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A One-Step-Two-Mappings Iterative Scheme for Multi-Valued Maps in W-Hyperbolic Spaces

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Abstract. In this paper, we study a one-step iterative scheme for two multi-valued nonexpansive maps in W-hyperbolic spaces. We establish strong and $\Delta-$convergence theorems for the proposed algorithm in a uniformly convex W-hyperbolic space which improve and extend the corresponding known results in uniformly convex Banach spaces as well as CAT(0) spaces. Our new results are also valid in geodesic spaces.

1. Introduction and preliminaries

Throughout the paper, $\mathbb{N}$ denotes the set of positive integers. Let X be a metric space. A subset K is called proximinal if for each $x \in X$, there exists an element $k \in K$ such that

$$d(x, K) = \inf\{d(x, y) : y \in K\} = d(x, k).$$

We shall denote the family of nonempty bounded proximinal subsets of K by $P(K)$ and the family of nonempty compact subsets of K by $C(K)$. Consistent with [1], let $CB(K)$ be the class of all nonempty bounded and closed subsets of K. Let H be a Hausdorff metric induced by the metric d of X, that is

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}$$

for every $A, B \in CB(X)$. A multivalued mapping $T : K \longrightarrow P(K)$ is said to be nonexpansive if $H(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$. A point $x \in K$ is called a fixed point of T if $x \in Tx$. The set of fixed points of T is denoted by $F(T)$. The mapping $T : K \longrightarrow P(K)$ is called quasi-nonexpansive if $F(T) \neq \emptyset$ and $H(Tx, p) \leq d(x, p)$ for all $x \in K$ and all $p \in F(T)$.

It is clear that every nonexpansive multi-valued mapping T with $F(T) \neq \emptyset$ is quasi-nonexpansive. But there exist quasi-nonexpansive mappings that are not nonexpansive (see [8]).

The study of fixed points for multivalued contractions and nonexpansive mappings using the Hausdorff metric was initiated by Markin [2]. Some classical fixed point theorems for single-valued maps were

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extended to multi-valued maps; for example, Nadler [1] extended Banach Contraction Principle for multi-valued contractive maps in complete metric spaces. Existence of fixed points of multi-valued nonexpansive maps was established in certain convex metric spaces by Shimizu and Takahashi [3]. Later, an interesting and rich fixed point theory for such maps was developed which has applications in control theory, convex optimization, differential inclusion and economics (see, [4] and references cited therein).

The theory of multivalued nonexpansive mappings is harder than the corresponding theory of single valued nonexpansive mappings. Different iterative techniques for approximating fixed points of non-expansive single-valued mappings have been investigated by various authors using the Mann iteration scheme or the Ishikawa iteration scheme. Multivalued results have been given by numerous authors in different underlying spaces using different iterative schemes for different type of mappings. For example, see [21–25] among others. In 2005, Sastry and Babu [5] defined the Mann and Ishikawa iteration schemes for multi-valued mappings T with a fixed point p and proved that these schemes converge to a fixed point q of T under certain conditions. Moreover, they illustrated that fixed point q may be different from p.

In 2007, Panyanak [6] generalized results of Sastry and Babu [5] to uniformly convex Banach spaces and proved a convergence theorem for a mapping defined on a noncompact domain. Furthermore, he gave an open question which was answered by Song and Wang [5]. Actually, Song and Wang [7] showed that his process was not well-defined.

In 2009, Shahzad and Zegeye [8] extended and improved the results of Panyanak [6], Sastry and Babu [5] and Song and Wang [7] to a quasi-nonexpansive multi-valued mappings. They also relaxed compactness of the domain of T and constructed an iteration scheme which removes the restriction of T namely $Tp = \{p\}$ for any $p \in F(T)$. In order to do this, they defined $P_T := \{y \in Tx : \left\|x - y\right\| = d(x, Tx)\}$ for a multivalued mapping $T : K \to P(K)$. The results provided an affirmative answer to Panyanak's question [6] in a more general setting.

In 2010, Khan et al. [9] proved weak and strong convergence theorems of a one-step iterative scheme for two multivalued nonexpansive mappings, say S and T. Although this scheme is simpler, yet it needs the so-called condition (C) : $d(x, y) \leq d(z, y)$ for $y \in Sx$ and $z \in Tx$.

In 2011, Abbas et al. [10] established weak and strong convergence theorems for a new one-step iterative process under some basic boundary conditions in a real uniformly convex Banach space. Let $S, T : K \to CB(K)$ be two multivalued nonexpansive mappings. They introduced the following iterative scheme:

$$\begin{aligned} x_1 &\in K, \\ x_{n+1} &= a_n x_n + b_n y_n + c_n z_n, n \in \mathbb{N}, \end{aligned} \tag{1}$$

where $y_n \in Tx_n$ and $z_n \in Sx_n$ such that $\|y_n - p\| \leq d(p, Sx_n)$ and $\|z_n - p\| \leq d(p, Tx_n)$ whenever p is fixed point of any one of the mappings S and T, and $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences of numbers in $(0, 1)$ satisfying $a_n + b_n + c_n \leq 1$.

Note that this scheme includes Mann's iteration scheme and is independent of and simpler than the Ishikawa scheme used by Panyanak [6], Sastry and Babu [5], Song and Wang [7], and Shahzad and Zegeye [8].

Inspired and motivated by the work of Abbas et al. [10], we translate scheme (1) for the multi-valued nonexpansive maps in the general setup of W-hyperbolic spaces and approximate a common fixed point of two multi-valued nonexpansive maps.

Kohlenbach [11] introduced a general setup known as W-hyperbolic spaces.

A W-hyperbolic space [11] is a triple (X, d, W) where (X, d) is a metric space and $W : X^2 \times [0, 1] \to X$ is such that

W1: $d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1 - \alpha) d(u, y)$

W2: $d(W(x, y, \alpha), W(x, y, \beta)) = \left|\alpha - \beta\right| d(x, y)$

W3: $W(x, y, \alpha) = W(y, x, (1 - \alpha))$

W4: $d(W(x, z, \alpha), W(y, w, \alpha)) \leq \alpha d(x, y) + (1 - \alpha) d(z, w)$

for all $x, y, z, u, w \in X$ and $\alpha, \beta \in [0, 1]$.

If (X, d, W) satisfies only (W1), then it coincides with the convex metric space introduced by Takahashi [13]. A subset K of a hyperbolic space X is convex if $W(x, y, \alpha) \in K$ for all $x, y \in K$ and $\alpha \in [0, 1]$. The class of W-hyperbolic spaces contains normed spaces and their convex subsets as subclasses and CAT(0) spaces form a very special subclass of the class of W-hyperbolic spaces with unique geodesic paths.

A hyperbolic space (X, d, W) is said to be uniformly convex [3] if for all $u, x, y \in X$, $r > 0$ and $\varepsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that

$$\left.\begin{array}{r} d(x,u) \leq r \\ d(y,u) \leq r \\ d(x,y) \geq \varepsilon r \end{array}\right\} \Rightarrow d\left(W\left(x, y, \frac{1}{2}\right), u\right) \leq (1-\delta)\, r.$$

A map $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ which provides such a $\delta = \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$, is called modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ε).

It has been shown in [14] that CAT(0) spaces are uniformly convex W-hyperbolic spaces with modulus of uniform convexity $\eta(r, \varepsilon) = \frac{\varepsilon^2}{8}$. Thus, uniformly convex W-hyperbolic spaces are a natural generalization of both uniformly convex Banach spaces and CAT(0) spaces. The class of W-hyperbolic spaces also contains Hadamard manifolds, $\mathbb{R}$-trees, Hilbert ball equipped with the hyperbolic metric [12] and Cartesian products of Hilbert balls, as special cases.

We translate the algorithm (1) for multivalued nonexpansive maps in a W-hyperbolic space as follows:

Let T and S be two multivalued nonexpansive maps from K into $CB(K)$ where K is a convex subset of a hyperbolic space. Let $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers satisfying $0 < a \leq \alpha_n, \beta_n \leq b < 1$ and $\alpha_n + \beta_n < 1$. Then for $x_1 \in K$, generate $\{x_n\}$ as

$$x_{n+1} = W\left(y_n, W\left(x_n, z_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right) \tag{2}$$

where $y_n \in Tx_n$ and $z_n \in Sx_n$ such that $d(y_n, p) \leq d(p, Sx_n)$ and $d(z_n, p) \leq d(p, Tx_n)$ whenever p is fixed point of any one of the mappings S and T.

Note that the algorithm (2) coincides with the algorithm (1) when $W(x, y, \alpha) = \alpha x + (1-\alpha) y$, T and S are multi-valued nonexpansive maps and X is a Banach space. Moreover, it coincides with the algorithm of Abbas and Khan [15] when T and S are single-valued nonexpansive maps and X is a CAT(0) space.

Another one step fixed point iteration scheme was studied by Khan et al. [20] in hyperbolic spaces. Our iteration scheme is different from their iteration scheme in two ways : i) Their scheme is implicit, but our scheme is explicit. ii) They used the idea of Shahzad [8] to guarantee well definedness of their scheme, but we used above different conditions. It is remarkable that implicit iteration schemes are not used unless explicit iteration schemes are inefficient. Thus our study is better than Khan et al. [20] scheme.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X. For $x \in X$, define a continuous functional $r(., \{x_n\}) : X \to [0, \infty)$ by $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. The asymptotic radius $\rho = r(\{x_n\})$ of $\{x_n\}$ is given by $\rho = \inf\{r(x, \{x_n\}) : x \in X\}$. The asymptotic center of a bounded sequence $\{x_n\}$ with respect to a subset K of X is defined as follows:

$$A_K(\{x_n\}) = \{x \in X : r(x, \{x_n\}) \leq r(y, \{x_n\}) \text{ for any } y \in K\}.$$

The set of all asymptotic centers of $\{x_n\}$ is denoted by $A(\{x_n\})$. In general, $A(\{x_n\})$ may be empty or may contain infinitely many points. It is remarkable that bounded sequences have unique asymptotic center with respect to closed convex subsets in a complete and uniformly convex hyperbolic space with monotone modulus of uniform convexity [14].

The concept of Δ-convergence in a metric space was introduced by Lim [16] and its analogue in CAT(0) spaces has been investigated by Dhompongsa and Panyanak [17]. They showed that Δ-convergence coincides with weak convergence in Banach spaces satisfying the Opial condition and both concepts have many common properties.

A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write Δ -$\lim_n x_n = x$.

We need the following useful lemmas which play very important role in proving strong and Δ-convergence of our algorithm.

Lemma 1.1. *[18] Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \leq r$, $\limsup_{n\to\infty} d(y_n, x) \leq r$ and $\lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = r$ for some $r \geq 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

Lemma 1.2. *[18] Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space and $\{x_n\}$ a bounded sequence in K such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in K such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.*

2. Main Results

In the sequel, $F = F(S) \cap F(T)$ denotes the set of all common fixed points of the multivalued maps S and T.

Lemma 2.1. *Let X be a uniformly convex W-hyperbolic space with monotone modulus of uniform convexity η and K a nonempty closed convex subset of X. Let $S, T : K \to C(K)$ be multivalued nonexpansive mappings and $F \neq \emptyset$. Then for the sequence $\{x_n\}$ in (2), we have $\lim_{n\to\infty} d(x_n, Sx_n) = 0 = \lim_{n\to\infty} d(x_n, Tx_n)$.*

Proof. Let $p \in F$. It follows from (2) that

$$\begin{aligned} d(x_{n+1}, p) &= d\left(W\left(y_n, W\left(x_n, z_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), p\right) \\ &\leq \alpha_n d(y_n, p) + (1-\alpha_n)\, d\left(W\left(x_n, z_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) \\ &\leq \alpha_n d(p, Sx_n) + \beta_n d(x_n, p) + (1-\alpha_n-\beta_n)\, d(p, Tx_n) \\ &\leq \alpha_n H(Sx_n, \{p\}) + \beta_n d(x_n, p) + (1-\alpha_n-\beta_n)\, H(Tx_n, \{p\}) \\ &\leq \alpha_n H(Sx_n, Sp) + \beta_n d(x_n, p) + (1-\alpha_n-\beta_n)\, H(Tx_n, Tp) \\ &\leq \alpha_n d(x_n, p) + \beta_n d(x_n, p) + (1-\alpha_n-\beta_n)\, d(x_n, p) \\ &= d(x_n, p). \end{aligned} \tag{3}$$

This gives that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. We suppose that $\lim_{n\to\infty} d(x_n, p) = c$ for some $c \geq 0$. Then

$$\lim_{n\to\infty} d(x_{n+1}, p) = \lim_{n\to\infty} d\left(W\left(y_n, W\left(x_n, z_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), p\right) = c. \tag{4}$$

Since S, T are nonexpansive mappings and $F \neq \emptyset$, we have $d(y_n, p) \leq d(Sx_n, p) \leq H(Sx_n, Sp) \leq d(x_n, p)$ for each $p \in F$. Taking lim sup on both sides, we obtain

$$\limsup_{n\to\infty} d(y_n, p) \leq c. \tag{5}$$

Similarly,

$$\limsup_{n\to\infty} d(z_n, p) \leq c. \tag{6}$$

Now,

$$\begin{aligned} d\left(W\left(x_n, z_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) &\leq \frac{\beta_n}{1-\alpha_n} d(x_n, p) + \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d\left(z_n, p\right) \\ &\leq \frac{\beta_n}{1-\alpha_n} d(x_n, p) + \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d\left(x_n, p\right) \\ &= d\left(x_n, p\right), \end{aligned}$$

gives

$$\limsup_{n\to\infty} d\left(W\left(x_n, z_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) \leq c. \tag{7}$$

Using (3)-(7) and Lemma 1.1, we get

$$\lim_{n\to\infty} d\left(y_n, W\left(x_n, z_n, \frac{\beta_n}{1-\alpha_n}\right)\right) = 0. \tag{8}$$

It follows that

$$\begin{aligned} d\left(x_{n+1}, y_n\right) &= d\left(W\left(y_n, W\left(x_n, z_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), y_n\right) \\ &\leq \alpha_n d\left(y_n, y_n\right) + (1-\alpha_n)\, d\left(y_n, W\left(x_n, z_n, \frac{\beta_n}{1-\alpha_n}\right)\right) \\ &\to 0,\ (n \to \infty). \end{aligned} \tag{9}$$

Also,

$$\begin{aligned} d\left(x_{n+1}, p\right) &= d\left(W\left(y_n, W\left(x_n, z_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), p\right) \\ &\leq \alpha_n d\left(y_n, p\right) + (1-\alpha_n)\, d\left(W\left(x_n, z_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) \\ &\leq \alpha_n d\left(x_n, p\right) + (1-\alpha_n)\, d\left(W\left(x_n, z_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) \end{aligned}$$

so,

$$c \leq \liminf_{n\to\infty} d\left(W\left(x_n, z_n, \frac{\beta_n}{1-\alpha_n}\right), p\right). \tag{10}$$

Combining (7) and (10), we get

$$\lim_{n\to\infty} d\left(W\left(x_n, z_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) = c. \tag{11}$$

By (6), (11) and Lemma 1.1, we get

$$\lim_{n\to\infty} d\left(x_n, z_n\right) = 0. \tag{12}$$

Observe

$$\begin{aligned} d\left(x_{n+1}, x_n\right) &= d\left(W\left(y_n, W\left(x_n, z_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), x_n\right) \\ &\leq \alpha_n d\left(y_n, x_n\right) + (1-\alpha_n) d\left(W\left(x_n, z_n, \frac{\beta_n}{1-\alpha_n}\right), x_n\right) \\ &\leq \alpha_n \left\{d\left(y_n, x_{n+1}\right) + d\left(x_{n+1}, x_n\right)\right\} + (1-\alpha_n-\beta_n)\, d\left(x_n, z_n\right). \end{aligned}$$

Re-arranging the terms in the above inequality, we have

$$d(x_{n+1}, x_n) \leq \frac{b}{1-b} d(y_n, x_{n+1}) + \frac{1}{1-b} d(x_n, z_n).$$

Taking limsup on both the sides in the above inequality and then using (9) and (12), we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{13}$$

By (9) and (13), we have

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, y_n) \\ &\to 0, \ (n \to \infty). \end{aligned}$$

Now

$$d(x_n, Tx_n) \leq d(x_n, z_n)$$

and

$$d(x_n, Sx_n) \leq d(x_n, y_n)$$

give $d(x_n, Tx_n) \to 0$ and $d(x_n, Sx_n) \to 0$ as $n \to \infty$. □

Our next result deals with $\Delta-$convergence of the algorithm (2).

Theorem 2.2. *Let X be a complete uniformly convex W-hyperbolic space with monotone modulus of uniform convexity η. Let K be a nonempty closed convex subset of X and S, T and $\{x_n\}$ be as in Lemma 2.1. Then $\{x_n\}$ $\Delta-$converges to a common fixed point of S and T.*

Proof. As $\lim_{n\to\infty} d(x_n, p)$ exists, therefore $\{x_n\}$ is bounded. Thus $A(\{x_n\}) = \{x\}$, that is, $\{x_n\}$ has a unique asymptotic centre. Let $\{v_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{v_n\}) = \{v\}$. From Lemma 2.1, we have $\lim_{n\to\infty} d(v_n, Sv_n) = 0$ and $\lim_{n\to\infty} d(v_n, Tv_n) = 0$. Let $v \in \omega_\omega(x_n) = \cup A(\{v_n\})$, where union is taken over all subsequences $\{v_n\}$ of $\{x_n\}$. We claim $v \in Tv$. To prove this, we take a sequence $z_{n_k} \in Tv$ such that

$$\begin{aligned} d(z_{n_k}, v_n) &\leq d(z_{n_k}, Tv_n) + d(Tv_n, v_n) \\ &\leq H(Tv, Tv_n) + d(Tv_n, v_n) \\ &\leq d(v, v_n) + d(Tv_n, v_n). \end{aligned}$$

Thus, we get

$$r(z_{n_k}, \{v_n\}) = \limsup_{n\to\infty} d(z_{n_k}, v_n) \leq \limsup_{n\to\infty} d(v, v_n) = r(v, \{v_n\}).$$

This implies that $\left| r(z_{n_k}, \{v_n\}) - r(v, \{v_n\}) \right| \to 0$ as $k \to \infty$. It follows from Lemma 1.2 that $\lim_{n\to\infty} z_{n_k} = v$. Since Tv is closed, therefore $v \in Tv$ and so $v \in F(T)$. Similarly, $v \in F(S)$. Hence $v \in F$. Moreover, $\lim_{n\to\infty} d(x_n, p)$ exists by Lemma 2.1. Next, we show that $\omega_\omega(x_n)$ is singleton.

Suppose $x \neq v$. By the uniqueness of asymptotic centers,

$$\begin{aligned} \limsup_{n\to\infty} d(v_n, v) &\leq \limsup_{n\to\infty} d(v_n, x) \\ &\leq \limsup_{n\to\infty} d(x_n, x) \\ &< \limsup_{n\to\infty} d(x_n, v) \\ &= \limsup_{n\to\infty} d(v_n, v), \end{aligned}$$

a contradiction. Hence $x = v$. This proves that $\{x_n\}$ Δ-converges to a common fixed point of S and T. □

Recall that a multi-valued map $T : K \to CB(K)$ is hemi-compact if any bounded sequence $\{x_n\}$ satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$, has a convergent subsequence.

Two multi-valued maps $T, S : K \to CB(K)$ are said to satisfy condition (II) if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0, \infty)$ such that

either $d(x, Tx) \geq f(d(x, F))$ or $d(x, Sx) \geq f(d(x, F))$

holds for all $x \in K$.

Following result is a necessary and sufficient condition for the strong convergence of the algorithm (2) in a complete W-hyperbolic space.

Theorem 2.3. *Let X be a complete uniformly convex W-hyperbolic space with monotone modulus of uniform convexity η. Let K be a nonempty closed convex subset of X and S, T and $\{x_n\}$ be as in Lemma 2.1. Then $\{x_n\}$ converges strongly to a common fixed point of S and T if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Proof. The necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. By (3) we have,

$$d(x_{n+1}, p) \leq d(x_n, p)$$

and so,

$$d(x_{n+1}, F) \leq d(x_n, F).$$

Hence $\lim_{n\to\infty} d(x_n, F)$ exists. By hypothesis, $\liminf_{n\to\infty} d(x_n, F) = 0$ so we must have $\lim_{n\to\infty} d(x_n, F) = 0$.

Next, we claim that $\{x_n\}$ is a Cauchy sequence in K. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, therefore there exists n_0 such that for all $n \geq n_0$, we have $d(x_n, F) < \frac{\varepsilon}{4}$. That is, $\inf\{d(x_{n_0}, p) : p \in F\} < \frac{\varepsilon}{4}$. Thus there must exist $p^* \in F$ such that $d(x_{n_0}, p^*) < \frac{\varepsilon}{2}$. Now for $m, n \geq n_0$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p^*) + d(p^*, x_n) \\ &\leq 2d(x_{n_0}, p^*) \\ &\leq 2\frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in a K. The completeness of X guarantees that $\{x_n\}$ converges, say to q. Now

$$\begin{aligned} d(q, Sq) &\leq d(q, x_n) + d(x_n, Sx_n) + H(Sx_n, Sq) \\ &\leq d(q, x_n) + d(x_n, y_n) + d(x_n, q) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore, we have $d(q, Sq) = 0$. Similarly, we can show that $d(q, Tq) = 0$. Hence $q \in F$. □

The following strong convergence result can be easily proved by using Lemma 2.1 as an application of above theorem.

Theorem 2.4. *Let X be a complete uniformly convex W-hyperbolic space with monotone modulus of uniform convexity η. Let K be a nonempty closed convex subset of X and S, T and $\{x_n\}$ be as in Lemma 2.1. Suppose that one of the map S, T is hemi-compact or S and T satisfy condition (II), then the sequence $\{x_n\}$ converges strongly to $p \in F$.*

Below gives us an example of two multivalued nonexpansive mappings with a common fixed point set. This example shows that our results is applicable.

Example 2.5. *[20] Let $K = [0, 1]$ be endowed with the Euclidean metric. Let $S, T : K \to CB(K)$ be defined by $Tx = \left[0, \frac{x}{6}\right]$ and $Sx = \left[0, \frac{x}{3}\right]$. For any $x, y \in K$, if $x < y$, $\sup_{x \in Tx} d(x, Ty) = \frac{y}{6} - \frac{x}{6}$, $\sup_{y \in Ty} d(y, Tx) = 0$ and if $y < x$, $\sup_{x \in Tx} d(x, Ty) = \frac{x}{6} - \frac{y}{6}$, $\sup_{y \in Ty} d(y, Tx) = 0$. Thus*

$$H(Tx, Ty) = \max \left\{ \left| \frac{x}{6} - \frac{y}{6} \right|, 0 \right\} \leq \left| \frac{x}{6} - \frac{y}{6} \right| \leq \left| x - y \right|.$$

In a similar way, we have

$$H(Sx, Sy) = \max \left\{ \left| \frac{x}{3} - \frac{y}{3} \right|, 0 \right\} \leq \left| \frac{x}{3} - \frac{y}{3} \right| \leq \left| x - y \right|.$$

These show that T and S are multivalued nonexpansive mappings with $F(T) \cap F(S) = \{0\}$.

3. Conclusions

(1) Theorem 2.2 sets analogue of [9, Theorem 2] and [10, Theorem 2] for multi-valued nonexpansive maps in a uniformly convex hyperbolic space.

(2) Our theorems extends the corresponding results of Abbas et al. [10] from Banach space to general setup of W-hyperbolic spaces.

(3) Our theorems extends the corresponding results of Gunduz and Akbulut [19] from single-valued nonexpansive mappings to multi-valued nonexpansive maps.

(4)Theorem 2.2 and Theorem 2.4 extends the corresponding results of Khan and Abbas [15] in two ways: (i) from two single-valued nonexpansive mappings to two multi-valued nonexpansive maps (ii) from $CAT(0)$ space to general setup of W-hyperbolic spaces.

(5) In view of simplicity of the iterative process (2) as compared with Ishikawa iteration scheme, our results improve and generalize the results of Panyanak [6], Sastry and Babu [5], Song and Wang [7], and Shahzad and Zegeye [8].

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