2.2. *For all $\lambda \in \mathbb{R} \setminus \{0\}$, the scattering function satisfies*

$$\mathcal{S}(-\lambda) = \overline{\mathcal{S}(\lambda)} = \mathcal{S}^{-1}(\lambda).$$

Proof. From the definition of $\mathcal{S}(\lambda)$, we have

$$\mathcal{S}(-\lambda) = \frac{\overline{E(0,-\lambda)}}{E(0,-\lambda)}. \tag{20}$$

Since $\overline{E(0,-\lambda)} = E(0,\lambda)$ and $E(0,-\lambda) = \overline{E(0,\lambda)}$, using (20), we get

$$\mathcal{S}(-\lambda) = \mathcal{S}^{-1}(\lambda) = \overline{\mathcal{S}(\lambda)}.$$

It completes the proof. □

By the definition of the wronskian, we have

$$W[E(x,\lambda), F(x,\lambda)] = \begin{cases} a(\lambda) & , \quad x \in [0,1) \\ \alpha\beta a(\lambda) & , \quad x \in (1,\infty) \end{cases}$$

for all $\lambda \in \mathbb{R} \setminus \{0\}$. We will denote the set of eigenvalues of (7)-(9) by σ_d.

Theorem 2.3. *The following equation holds:*

$$\sigma_d = \{\mu : \mu = \lambda^2,\ \lambda \in \mathbb{C}_+,\ a(\lambda) = 0\} \tag{21}$$

Proof. Let $\breve{e}(x,\lambda)$ denote the solution of the equation (7) in $(1,\infty)$, subjecting the conditions ([14])

$$\lim_{x\to\infty}\breve{e}(x,\lambda)e^{i\lambda x}=1,\quad \lim_{x\to\infty}\breve{e}'(x,\lambda)e^{i\lambda x}=-i\lambda,\quad \lambda\in\overline{\mathbb{C}}_+.$$

Note that $\breve{e}(x,\lambda)$ is the unbounded solution of (7) in $(1,\infty)$. It is evident that

$$W[e(x,\lambda),\breve{e}(x,\lambda)]=-2i\lambda,\quad x\in(1,\infty)\quad \lambda\in\overline{\mathbb{C}}_+.$$

For all $\overline{\mathbb{C}}_+\setminus\{0\}$, we will consider the following solution of (7)

$$G(x,\lambda)=\begin{cases} S\left(x,\lambda^2\right) & , \quad x\in[0,1) \\ \gamma(\lambda)e\left(x,\lambda\right)+\delta(\lambda)\breve{e}(x,\lambda) & , \quad x\in(1,\infty)\,. \end{cases} \tag{22}$$

Using the impulsive condition (9), we find that

$$\delta(\lambda)=-\frac{\alpha\beta}{2i\lambda}a(\lambda) \tag{23}$$

and

$$\gamma(\lambda)=-\frac{1}{2i\lambda}[\alpha S(1,\lambda^2)\breve{e}'(1,\lambda)-\beta S'\left(1,\lambda^2\right)\breve{e}\left(1,\lambda\right)], \tag{24}$$

for all $\lambda\in\overline{\mathbb{C}}_+\setminus\{0\}$. The function $G(x,\lambda)$ is the unbounded solution of impulsive boundary value problem (7)-(9). It follows from (22), (23) and the definition of eigenvalues that [14]

$$\sigma_d=\{\mu=\lambda^2:\lambda\in\mathbb{C}_+,\ \delta(\lambda)=0\}$$

or

$$\sigma_d=\{\mu=\lambda^2:\lambda\in\mathbb{C}_+,\ a(\lambda)=0\}.$$

It completes the proof. □

Using (10) and (22), we obtain

$$W[E(x,\lambda),G(x,\lambda)]=\begin{cases} a\left(\lambda\right) & , \quad x\in[0,1) \\ \alpha\beta a\left(\lambda\right) & , \quad x\in(1,\infty)\,, \end{cases}$$

for all $\overline{\mathbb{C}}_+\setminus\{0\}$. Theorem 2.3 shows that, in order to investigate the quantitative properties of the eigenvalues of impulsive boundary value problem (7)-(9), we need to discuss the quantitative properties of the zeros of the function $a(\lambda)$ in $\mathbb{C}_+$.

Theorem 2.4. *Under the condition* (3)*, the Jost function of* (7)-(9) *satisfies*

$$a(\lambda)=e^{i\lambda}\left(\frac{1}{\alpha}\cos\lambda-\frac{i}{\beta}\sin\lambda\right)+o(1),\ \lambda\in\overline{\mathbb{C}}_+,\ |\lambda|\to\infty. \tag{25}$$

Proof. It is clear that the solution $S(x,\lambda^2)$ has an integral representation

$$S(x,\lambda^2)=\frac{\sin\lambda x}{\lambda}+\int_0^x B(x,t)\frac{\sin\lambda t}{\lambda}\mathrm{d}t,\ \lambda\in\mathbb{C}, \tag{26}$$

where the kernel $B(x,t)$ may be expressed in terms of the potential function q ([5]). We easily find from (5) and (26) that

$$e(1,\lambda)=e^{i\lambda}+\int_1^\infty K(1,t)e^{i\lambda t}\mathrm{d}t,\ \lambda\in\overline{\mathbb{C}}_+, \tag{27}$$

$$e'(1,\lambda) = i\lambda - K(1,1)e^{i\lambda} + \int_1^\infty K_x(1,t)e^{i\lambda t}\mathrm{d}t,\ \lambda \in \overline{\mathbb{C}}_+, \tag{28}$$

$$S(1,\lambda^2) = \frac{\sin\lambda}{\lambda} + \int_0^1 B(1,t)\frac{\sin\lambda t}{\lambda}\mathrm{d}t,\ \lambda \in \mathbb{C}, \tag{29}$$

and

$$S'(1,\lambda^2) = \cos\lambda + B(1,1)\frac{\sin\lambda}{\lambda} + \int_0^1 B_x(1,t)\frac{\sin\lambda t}{\lambda}\mathrm{d}t,\ \lambda \in \mathbb{C}. \tag{30}$$

Using (13), (27)-(30), we see that the Jost function $a(\lambda)$ satisfies the asymptotic equation (24). □

3. Examples

In this Section, we will find the Jost function, scattering function and eigenvalues of two different impulsive Sturm–Liouville boundary value problems.

Example 3.1. *Let us consider the following impulsive Sturm–Liouville problem*

$$\begin{cases} -y'' = \lambda^2 y, & x \in [0,1)\cup(1,\infty) \\ y(0) = 0 & \\ y(1^+) = \alpha y(1^-) & \\ y'(1^+) = \beta y'(1^-), & \end{cases} \tag{31}$$

where $\alpha,\beta \in \mathbb{R}$ and $\alpha\beta \neq 0$. It is evident that

$$e(x,\lambda) = e^{i\lambda x},\ S(x,\lambda^2) = \frac{\sin\lambda x}{\lambda},\ C(x,\lambda^2) = \cos\lambda x,$$

and

$$E(x,\lambda) = \begin{cases} e^{i\lambda}m(\lambda)\cos\lambda x + \lambda e^{i\lambda}n(\lambda)\frac{\sin\lambda x}{\lambda} & ,\quad x \in [0,1) \\ e^{i\lambda x} & ,\quad x \in (1,\infty), \end{cases} \tag{32}$$

where $m(\lambda) = \left(\frac{\cos\lambda}{\alpha} - \frac{i\sin\lambda}{\beta}\right)$ and $n(\lambda) = \left(\frac{\sin\lambda}{\alpha} + \frac{i\cos\lambda}{\beta}\right)$. From (32)*, we obtain the Jost function and scattering function of* (31) *as*

$$E(0,\lambda) = e^{i\lambda}\left(\frac{\cos\lambda}{\alpha} - \frac{i\sin\lambda}{\beta}\right),\ \lambda \in \overline{\mathbb{C}}_+, \tag{33}$$

and

$$\mathcal{S}(\lambda) = e^{-2i\lambda}\left(\frac{\beta\cos\lambda + i\alpha\sin\lambda}{\beta\cos\lambda - i\alpha\sin\lambda}\right),\ \lambda \in \mathbb{R}\setminus\{0\},$$

respectively. Now, we can write the set of eigenvalues of (31) *using the Theorem 2.4*

$$\sigma_d = \{\mu = \lambda^2 :\ \lambda \in \mathbb{C}_+,\ E(0,\lambda) = 0\}.$$

Since $E(0,\lambda) = 0$, it follows from (33) *that $e^{i\lambda}\left(\frac{\cos\lambda}{\alpha} - \frac{i\sin\lambda}{\beta}\right) = 0$. Using the last equation, we find*

$$\lambda_k = -\frac{i}{2}\ln\left|\frac{1+A}{1-A}\right| + \frac{1}{2}Arg\left(\frac{1+A}{1-A}\right) + k\pi,\ k \in \mathbb{Z} = 0,\pm1,\pm2,...,$$

where $A = \frac{\beta}{\alpha}$.
Case1: *For* $0 < A < 1$, *we see that*

$$\lambda_k = -\frac{i}{2}\ln\frac{1+A}{1-A} + k\pi,\ k \in \mathbb{Z}.$$

Since $\lambda_k \in \mathbb{C}_- := \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda < 0\}$ *in this case,* (31) *has no eigenvalues.*
Case2: *For* $1 < A < \infty$, *we find*

$$\lambda_k = -\frac{i}{2}\ln\left|\frac{1+A}{1-A}\right| + (k+1)\pi,\ k \in \mathbb{Z}.$$

Similarly to the Case1, $\lambda_k \in \mathbb{C}_-$ *and in this case again, the eigenvalues of* (31) *are not existing.*
Case3: *For* $A \in (-1, 0)$, *we obtain that*

$$\lambda_k = \frac{i}{2}\ln\frac{1-A}{1+A} + k\pi,\ k \in \mathbb{Z},$$

here $\lambda_k \in \mathbb{C}_+$ *and* $\mu_k = \lambda_k^2$, $k \in \mathbb{Z}$ *are the eigenvalues of the impulsive boundary value problem* (31).
Case4: *For* $A \in (-\infty, -1)$, *we find*

$$\lambda_k = \frac{i}{2}\ln\left|\frac{1-A}{1+A}\right| + (k+1)\pi,\ k \in \mathbb{Z},$$

and similar to the Case3, the numbers $\mu_k = \lambda_k^2$, $k \in \mathbb{Z}$ *are the eigenvalues of* (31).

Example 3.2. *Let us consider the Sturm–Liouville problem*

$$\begin{cases} -y'' = \lambda^2 \rho(x) y, & 0 \le x < \infty \\ y(0) = 0, & \end{cases} \tag{34}$$

where

$$\rho(x) = \begin{cases} w^2, & 0 \le x \le 1 \\ 1, & 1 < x < \infty \end{cases} \tag{35}$$

and $w \in \mathbb{R} \setminus \{-1, 0, 1\}$. *Note that* (34)-(35) *boundary value problem can be stated as an impulsive Sturm–Liouville boundary value problem*

$$\begin{cases} -y'' = \lambda^2 \rho(x) y, & 0 \le x < \infty \\ y(0) = 0 & \\ y(1^-) = y(1^+) & \\ y'(1^-) = y'(1^+). & \end{cases} \tag{36}$$

It can be easily seen that

$$e(x,\lambda) = e^{i\lambda x},\ S(x,\lambda^2) = \frac{\sin(\lambda w x)}{\lambda w},\ C(x,\lambda^2) = \cos(\lambda w x),$$

and

$$E(x,\lambda) = \begin{cases} e^{i\lambda} a_0(\lambda)\cos(\lambda w x) + \lambda e^{i\lambda} b_0(\lambda)\frac{\sin(\lambda w x)}{\lambda w} & , \quad x \in [0,1] \\ e^{i\lambda x} & , \quad x \in (1,\infty), \end{cases}$$

where $a_0(\lambda) = \left[\cos(\lambda w) - i\frac{\sin \lambda w}{w}\right]$ *and* $b_0(\lambda) = [sin(\lambda w) + i\cos(\lambda w)]$ *For this problem, we get the Jost function and scattering function of* (36) *(or* (34)-(35)*) as*

$$E(0,\lambda) = e^{i\lambda}\left(\cos\lambda w - i\frac{\sin\lambda w}{w}\right),$$

and

$$\mathcal{S}(\lambda) = e^{-2i\lambda} \frac{w \cos \lambda w + i \sin \lambda w}{w \cos \lambda w - i \sin \lambda w}.$$

Using Theorem 2.4, we obtain the eigenvalues of impulsive Sturm-Liouville boundary value problem (34)-(35)

$$\sigma_d = \{\mu = \lambda^2 : \ \lambda \in \mathbb{C}_+, \ \left(\cos \lambda w - i \frac{\sin \lambda w}{w}\right) = 0\},$$

it follows from that $\left(\cos \lambda w - i \frac{\sin \lambda w}{w}\right) = 0$ *and*

$$\lambda_k = -\frac{i}{2} \ln \left| \frac{w+1}{w-1} \right| + Arg\left(\frac{1+w}{1-w}\right) + k\pi, \ k \in \mathbb{Z}.$$

Case1: *If* $w \in (0, 1)$, *then*

$$\lambda_k = -\frac{i}{2} \ln \frac{1+w}{1-w} + k\pi, \ k \in \mathbb{Z}.$$

In this case, $\lambda_k \in \mathbb{C}_-$, *so the impulsive Sturm-Liouville boundary value problem* (34)-(35) *has no eigenvalues.*
Case2: *If* $w \in (1, \infty)$, *then*

$$\lambda_k = -\frac{i}{2} \ln \left| \frac{1+w}{1-w} \right| + (k+1)\pi, \ k \in \mathbb{Z}.$$

In this case, again there is no eigenvalues of impulsive Sturm-Liouville boundary value problem (34)-(35).
Case3: *If* $w \in (-1, 0)$, *we obtain that*

$$\lambda_k = \frac{i}{2} \ln \frac{1+w}{1-w} + k\pi, \ k \in \mathbb{Z},$$

here $\mu_k = \lambda_k^2, \ k \in \mathbb{Z}$ *are the eigenvalues of* (34)-(35).
Case4: *For* $w \in (-\infty, -1)$, *we find that* $\mu_k = \lambda_k^2, \ k \in \mathbb{Z}$ *are the eigenvalues of* (36), *where*

$$\lambda_k = \frac{i}{2} \ln \left| \frac{1+w}{1-w} \right| + (k+1)\pi, \ k \in \mathbb{Z}.$$

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