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On Warped Product Gradient η-Ricci Solitons

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Abstract. If the potential vector field of an η-Ricci soliton is of gradient type, using Bochner formula, we derive from the soliton equation a nonlinear second order PDE. In a particular case of irrotational potential vector field we prove that the soliton is completely determined by f. We give a way to construct a gradient η-Ricci soliton on a warped product manifold and show that if the base manifold is oriented, compact and of constant scalar curvature, the soliton on the product manifold gives a lower bound for its scalar curvature.

1. Introduction

Ricci flow, introduced by R. S. Hamilton [15], deforms a Riemannian metric g by the evolution equation $\frac{\partial}{\partial t}g = -2S$, called the "heat equation" for Riemannian metrics, towards a canonical metric. Modeling the behavior of the Ricci flow near a singularity, *Ricci solitons* [14] have been studied in the contexts of complex, contact and paracontact geometries [2].

The more general notion of *η-Ricci soliton* was introduced by J. T. Cho and M. Kimura [10] and was treated by C. Călin and M. Crasmareanu on Hopf hypersurfaces in complex space forms [9]. We also discussed some aspects of η-Ricci solitons in paracontact [5], [6] and Lorentzian para-Sasakian geometry [4].

A particular case of soliton arises when the potential vector field is the gradient of a smooth function. The gradient vector fields play a central rôle in the Morse-Smale theory [21]. G. Y. Perelman showed that if the manifold is compact, then the Ricci soliton is gradient [17]. In [13], R. S. Hamilton conjectured that a compact gradient Ricci soliton on a manifold M with positive curvature operator implies that M is Einstein manifold. In [11], S. Deshmukh proved that a Ricci soliton of positive Ricci curvature and whose potential vector field is of Jacobi-type, is compact and therefore, a gradient Ricci soliton. Different aspects of gradient Ricci solitons were studied in various papers. In [1], N. Basu and A. Bhattacharyya treated gradient Ricci solitons in Kenmotsu manifolds having Killing potential vector field. P. Petersen and W. Wylie discussed the rigidity of gradient Ricci solitons [19] and gave a classification imposing different curvature conditions [18].

The aim of our paper is to investigate some properties of gradient η-Ricci solitons. After deducing some results derived from the Bochner formula, we construct a gradient η-Ricci soliton on a warped product manifold and for the particular case of product manifolds, we show that if the base is oriented and of constant scalar curvature, then we obtain a lower bound for the scalar curvature of the product manifold.

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2. Bochner Formula Revisited

Let (M, g) be an m-dimensional Riemannian manifold and consider ξ a gradient vector field on M. If $\xi := grad(f)$, for f a smooth function on M, then the g-dual 1-form η of ξ is closed, as $\eta(X) := g(X, \xi) = df(X)$. Then $0 = (d\eta)(X, Y) := X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]) = g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X)$, hence:

$$g(\nabla_X \xi, Y) = g(\nabla_Y \xi, X), \tag{1}$$

for any $X, Y \in \chi(M)$, where ∇ is the Levi-Civita connection of g.

Also:

$$div(\xi) = \Delta(f) \tag{2}$$

and

$$div(\eta) := trace(Z \mapsto \sharp((\nabla\eta)(Z, \cdot))) = \sum_{i=1}^{m} (\nabla_{E_i}\eta)E_i = \sum_{i=1}^{m} g(E_i, \nabla_{E_i}\xi) := div(\xi), \tag{3}$$

for $\{E_i\}_{1\leq i\leq m}$ a local orthonormal frame field with $\nabla_{E_i}E_j = 0$ in a point. From now on, whenever we make a local computation, we will consider this frame.

In this case, the Bochner formula becomes:

$$\frac{1}{2}\Delta(|\xi|^2) = |\nabla\xi|^2 + S(\xi, \xi) + \xi(div(\xi)), \tag{4}$$

where S is the Ricci curvature of g. Indeed:

$$\begin{aligned}(div(\mathcal{L}_\xi g))(X) &:= trace(Z \mapsto \sharp((\nabla(\mathcal{L}_\xi g))(Z, \cdot, X))) = \sum_{i=1}^{m} (\nabla_{E_i}(\mathcal{L}_\xi g))(E_i, X) = \\ &= \sum_{i=1}^{m} \{E_i((\mathcal{L}_\xi g)(E_i, X)) - (\mathcal{L}_\xi g)(E_i, \nabla_{E_i}X)\} = 2\sum_{i=1}^{m} g(\nabla_{E_i}\nabla_X\xi - \nabla_{\nabla_{E_i}X}\xi, E_i) := \\ &:= 2\sum_{i=1}^{m} g(\nabla^2_{E_i,X}\xi, E_i) = 2\sum_{i=1}^{m} g(\nabla^2_{X,E_i}\xi + R(E_i, X)\xi, E_i) := \\ &:= 2\sum_{i=1}^{m} g(\nabla^2_{X,E_i}\xi, E_i) + 2trace(Z \mapsto R(Z, X)\xi) := 2\sum_{i=1}^{m} g(\nabla_X\nabla_{E_i}\xi - \nabla_{\nabla_X E_i}\xi, E_i) + 2S(X, \xi) = \\ &= 2\sum_{i=1}^{m} g(\nabla_X\nabla_{E_i}\xi, E_i) + 2S(X, \xi) = 2\sum_{i=1}^{m} X(g(\nabla_{E_i}\xi, E_i)) + 2S(X, \xi) = 2X(div(\xi)) + 2S(X, \xi),\end{aligned} \tag{5}$$

where R is the Riemann curvature and S is the Ricci curvature tensor fields of the metric g and the relation (5), for $X := \xi$, becomes:

$$(div(\mathcal{L}_\xi g))(\xi) = 2\xi(div(\xi)) + 2S(\xi, \xi). \tag{6}$$

But the Bochner formula states that for any vector field X [19]:

$$(div(\mathcal{L}_X g))(X) = \frac{1}{2}\Delta(|X|^2) - |\nabla X|^2 + S(X, X) + X(div(X)) \tag{7}$$

and from (6) and (7) we deduce that:

$$\Delta(|\xi|^2) - 2|\nabla\xi|^2 = 2S(\xi, \xi) + 2\xi(div(\xi)). \tag{8}$$

Remark that (5) can be written in terms of $(1, 1)$-tensor fields:

$$div(L_\xi g) = 2d(div(\xi)) + 2i_{Q\xi}g, \tag{9}$$

where Q is the Ricci operator defined by $g(QX, Y) := S(X, Y)$.

3. Gradient η-Ricci Solitons

Consider now the equation:

$$\mathcal{L}_{\xi}g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{10}$$

where g is a Riemannian metric, S its Ricci curvature, η a 1-form and λ and μ are real constants. The data (g, ξ, λ, μ) which satisfy the equation (10) is said to be an *η-Ricci soliton* on M [10]; in particular, if $\mu = 0$, (g, ξ, λ) is a *Ricci soliton* [14]. If the potential vector field ξ is of gradient type, $\xi = grad(f)$, for f a smooth function on M, then (g, ξ, λ, μ) is called *gradient η-Ricci soliton*.

Proposition 3.1. *Let (M, g) be a Riemannian manifold. If (10) defines a gradient η-Ricci soliton on M with the potential vector field $\xi := grad(f)$ and η is the g-dual 1-form of ξ, then:*

$$(\nabla_X Q)Y - (\nabla_Y Q)X = -\nabla^2_{X,Y}\xi + \nabla^2_{Y,X}\xi + \mu(df \otimes \nabla\xi - \nabla\xi \otimes df)(X, Y), \tag{11}$$

for any X, $Y \in \chi(M)$, where Q stands for the Ricci operator.

Proof. As $g(QX, Y) := S(X, Y)$, follows:

$$\nabla\xi + Q + \lambda I_{\chi(M)} + \mu df \otimes \xi = 0. \tag{12}$$

Then:

$$(\nabla_X Q)Y = -(\nabla_X\nabla_Y\xi - \nabla_{\nabla_X Y}\xi) - \mu\{g(Y, \nabla_X\xi)\xi + df(Y)\nabla_X\xi\} :=:= -\nabla^2_{X,Y}\xi - \mu\{g(Y, \nabla_X\xi)\xi + df(Y)\nabla_X\xi\} \tag{13}$$

and using (1) we get the required relation. □

Theorem 3.2. *If (10) defines a gradient η-Ricci soliton on the m-dimensional Riemannian manifold (M, g) and η is the g-dual 1-form of the gradient vector field $\xi := grad(f)$, then:*

$$\frac{1}{2}(\Delta - \nabla_{\xi})(|\xi|^2) = |Hess(f)|^2 + \lambda|\xi|^2 + \mu|\xi|^2\{|\xi|^2 - 2\Delta(f)\}. \tag{14}$$

Proof. First remark that if $\xi = \sum_{i=1}^{m} \xi^i E_i$, for $\{E_i\}_{1\leq i\leq m}$ a local orthonormal frame field with $\nabla_{E_i}E_j = 0$ in a point, then:

$$trace(\eta \otimes \eta) = \sum_{i=1}^{m}[df(E_i)]^2 = \sum_{1\leq i,j,k\leq m} \xi^j\xi^k g(E_i, E_j)g(E_i, E_k) = \sum_{i=1}^{m}(\xi^i)^2 == \textstyle\sum_{1\leq i,j\leq m} \xi^i\xi^j g(E_i, E_j) = |\xi|^2. \tag{15}$$

Taking the trace of the equation (10), we obtain:

$$div(\xi) + scal + m\lambda + \mu|\xi|^2 = 0 \tag{16}$$

and differentiating it:

$$d(div(\xi)) + d(scal) + \mu d(|\xi|^2) = 0. \tag{17}$$

Then taking the divergence of the same equation, we get:

$$div(\mathcal{L}_{\xi}g) + 2div(S) + 2\mu \cdot div(df \otimes df) = 0. \tag{18}$$

Substracting the relations (18) and (17) computed in ξ, considering (6), (8) and using the fact that the scalar and the Ricci curvatures satisfy [19]:

$$d(scal) = 2div(S), \tag{19}$$

we obtain:

$$\frac{1}{2}\Delta(|\xi|^2) - |\nabla\xi|^2 + S(\xi,\xi) + \mu\{2(div(df\otimes df))(\xi) - \xi(|\xi|^2)\} = 0. \tag{20}$$

As

$$(div(df\otimes df))(\xi) := \sum_{i=1}^{m}\{E_i(df(E_i)df(\xi)) - df(E_i)df(\nabla_{E_i}\xi)\} = \sum_{i=1}^{m}\{g(E_i,\xi)g(\nabla_{E_i}\xi,\xi) + g(\xi,\xi)g(E_i,\nabla_{E_i}\xi)\} = \tag{21}$$

$$= g(\nabla_\xi\xi,\xi) + |\xi|^2\sum_{i=1}^{m} g(\nabla_{E_i}\xi, E_i) := \frac{1}{2}\xi(|\xi|^2) + |\xi|^2 div(\xi),$$

the equation (20) becomes:

$$\frac{1}{2}\Delta(|\xi|^2) - |\nabla\xi|^2 + S(\xi,\xi) + 2\mu|\xi|^2 div(\xi) = 0. \tag{22}$$

From the η-soliton equation (10), we get:

$$S(\xi,\xi) = -\frac{1}{2}\xi(|\xi|^2) - \lambda|\xi|^2 - \mu|\xi|^4, \tag{23}$$

and the equation (22) becomes:

$$\frac{1}{2}\Delta(|\xi|^2) = |\nabla\xi|^2 + \frac{1}{2}\xi(|\xi|^2) + \lambda|\xi|^2 + \mu|\xi|^4 - 2\mu|\xi|^2 div(\xi). \tag{24}$$

As $\xi := grad(f)$ follows $Hess(f) = \nabla(df)$ and $|\nabla\xi|^2 = |Hess(f)|^2$. □

Remark 3.3. *For $\mu = 0$ in Theorem 3.2, we obtain the relation for the particular case of gradient Ricci soliton [19].*

Remark 3.4. *i) Assume that $\mu \neq 0$. Denoting by $\Delta_\xi := \Delta - \nabla_\xi$, the equation (14) can be written:*

$$\frac{1}{2}\Delta_\xi(|\xi|^2) = |Hess(f)|^2 + |\xi|^2\{\lambda + \mu[|\xi|^2 - 2\Delta(f)]\},$$

where $\xi := grad(f)$. If $\lambda \geq \mu[2\Delta(f) - |\xi|^2]$, then $\Delta_\xi(|\xi|^2) \geq 0$ and from the maximum principle follows that $|\xi|^2$ is constant in a neighborhood of any local maximum. If $|\xi|$ achieve its maximum, then M is quasi-Einstein. Indeed, since $Hess(f) = 0$, from (10) we have $S = -\lambda g - \mu df \otimes df$. Moreover, in this case, $|\xi|^2\{\lambda + \mu[|\xi|^2 - 2\Delta(f)]\} = 0$, which implies either $\xi = 0$, so M is Einstein, or $|\xi|^2 = 2\Delta(f) - \frac{\lambda}{\mu} \geq 0$. Since $\Delta(f) = -scal - m\lambda - \mu|\xi|^2$ we get $\mu(2\mu+1)|\xi|^2 = -(2\mu\cdot scal + 2m\lambda\mu + \lambda)$. If $\mu = -\frac{1}{2}$, the scalar curvature equals to $\lambda(1-m)$ and if $\mu \neq -\frac{1}{2}$, it is either locally upper (or lower) bounded by $-\frac{\lambda(1+2m\mu)}{2\mu}$, for $\mu < -\frac{1}{2}$ ($\mu > -\frac{1}{2}$, respectively). On the other hand, if the potential vector field is of constant length, then $2\mu\Delta(f) \geq \lambda + \mu|\xi|^2$ equivalent to $\mu(2\mu+1)|\xi|^2 + (2\mu\cdot scal + 2m\lambda\mu + \lambda) \leq 0$ with equality for $\Delta(f) = \frac{\lambda}{2\mu} + \frac{|\xi|^2}{2} \geq \frac{\lambda}{2\mu}$ and $Hess(f) = 0$ which yields the quasi-Einstein case.

ii) For $\mu = 0$, we get the Ricci soliton case [19].

Proposition 3.5. *Let (M, g) be an m-dimensional Riemannian manifold and η be the g-dual 1-form of the gradient vector field $\xi := grad(f)$. If ξ satisfies $\nabla\xi = I_{\chi(M)} - \eta\otimes\xi$, where ∇ is the Levi-Civita connection associated to g, then:*

1. *$Hess(f) = g - \eta\otimes\eta$;*
2. *$R(X,Y)\xi = \eta(X)Y - \eta(Y)X$, for any $X, Y \in \chi(M)$;*
3. *$S(\xi,\xi) = (1-m)|\xi|^2$.*

The condition satisfied by the potential vector field ξ, namely, $\nabla\xi = I_{\chi(M)} - \eta\otimes\xi$, naturally arises if (M,φ,ξ,η,g) is for example, Kenmotsu manifold [16]. In this case, M is a quasi-Einstein manifold.

Example 3.6. *Let $M = \{(x, y, z) \in \mathbb{R}^3, z > 0\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. Set*

$$\varphi := -\frac{\partial}{\partial y} \otimes dx + \frac{\partial}{\partial x} \otimes dy, \quad \xi := -z\frac{\partial}{\partial z}, \quad \eta := -\frac{1}{z}dz,$$

$$g := \frac{1}{z^2}(dx \otimes dx + dy \otimes dy + dz \otimes dz).$$

Then (φ, ξ, η, g) is a Kenmotsu structure on M.

Consider the linearly independent system of vector fields:

$$E_1 := z\frac{\partial}{\partial x}, \quad E_2 := z\frac{\partial}{\partial y}, \quad E_3 := -z\frac{\partial}{\partial z}.$$

Follows

$$\varphi E_1 = -E_2, \quad \varphi E_2 = E_1, \quad \varphi E_3 = 0,$$
$$\eta(E_1) = 0, \quad \eta(E_2) = 0, \quad \eta(E_3) = 1,$$
$$[E_1, E_2] = 0, \quad [E_2, E_3] = E_2, \quad [E_3, E_1] = -E_1$$

and the Levi-Civita connection ∇ is deduced from Koszul's formula

$$2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]),$$

precisely

$$\nabla_{E_1} E_1 = -E_3, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = E_1,$$
$$\nabla_{E_2} E_1 = 0, \quad \nabla_{E_2} E_2 = -E_3, \quad \nabla_{E_2} E_3 = E_2,$$
$$\nabla_{E_3} E_1 = 0, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = 0.$$

Then the Riemann and the Ricci curvature tensor fields are given by:

$$R(E_1, E_2)E_2 = -E_1, \quad R(E_1, E_3)E_3 = -E_1, \quad R(E_2, E_1)E_1 = -E_2,$$
$$R(E_2, E_3)E_3 = -E_2, \quad R(E_3, E_1)E_1 = -E_3, \quad R(E_3, E_2)E_2 = -E_3,$$
$$S(E_1, E_1) = S(E_2, E_2) = S(E_3, E_3) = -2.$$

From (10) computed in (E_i, E_i):

$$2[g(E_i, E_i) - \eta(E_i)\eta(E_i)] + 2S(E_i, E_i) + 2\lambda g(E_i, E_i) + 2\mu\eta(E_i)\eta(E_i) = 0,$$

for all $i \in \{1, 2, 3\}$, we have:

$$2(1 - \delta_{i3}) - 4 + 2\lambda + 2\mu\delta_{i3} = 0 \quad \Longleftrightarrow \quad \lambda - 1 + (\mu - 1)\delta_{i3} = 0,$$

for all $i \in \{1, 2, 3\}$. Therefore, $\lambda = \mu = 1$ define an η-Ricci soliton on $(M, \varphi, \xi, \eta, g)$. Moreover, it is a gradient η-Ricci soliton, as the potential vector field ξ is of gradient type, $\xi = grad(f)$, where $f(x, y, z) := -\ln z$.

Assume now that (10) defines a gradient η-Ricci soliton on (M, g) with $\mu \neq 0$. Under the hypotheses of the Proposition 3.5, the equation (24) simplifies a lot. Compute:

$$|\nabla\xi|^2 := \sum_{i=1}^{m} g(\nabla_{E_i}\xi, \nabla_{E_i}\xi) = \sum_{i=1}^{m} \{1 + (|\xi|^2 - 2)[\eta(E_i)]^2\} = m + |\xi|^2(|\xi|^2 - 2), \tag{25}$$

for $\{E_i\}_{1 \leq i \leq m}$ a local orthonormal frame field with $\nabla_{E_i} E_j = 0$ in a point,

$$\xi(|\xi|^2) = \xi(g(\xi, \xi)) = 2g(\nabla_\xi \xi, \xi) = 2(|\xi|^2 - |\xi|^4), \tag{26}$$

$$\xi(|\xi|^4) = 2|\xi|^2\xi(|\xi|^2) = 4(|\xi|^4 - |\xi|^6). \tag{27}$$

From the equation (10) we obtain:

$$S(\xi,\xi) = -(\lambda+1)|\xi|^2 - (\mu-1)|\xi|^4. \tag{28}$$

Using Proposition 3.5 and the relation (28), we get:

$$|\xi|^2 = (m-1-\lambda)|\xi|^2 - (\mu-1)|\xi|^4, \tag{29}$$

so $|\xi|^2(\mu-1) = m-2-\lambda$ i.e. ξ is of constant length. Using (26) we get $|\xi| = 1$. It follows $\lambda+\mu = m-1$ and we deduce:

Theorem 3.7. *Under the hypotheses of the Proposition 3.5, if (10) defines a gradient η-Ricci soliton on (M, g) with $\mu \neq 0$, then the Laplacian equation (24) becomes:*

$$\Delta(f) = \frac{m-1}{\mu}. \tag{30}$$

Therefore, the existence of a gradient η-Ricci soliton defined by (10) with the potential vector field $\xi := grad(f)$, yields the Laplacian equation (30), and the soliton is completely determined by f.

4. Warped Product η-Ricci Solitons

Consider (B, g_B) and (F, g_F) two Riemannian manifolds of dimensions n and m, respectively. Denote by π and σ the projection maps from the product manifold $B\times F$ to B and F and by $\widetilde{\varphi} := \varphi\circ\pi$ the lift to $B\times F$ of a smooth function φ on B. In this context, we shall call B *the base* and F *the fiber* of $B\times F$, the unique element $\widetilde{X}$ of $\chi(B\times F)$ that is π-related to $X \in \chi(B)$ and to the zero vector field on F, the *horizontal lift of* X and the unique element $\widetilde{V}$ of $\chi(B\times F)$ that is σ-related to $V \in \chi(F)$ and to the zero vector field on B, the *vertical lift of* V. For simplicity, we shall simply denote by X the horizontal lift of $X \in \chi(B)$ and by V the vertical lift of $V \in \chi(F)$. Also, denote by $\mathcal{L}(B)$ the set of all horizontal lifts of vector fields on B, by $\mathcal{L}(F)$ the set of all vertical lifts of vector fields on F, by $\mathcal{H}$ the orthogonal projection of $T_{(p,q)}(B\times F)$ onto its horizontal subspace $T_{(p,q)}(B\times\{q\})$ and by $\mathcal{V}$ the orthogonal projection of $T_{(p,q)}(B\times F)$ onto its vertical subspace $T_{(p,q)}(\{p\}\times F)$.

Let $\varphi > 0$ be a smooth function on B and

$$g := \pi^* g_B + (\varphi\circ\pi)^2\sigma^* g_F \tag{31}$$

be a Riemannian metric on $B\times F$.

Definition 4.1. *[3] The product manifold of B and F together with the Riemannian metric g defined by (31) is called the warped product of B and F by the warping function φ (and is denoted by $(M := B\times_\varphi F, g)$).*

If φ is constant equal to 1, the warped product becomes the usual product of the Riemannian manifolds.

Due to a result of J. Case, Y.-J. Shu and G. Wei [7], we know that for a gradient η-Ricci soliton $(g, \xi := grad(f), \lambda, \mu)$ with $\mu \in (-\infty, 0)$ and $\eta = df$ the g-dual of ξ, on a connected n-dimensional Riemannian manifold (M, g), $e^{2\mu f}[\Delta(f) - |\xi|^2 - \frac{\lambda}{\mu}]$ is constant. Choosing properly an Einstein manifold, a smooth function and considering the warped product manifold, we can characterize the gradient η-Ricci soliton on the base manifold as follows [7]. Let (B, g_B) be an n-dimensional connected Riemannian manifold, λ and μ real constants such that $-\frac{1}{\mu}$ is a natural number, f a smooth function on B, $k := \mu e^{2\mu f}[\Delta(f) - |\xi|^2 - \frac{\lambda}{\mu}]$ and (F, g_F) an m-dimensional Riemannian manifold with $m = -\frac{1}{\mu}$ and $S_F = kg_F$. Then $(g, \xi := grad(f), \lambda, \mu)$ is a gradient η-Ricci soliton on (B, g_B) with $\eta = df$ the g-dual of ξ, if and only if the warped product manifold $(M := B\times_\varphi F, g)$ with the warping function $\varphi = e^{-\frac{f}{m}}$ is Einstein manifold with $S = \lambda g$.

Let S, S_B, S_F the Ricci tensors on M, B and F and $\widetilde{S_B}$, $\widetilde{S_F}$ the lift on M of S_B and S_F, which satisfy:

Lemma 4.2. *[3] If $(M := B \times_{\varphi} F, g)$ is the warped product of B and F by the warping function φ and $m > 1$, then for any $X, Y \in \mathcal{L}(B)$ and any $V, W \in \mathcal{L}(F)$, we have:*

1. $S(X,Y) = \widetilde{S_B}(X,Y) - \frac{m}{\varphi} H^{\varphi}(X,Y)$, *where H^{φ} is the lift on M of $Hess(\varphi)$;*
2. $S(X,V) = 0$;
3. $S(V,W) = \widetilde{S_F}(V,W) - \pi^*[\frac{\Delta(\varphi)}{\varphi} + (m-1)\frac{|grad(\varphi)|^2}{\varphi^2}]|_F g(V,W)$.

Notice that the lift on M of the gradient and the Hessian of any smooth function f on B satisfy:

$$grad(\widetilde{f}) = \widetilde{grad(f)}, \tag{32}$$

$$(Hess(\widetilde{f}))(X,Y) = (\widetilde{Hess(f))}(X,Y), \ \ for\ any\ \ X, Y \in \mathcal{L}(B). \tag{33}$$

We shall construct a gradient η-Ricci soliton on a warped product manifold following [12].

Let (B, g_B) be a Riemannian manifold, $\varphi > 0$ and f two smooth functions on B such that:

$$S_B + Hess(f) - \frac{m}{\varphi} Hess(\varphi) + \lambda g_B + \mu df \otimes df = 0, \tag{34}$$

where λ, μ and $m > 1$ are real constants.

Take (F, g_F) an m-dimensional manifold with $S_F = k g_F$, for $k := \pi^*[-\lambda\varphi^2 + \varphi\Delta(\varphi) + (m-1)|grad(\varphi)|^2 - \varphi(grad(f))(\varphi)]|_F$, where π and σ be the projection maps from the product manifold $B \times F$ to B and F, respectively, and $g := \pi^* g_B + (\varphi \circ \pi)^2 \sigma^* g_F$ a Riemannian metric on $B \times F$. Then, for $\xi := grad(f \circ \pi)$, if consider $\mu = 0$ in (34), (g, ξ, λ) is a gradient Ricci soliton on $B \times_{\varphi} F$ called the *warped product Ricci soliton* [12].

With the above notations, we prove that:

Theorem 4.3. *Let (B, g_B) be a Riemannian manifold, $\varphi > 0$, f two smooth functions on B, let $m > 1$, λ, μ be real constants satisfying (34) and (F, g_F) an m-dimensional Riemannian manifold. Then (g, ξ, λ, μ) is a gradient η-Ricci soliton on the warped product manifold $(B \times_{\varphi} F, g)$, where $\xi = grad(\widetilde{f})$ and the 1-form η is the g-dual of ξ, if and only if:*

$$S_B = -Hess(f) + \frac{m}{\varphi} Hess(\varphi) - \lambda g_B - \mu df \otimes df \tag{35}$$

and

$$S_F = k g_F, \tag{36}$$

where $k := \pi^*[-\lambda\varphi^2 + \varphi\Delta(\varphi) + (m-1)|grad(\varphi)|^2 - \varphi(grad(f))(\varphi)]|_F$.

Proof. The gradient η-Ricci soliton (g, ξ, λ, μ) on $(B \times_{\varphi} F, g)$ is given by:

$$Hess(\widetilde{f}) + S + \lambda g + \mu \eta \otimes \eta = 0. \tag{37}$$

Then for any $X, Y \in \mathcal{L}(B)$ and for any $V, W \in \mathcal{L}(F)$, from Lemma 4.2 we get:

$$H^f(X,Y) + \widetilde{S_B}(X,Y) - \frac{m}{\widetilde{\varphi}} H^{\varphi}(X,Y) + \lambda g_B(X,Y) + \mu df(X) df(Y) = 0$$

$$H^f(V,W) + \widetilde{S_F}(V,W) - \pi^*[\varphi\Delta(\varphi) + (m-1)|grad(\varphi)|^2 - \lambda\varphi^2]|_F g(V,W) = 0$$

and using the fact that

$$H^f(V,W) = (Hess(\widetilde{f}))(V,W) = g(\nabla_V(grad(\widetilde{f})), W) = \pi^*[\frac{(grad(f))(\varphi)}{\varphi}]|_F \widetilde{\varphi}^2 g_F(V,W),$$

we obtain:

$$\widetilde{S_F}(V,W) = \pi^*[\varphi\Delta(\varphi) + (m-1)|grad(\varphi)|^2 - \varphi(grad(f))(\varphi) - \lambda\varphi^2]|_F g_F(V,W).$$

Conversely, notice that the left-hand side term in (37) computed in (X,V), for $X \in \mathcal{L}(B)$ and $V \in \mathcal{L}(F)$ vanishes identically and using again Lemma 4.2, for each situation (X,Y) and (V,W), we can recover the equation (37) from (35) and (36). □

Remark 4.4. *In the case of product manifold (for $\varphi = 1$), notice that the equation (34) defines a gradient η-Ricci soliton on B and the chosen manifold (F, g_F) is Einstein ($S_F = -\lambda g_F$), so a gradient η-Ricci soliton on the product manifold $B \times F$ can be naturally obtained by "lifting" a gradient η-Ricci soliton on B.*

Remark 4.5. *If for the function φ and f on B there exists two constants a and b such that $\nabla(grad(\varphi)) = \varphi[aI_{\chi(B)} + bdf \otimes grad(f)]$, then $Hess(\varphi) = \varphi(ag_B + bdf \otimes df)$ and $(g_B, grad(f), \lambda - ma, \mu - mb)$ is a gradient η-Ricci soliton on B.*

Let us make some remark on the class of manifolds that satisfy the condition (34):

$$S_B + Hess(f) - \frac{m}{\varphi}Hess(\varphi) + \lambda g_B + \mu df \otimes df = 0, \tag{38}$$

for $\varphi > 0$, f smooth functions on the oriented and compact Riemannian manifold (B, g_B), λ, μ and $m > 1$ real constants. Denote by $\xi := grad(f)$.

Taking the trace of (38), we obtain:

$$scal_B + \Delta(f) - m\frac{\Delta(\varphi)}{\varphi} + n\lambda + \mu|\xi|^2 = 0. \tag{39}$$

Remark that:

$$|Hess(f) - \frac{\Delta(f)}{n}g_B|^2 := \sum_{1\leq i,j\leq n}[Hess(f)(E_i, E_j) - \frac{\Delta(f)}{n}g_B(E_i, E_j)]^2 = \tag{40}$$

$$= |Hess(f)|^2 - 2\frac{\Delta(f)}{n}\sum_{i=1}^{n} g_B(\nabla_{E_i}\xi, E_i) + \frac{(\Delta(f))^2}{n} = |Hess(f)|^2 - \frac{(\Delta(f))^2}{n}.$$

Also:

$$(div(Hess(f)))(\xi) := \sum_{i=1}^{n}(\nabla_{E_i}(Hess(f)))(E_i, \xi) = \sum_{i=1}^{n}[E_i(Hess(f)(E_i, \xi)) - Hess(f)(E_i, \nabla_{E_i}\xi)] =$$

$$= \sum_{i=1}^{n} E_i(g_B(\nabla_{E_i}\xi, \xi)) - \sum_{i=1}^{n} g_B(\nabla_{E_i}\xi, \nabla_{E_i}\xi) = \sum_{i=1}^{n} g_B(\nabla_{E_i}\nabla_{\xi}\xi, E_i) - |\nabla\xi|^2 := div(\nabla_{\xi}\xi) - |Hess(f)|^2$$

and

$$div(\nabla_{\xi}\xi) := \sum_{i=1}^{n} g_B(\nabla_{E_i}\nabla_{\xi}\xi, E_i) = \sum_{i=1}^{n} E_i(g_B(\nabla_{\xi}\xi, E_i)) = \sum_{i=1}^{n} E_i(Hess(f)(\xi, E_i)) =$$

$$= \sum_{i=1}^{n}(\nabla_{E_i}(Hess(f)(\xi)))E_i := div(Hess(f)(\xi)),$$

therefore:

$$(div(Hess(f)))(\xi) = div(Hess(f)(\xi)) - |Hess(f)|^2. \tag{41}$$

Applying the divergence to (38), computing it in ξ and considering (21), we get:

$$(div(Hess(f)))(\xi) = -(div(S_B))(\xi) + m(div(\frac{Hess(\varphi)}{\varphi}))(\xi) - \mu(\frac{1}{2}d(|\xi|^2) + \Delta(f)df)(\xi) = \tag{42}$$

$$= -\frac{d(scal_B)(\xi)}{2} + \frac{m}{\varphi}(div(Hess(\varphi)))(\xi) - \frac{m}{\varphi^2}Hess(\varphi)(grad(\varphi), \xi) - \mu[\frac{1}{2}d(|\xi|^2)(\xi) + \Delta(f)|\xi|^2] =$$

$$= -\frac{d(scal_B)(\xi)}{2} + m \cdot div(Hess(\varphi)(\frac{\xi}{\varphi})) - \frac{m}{\varphi}\langle Hess(f), Hess(\varphi)\rangle - \mu[\frac{1}{2}d(|\xi|^2)(\xi) + \Delta(f)|\xi|^2].$$

From (39), (40), (41) and (42), we obtain:

$$div(Hess(f)(\xi)) = |Hess(f) - \frac{\Delta(f)}{n}g_B|^2 - \frac{scal_B}{n}\Delta(f) + \frac{m}{n}\frac{\Delta(\varphi)}{\varphi}\Delta(f) - div(\lambda\xi) - \tag{43}$$
$$-\frac{d(scal_B)(\xi)}{2} + m \cdot div(Hess(\varphi)(\frac{\xi}{\varphi})) - \frac{m}{\varphi}\langle Hess(f), Hess(\varphi)\rangle - \frac{\mu}{2}d(|\xi|^2)(\xi) - \frac{n+1}{n}\mu|\xi|^2\Delta(f).$$

Integrating with respect to the canonical measure on B, we have:

$$\int_B d(scal_B)(\xi) = \int_B \langle grad(scal_B), \xi\rangle = -\int_B \langle scal_B, div(\xi)\rangle = -\int_B scal_B \cdot \Delta(f)$$

and similarly:

$$\int_B d(|\xi|^2)(\xi) = \int_B \langle grad(|\xi|^2), \xi\rangle = -\int_B \langle |\xi|^2, div(\xi)\rangle = -\int_B |\xi|^2 \cdot \Delta(f).$$

Using:

$$|\xi|^2 \cdot div(\xi) = div(|\xi|^2\xi) - |\xi|^2$$

and integrating (43) on B, from the above relations and the divergence theorem, we obtain:

$$\frac{n-2}{2n}\int_B \langle grad(scal_B), \xi\rangle = \int_B |Hess(f) - \frac{\Delta(f)}{n}g_B|^2 - m\int_B \frac{1}{\varphi}\langle Hess(f), Hess(\varphi)\rangle + \tag{44}$$
$$+\frac{m}{n}\int_B \frac{\Delta(\varphi)}{\varphi}\Delta(f) + \frac{n+2}{2n}\mu\int_B |\xi|^2.$$

Proposition 4.6. *Let (B, g_B) be an oriented and compact Riemannian manifold, f a smooth function on B, let $m > 1$, λ, μ be real constants satisfying (34) (for $\varphi = 1$) and (F, g_F) be an m-dimensional Einstein manifold with $S_F = -\lambda g_F$. If (g, ξ, λ, μ) is a gradient η-Ricci soliton on the product manifold $(B \times F, g)$, where $\xi = grad(\widetilde{f})$ and the 1-form η is the g-dual of ξ, then:*

$$\frac{n-2}{2n}\int_B \langle grad(scal_B), \xi\rangle = \int_B |Hess(f) - \frac{\Delta(f)}{n}g_B|^2 + \frac{n+2}{2n}\mu\int_B |\xi|^2. \tag{45}$$

Let now consider the product manifold $B \times F$, in which case (39) (for $\varphi = 1$) becomes:

$$scal_B + \Delta(f) + n\lambda + \mu|\xi|^2 = 0 \tag{46}$$

and integrating it on B, we get:

$$\mu\int_B |\xi|^2 = -\int_B scal_B - n\lambda \cdot vol(B). \tag{47}$$

Replacing it in (45), we obtain:

$$\frac{n-2}{2n}\int_B \langle grad(scal_B), \xi\rangle + \frac{n+2}{2n}\int_B scal_B = \int_B |Hess(f) - \frac{\Delta(f)}{n}g_B|^2 - \frac{n+2}{2}\lambda \cdot vol(B). \tag{48}$$

Proposition 4.7. *Let (B, g_B) be an oriented, compact and complete n-dimensional ($n > 1$) Riemannian manifold of constant scalar curvature, $\varphi > 0$, f two smooth functions on B, let $m > 1$, λ, μ be real constants satisfying (38). If one of the following two conditions hold:*

1. *$\varphi = 1$ and $\lambda = -\frac{scal_B}{n}$;*
2. *there exists a positive function h on B such that $Hess(f) = -h \cdot Hess(\varphi)$ and $\mu \geq 0$,*

then B is conformal to a sphere in the $(n+1)$-dimensional Euclidean space.

Proof. 1. From (48) we obtain:

$$\int_B |Hess(f) - \frac{\Delta(f)}{n} g_B|^2 = \frac{n+2}{2}(\frac{scal_B}{n} + \lambda)\mu \cdot vol(B),$$

so $Hess(f) = \frac{\Delta(f)}{n} g_B$ which implies by [22] that B is conformal to a sphere in the $(n+1)$-dimensional Euclidean space.

2. From the condition $Hess(f) = -h \cdot Hess(\varphi)$ we obtain $\Delta(f) = -h\Delta(\varphi)$ and replacing them in (44), we get:

$$\int_B |Hess(f) - \frac{\Delta(f)}{n} g_B|^2 + \frac{n+2}{2n}\mu \int_B |\xi|^2 = 0.$$

From $\mu \geq 0$ we deduce that $Hess(f) = \frac{\Delta(f)}{n} g_B$ and according to [22], we get the conclusion.

□

Finally, we state a result on the scalar curvature of a product manifold admitting an η-Ricci soliton:

Proposition 4.8. *Let (B, g_B) be an oriented and compact Riemannian manifold of constant scalar curvature, f a smooth function on B, let $m > 1$, λ, μ be real constants satisfying (34) (for $\varphi = 1$) and (F, g_F) be an m-dimensional Einstein manifold with $S_F = -\lambda g_F$. If (g, ξ, λ, μ) is a gradient η-Ricci soliton on the product manifold $(B \times F, g)$, where $\xi = grad(\widetilde{f})$ and the 1-form η is the g-dual of ξ, then the scalar curvature of $B \times F$ is $\geq -(n+m)\lambda$.*

Proof. From (48) we deduce that $\frac{n+2}{2}(\frac{scal_B}{n} + \lambda) \cdot vol(B) = \int_B |Hess(f) - \frac{\Delta(f)}{n} g_B|^2 \geq 0$ and since $scal_F = -m\lambda$, we get the conclusion. □

We end these considerations by giving an example of gradient η-Ricci soliton on a product manifold.

Example 4.9. *Let $(g_M, \xi_M, 1, 1)$ be the gradient η-Ricci soliton on the Riemannian manifold $M = \{(x, y, z) \in \mathbb{R}^3, z > 0\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$, with the metric $g_M := \frac{1}{z^2}(dx \otimes dx + dy \otimes dy + dz \otimes dz)$ (given by Example 3.6) and let S^3 be the 3-sphere with the round metric g_S (which is Einstein with the Ricci tensor equals to $2g_S$). By Remark 4.4 we obtain the gradient η-Ricci soliton $(g, \xi, 1, 1)$ on the "generalized cylinder" $M \times S^3$, where $g = g_M + g_S$ and ξ is the lift on $M \times S^3$ of the gradient vector field $\xi_M = grad(f)$, where $f(x, y, z) := -\ln z$.*

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