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Soft Intersection Semigroups, Ideals and Bi-Ideals; a New Application on Semigroup Theory I

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Abstract. In this paper, we define soft intersection semigroups, soft intersection left (right, two-sided) ideals and bi-ideals of semigroups, give their properties and interrelations and we characterize regular, intra-regular, completely regular, weakly regular and quasi-regular semigroups in terms of these ideals.

1. Introduction

Since its inception by Molodtsov [28] in 1999, soft set theory has been regarded as a new mathematical tool for dealing with uncertainties and it has seen a wide-ranging applications in the mean of algebraic structures such as groups [6, 31], semirings [17], rings [1], BCK/BCI-algebras [21–23], BL-algebras [36], near-rings [33] and soft substructures and union soft substructures [7, 34].

Many related concepts with soft sets, especially soft set operations, have also undergone tremendous studies. Maji et al. [27] presented some definitions on soft sets and based on the analysis of several operations on soft sets Ali et al. [3] introduced several operations of soft sets and Sezgin and Atagün [35] and Ali et al. [2] studied on soft set operations as well.

The theory of soft set has also gone through remarkably rapid strides with a wide-ranging applications especially in soft decision making as in the following studies: [10, 11, 26] and some other fields as [4, 14–16, 18, 32].Soft set theory emphasizes a balanced coverage of both theory and practice. Nowadays, it has promoted a breadth of the discipline of Informations Sciences with intelligent systems, approximate reasoning, expert and decision support systems, self-adaptation and self-organizational systems, information and knowledge, modeling and computing with words.

In [5], the concept of soft ideals, soft quasi-ideals and soft bi-ideals over a given semigroup S are defined and some interesting properties of these ideals are obtained. In this paper, we make a new approach to the classical semigroup theory via soft sets, with the concept of soft intersection semigroup and soft intersection ideals of a semigroup. In the paper [5], the basic definitions are based on soft sets over a semigroup. That

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is to say, the parameter set of the soft set may be any set, whereas the universe set is semigroup. In this paper, the parameter set of the soft set is semigroup, whereas the universe set is any set. This provides us to operate on sets easily with respect to inclusion relation and intersection of sets and also since the parameter set of the soft set is a semigroup, we can more focus on the elements of the semigroup. This make the new concept more functional in the mean of improving the semigroup theory with respect to soft set. The paper reads as follows: In Section 2, we remind some basic definitions about soft sets and semigroups. In Section 3, we define soft intersection product and soft characteristic function and obtain their basic properties. In Section 4, soft intersection semigroup, Section 5, soft intersection left (right, two-sided) ideals, Section 6, soft intersection bi-ideals and soft semiprime ideals are defined and study with respect to soft set operations and soft intersection product. In the following five sections, regular, intra-regular, completely regular, weakly regular and quasi-regular semigroups are characterized by the properties of these ideals, respectively.

2. Preliminaries

In this section, we recall some basic notions relevant to semigroups and soft sets. A *semigroup* S is a nonempty set with an associative binary operation. Note that throughout this paper, S denotes a semigroup.

A nonempty subset A of S is called a *subsemigroup* of S if $AA \subseteq A$ and is called a *right ideal* of S if $AS \subseteq A$ and is called a *left ideal* of S if $SA \subseteq A$. By *two-sided ideal* (or simply *ideal*), we mean a subset of S, which is both a left and right ideal of S. A subsemigroup X of S is called a *bi-ideal* of S if $XSX \subseteq X$. A subset P of a semigroup S is called *semiprime* if $\forall a \in S$, $a^2 \in P$ implies that $a \in P$. We denote by $L[a](R[a], J[a], B[a])$, the principal left ideal (right ideal, two-sided ideal, bi-ideal) of a semigroup S generated by $a \in S$, that is, $L[a] = \{a\} \cup Sa$, $R[a] = \{a\} \cup aS$, $J[a] = \{a\} \cup Sa \cup aS \cup SaS$, $B[a] = \{a\} \cup \{a^2\} \cup aSa$. A *semilattice* is a structure $S = (S, .)$, where "." is an infix binary operation, called the *semilattice operation*, such that "." is associative, commutative and idempotent. For all undefined concepts and notions about semigroups, we refer to [8, 19, 30]. Note that, throughout this paper the product of ordered pairs will be considered componentwise.

Definition 2.1. *([10, 28]) A soft set f_A over U is a set defined by*

$$f_A : E \to P(U) \text{ such that } f_A(x) = \emptyset \text{ if } x \notin A.$$

Here f_A is also called an approximate function. A soft set over U can be represented by the set of ordered pairs

$$f_A = \{(x, f_A(x)) : x \in E, f_A(x) \in P(U)\}.$$

It is clear to see that a soft set is a parametrized family of subsets of the set U. Note that the set of all soft sets over U will be denoted by $S(U)$.

Definition 2.2. *[10] Let $f_A, f_B \in S(U)$. Then, f_A is called a soft subset of f_B and denoted by $f_A \tilde{\subseteq} f_B$, if $f_A(x) \subseteq f_B(x)$ for all $x \in E$.*

Definition 2.3. *[10] Let f_A, $f_B \in S(U)$. Then, union of f_A and f_B, denoted by $f_A \tilde{\cup} f_B$, is defined as $f_A \tilde{\cup} f_B = f_{A \tilde{\cup} B}$, where $f_{A \tilde{\cup} B}(x) = f_A(x) \cup f_B(x)$ for all $x \in E$.*

Definition 2.4. *[10] Let f_A, $f_B \in S(U)$. Then, intersection of f_A and f_B, denoted by $f_A \tilde{\cap} f_B$, is defined as $f_A \tilde{\cap} f_B = f_{A \tilde{\cap} B}$, where $f_{A \tilde{\cap} B}(x) = f_A(x) \cap f_B(x)$ for all $x \in E$.*

Definition 2.5. *[10] Let f_A, $f_B \in S(U)$. Then, $\wedge$-product of f_A and f_B, denoted by $f_A \wedge f_B$, is defined as $f_A \wedge f_B = f_{A \wedge B}$, where $f_{A \wedge B}(x, y) = f_A(x) \cap f_B(y)$ for all $(x, y) \in E \times E$.*

Definition 2.6. *[12] Let f_A and f_B be soft sets over the common universe U and Ψ be a function from A to B. Then, soft image of f_A under Ψ, denoted by $\Psi(f_A)$, is a soft set over U by*

$$(\Psi(f_A))(b) = \begin{cases} \bigcup\{f_A(a) \mid a \in A \ \ and \ \ \Psi(a) = b\}, & \text{if } \ \Psi^{-1}(b) \neq \emptyset, \\ \emptyset, & otherwise \end{cases}$$

for all $b \in B$. And soft pre-image (or soft inverse image) of f_B under Ψ, denoted by $\Psi^{-1}(f_B)$, is a soft set over U by $(\Psi^{-1}(f_B))(a) = f_B(\Psi(a))$ for all $a \in A$.

Definition 2.7. *[13] Let f_A be a soft set over U and $\alpha \subseteq U$. Then, upper α-inclusion of f_A, denoted by $\mathcal{U}(f_A;\alpha)$, is defined as*

$$\mathcal{U}(f_A : \alpha) = \{x \in A \mid f_A(x) \supseteq \alpha\}.$$

3. Soft Intersection Product and Soft Characteristic Function

In this section, we define soft intersection product and soft characteristic function and study their properties.

Definition 3.1. *Let f_S and g_S be soft sets over the common universe U. Then, soft intersection product $f_S \circ g_S$ is defined by*

$$(f_S \circ g_S)(x) = \begin{cases} \bigcup_{x=yz}\{f_S(y) \cap g_S(z)\}, & \text{if } \exists y, z \in S \text{ such that } x = yz, \\ \emptyset, & \text{otherwise} \end{cases}$$

for all $x \in S$.

Note that soft intersection product is abbreviated by soft int-product in what follows.

Example 3.2. *Consider the semigroup $S = \{a, b, c, d\}$ defined by the following table:*

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Let $U = D_3 = \{< x, y >: x^3 = y^2 = e, xy = yx^2\} = \{e, x, x^2, y, yx, yx^2\}$ be the universal set. Let f_S and g_S be soft sets over U such that $f_S(a) = \{e, x, y, yx\}$, $f_S(b) = \{e, x, y^2\}$, $f_S(c) = \{e, y, yx^2\}$, $f_S(d) = \{e, x, x^2, y\}$ and $g_S(a) = \{e, y, y^2\}$, $g_S(b) = \{e, x, yx\}$, $g_S(c) = \{e, yx, yx^2\}$, $g_S(d) = \{e, y, yx\}$. Since $b = cc$, $b = dc$ and $b = dd$, then

$$(f_S \circ g_S)(b) = \{f_S(c) \cap g_S(c)\} \cup \{f_S(d) \cap g_S(c)\} \cup \{f_S(d) \cap g_S(d)\} = \{e, y, yx, yx^2\}$$

Similarly, $(f_S \circ g_S)(a) = \{e, x, y, yx\}$, $(f_S \circ g_S)(c) = (f_S \circ g_S)(d) = \emptyset$.

Theorem 3.3. *Let $f_S, g_S, h_S \in S(U)$. Then,*

i) $(f_S \circ g_S) \circ h_S = f_S \circ (g_S \circ h_S)$.

ii) $f_S \circ g_S \neq g_S \circ f_S$, *generally.*

iii) $f_S \circ (g_S \widetilde{\cup} h_S) = (f_S \circ g_S) \widetilde{\cup} (f_S \circ h_S)$ *and* $(f_S \widetilde{\cup} g_S) \circ h_S = (f_S \circ h_S) \widetilde{\cup} (g_S \circ h_S)$.

iv) $f_S \circ (g_S \widetilde{\cap} h_S) = (f_S \circ g_S) \widetilde{\cap} (f_S \circ h_S)$ *and* $(f_S \widetilde{\cap} g_S) \circ h_S = (f_S \circ h_S) \widetilde{\cap} (g_S \circ h_S)$.

v) If $f_S \widetilde{\subseteq} g_S$, *then* $f_S \circ h_S \widetilde{\subseteq} g_S \circ h_S$ *and* $h_S \circ f_S \widetilde{\subseteq} h_S \circ g_S$.

vi) If $t_S, l_S \in S(U)$ *such that* $t_S \widetilde{\subseteq} f_S$ *and* $l_S \widetilde{\subseteq} g_S$, *then* $t_S \circ l_S \widetilde{\subseteq} f_S \circ g_S$.

Proof. *i*) and *ii*) follows from Definition 3.1 and Example 3.2.

iii) Let $a \in S$. If a is not expressible as $a = xy$, then $(f_S \circ (g_S \widetilde{\cup} h_S))(a) = \emptyset$. Similarly,

$$((f_S \circ g_S) \widetilde{\cup} (f_S \circ h_S))(a) = (f_S \circ g_S)(a) \cup (f_S \circ h_S)(a) = \emptyset \cup \emptyset = \emptyset$$

Now, let there exist $x, y \in S$ such that $a = xy$. Then,

$$\begin{aligned}
(f_S \circ (g_S \widetilde{\cup} h_S))(a) &= \bigcup_{a=xy} (f_S(x) \cap (g_S \widetilde{\cup} h_S)(y)) \\
&= \bigcup_{a=xy} (f_S(x) \cap (g_S(y) \cup h_S(y)) \\
&= \bigcup_{a=xy} [(f_S(x) \cap g_S(y)) \cup (f_S(x) \cap h_S(y))] \\
&= [\bigcup_{a=xy} (f_S(x) \cap g_S(y))] \cup [\bigcup_{a=xy} (f_S(x) \cap h_S(y))] \\
&= (f_S \circ g_S)(a) \cup (f_S \circ h_S)(a) \\
&= [(f_S \circ g_S) \widetilde{\cup} (f_S \circ h_S)](a)
\end{aligned}$$

Thus, $(f_S \widetilde{\cup} g_S) \circ h_S = (f_S \circ h_S) \widetilde{\cup} (g_S \circ h_S)$ and (iv) can be proved similarly.

v) Let $x \in S$. If x is not expressible as $x = yz$, then $(f_S \circ h_S)(x) = (g_S \circ h_S)(x) = \emptyset$. Otherwise,

$$\begin{aligned}
(f_S \circ h_S)(x) &= \bigcup_{x=yz} (f_S(y) \cap h_S(z)) \\
&\subseteq \bigcup_{x=yz} (g_S(y) \cap h_S(z)) \ (since\ f_S(y) \subseteq g_S(y)) \\
&= (g_S \circ h_S)(x)
\end{aligned}$$

Similarly, one can show that $h_S \circ f_S \widetilde{\subseteq} h_S \circ g_S$.

(vi) can be proved similar to (v). □

Definition 3.4. *Let X be a subset of S. We denote by $\mathcal{S}_X$ the soft characteristic function of X and define as*

$$\mathcal{S}_X(x) = \begin{cases} U, & \text{if } x \in X, \\ \emptyset, & \text{if } x \notin X \end{cases}$$

It is obvious that the soft characteristic function is a soft set over U, that is,

$$\mathcal{S}_X : S \to P(U).$$

Theorem 3.5. *Let X and Y be nonempty subsets of a semigroup S. Then, the following properties hold:*

i) If $X \subseteq Y$, then $\mathcal{S}_X \widetilde{\subseteq} \mathcal{S}_Y$.

ii) $\mathcal{S}_X \widetilde{\cap} \mathcal{S}_Y = \mathcal{S}_{X\cap Y}$, $\mathcal{S}_X \widetilde{\cup} \mathcal{S}_Y = \mathcal{S}_{X\cup Y}$.

iii) $\mathcal{S}_X \circ \mathcal{S}_Y = \mathcal{S}_{XY}$.

Proof. i) is straightforward by Definition 3.4.

ii) Let s be any element of S. Suppose $s \in X \cap Y$. Then, $s \in X$ and $s \in Y$. Thus, we have

$$(\mathcal{S}_X \widetilde{\cap} \mathcal{S}_Y)(s) = \mathcal{S}_X(s) \cap \mathcal{S}_Y(s) = U \cap U = U = \mathcal{S}_{X\cap Y}(s)$$

Suppose $s \notin X \cap Y$. Then, $s \notin X$ or $s \notin Y$. Hence, we have

$$(\mathcal{S}_X \widetilde{\cap} \mathcal{S}_Y)(s) = \mathcal{S}_X(s) \cap \mathcal{S}_Y(s) = \emptyset = \mathcal{S}_{X\cap Y}(s)$$

Let s be any element of S. Suppose $s \in X \cup Y$. Then, $s \in X$ or $s \in Y$. Thus, we have

$$(\mathcal{S}_X \widetilde{\cup} \mathcal{S}_Y)(s) = \mathcal{S}_X(s) \cup \mathcal{S}_Y(s) = U = \mathcal{S}_{X \cup Y}(s)$$

Suppose $s \notin X \cup Y$. Then, $s \notin X$ and $s \notin Y$. Hence, we have

$$(\mathcal{S}_X \widetilde{\cup} \mathcal{S}_Y)(s) = \mathcal{S}_X(s) \cup \mathcal{S}_Y(s) = \emptyset = \mathcal{S}_{X \cup Y}(s)$$

iii) Let s be any element of S. Suppose $s \in XY$. Then, $s = xy$ for some $x \in X$ and $y \in Y$. Thus we have,

$$\begin{aligned}(\mathcal{S}_X \circ \mathcal{S}_Y)(s) &= \bigcup_{s=mn} (\mathcal{S}_X(m) \cap \mathcal{S}_Y(n)) \\ &\supseteq \mathcal{S}_X(x) \cap \mathcal{S}_Y(y) \\ &= U\end{aligned}$$

which implies that $(\mathcal{S}_X \circ \mathcal{S}_Y)(s) = U$. Since $s = xy \in XY$, $\mathcal{S}_{XY}(s) = U$. Thus, $\mathcal{S}_X \circ \mathcal{S}_Y = \mathcal{S}_{XY}$.

In another case, when $s \notin XY$, we have $s \neq xy$ for all $x \in X$ and $y \in Y$. If $s = mn$ for some $m, n \in S$, then we have,

$$(\mathcal{S}_X \circ \mathcal{S}_Y)(s) = \bigcup_{s=mn} (\mathcal{S}_X(m) \cap \mathcal{S}_Y(n)) = \emptyset = \mathcal{S}_{XY}(s)$$

If $s \neq mn$ for all $m, n \in S$, then $(\mathcal{S}_X \circ \mathcal{S}_Y)(s) = \emptyset = \mathcal{S}_{XY}(s)$. In any case, we have $\mathcal{S}_X \circ \mathcal{S}_Y = \mathcal{S}_{XY}$. □

4. Soft Intersection Semigroup

In this section, we define soft intersection semigroups, study their basic properties with respect to soft operations and soft int-product.

Definition 4.1. *Let S be a semigroup and f_S be a soft set over U. Then, f_S is called a soft intersection semigroup of S, if*

$$f_S(xy) \supseteq f_S(x) \cap f_S(y)$$

for all $x, y \in S$.

For the sake of brevity, soft intersection semigroup is abbreviated by *SI*-semigroup in what follows.

Example 4.2. *Let $S = \{a, b, c, d\}$ be the semigroup in Example 3.2 and f_S be a soft set over $U = S_3$, symmetric group. If we construct a soft set such that $f_S(a) = \{(1), (123), (132), (12)\}$, $f_S(b) = \{(123), (12)\}$, $f_S(c) = \{(12)\}$, $f_S(d) = \{(123)\}$ then, one can easily show that f_S is an SI-semigroup over U.*

Now, let $U = \left\{ \begin{bmatrix} x & 0 \\ x & 0 \end{bmatrix} \mid x, y \in \mathbb{Z}_4 \right\}$, 2×2 matrices with $\mathbb{Z}_4$ terms, be the universal set. We construct a soft set g_S over U by

$$g_S(a) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 3 & 0 \end{bmatrix} \right\},$$

$$g_S(b) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\},$$

$$g_S(c) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 3 & 0 \end{bmatrix} \right\}$$

$$g_S(d) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} \right\}.$$

Then, since

$$g_S(dc) = g_S(b) \not\supseteq g_S(d) \cap g_S(c),$$

g_S is not an SI-semigroup over U.

It is easy to see that if $f_S(x) = U$ for all $x \in S$, then f_S is an SI-semigroup over U. We denote such a kind of SI-semigroup by $\widetilde{\mathbb{S}}$. It is obvious that $\widetilde{\mathbb{S}} = \mathcal{S}_S$, i.e. $\widetilde{\mathbb{S}}(x) = U$ for all $x \in S$.

Lemma 4.3. *Let f_S be any SI-semigroup over U. Then, we have the followings:*

i) $\widetilde{\mathbb{S}} \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} \widetilde{\mathbb{S}}$.

ii) $f_S \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} \widetilde{\mathbb{S}}$ *and* $\widetilde{\mathbb{S}} \circ f_S \widetilde{\subseteq} \widetilde{\mathbb{S}}$.

iii) $f_S \widetilde{\cup} \widetilde{\mathbb{S}} = \widetilde{\mathbb{S}}$ *and* $f_S \widetilde{\cap} \widetilde{\mathbb{S}} = f_S$.

It is known that a nonempty subset A of S is a subsemigroup if and only if $AA \subseteq A$. It is natural to extend this property to SI-semigroups with the following:

Theorem 4.4. *Let f_S be a soft set over U. Then, f_S is an SI-semigroup over U if and only if*

$$f_S \circ f_S \widetilde{\subseteq} f_S$$

Proof. Assume that f_S is an SI-semigroup over U. Let $a \in S$. If $(f_S \circ f_S)(a) = \emptyset$, then it is obvious that

$$(f_S \circ f_S)(a) \subseteq f_S(a), \ \textit{thus} \ f_S \circ f_S \widetilde{\subseteq} f_S.$$

Otherwise, there exist elements $x, y \in S$ such that $a = xy$. Then, since f_S is an SI-semigroup over U, we have:

$$\begin{aligned}
(f_S \circ f_S)(a) &= \bigcup_{a=xy} (f_S(x) \cap f_S(y)) \\
&\subseteq \bigcup_{a=xy} f_S(xy) \\
&= \bigcup_{a=xy} f_S(a) \\
&= f_S(a)
\end{aligned}$$

Thus, $f_S \circ f_S \widetilde{\subseteq} f_S$.

Conversely, assume that $f_S \circ f_S \widetilde{\subseteq} f_S$. Let $x, y \in S$ and $a = xy$. Then, we have:

$$\begin{aligned}
f_S(xy) &= f_S(a) \\
&\supseteq (f_S \circ f_S)(a) \\
&= \bigcup_{a=xy} (f_S(x) \cap f_S(y)) \\
&\supseteq f_S(x) \cap f_S(y)
\end{aligned}$$

Hence, f_S is an SI-semigroup over U. This completes the proof. □

Theorem 4.5. *Let X be a nonempty subset of a semigroup S. Then, X is a subsemigroup of S if and only if $\mathcal{S}_X$ is an SI-semigroup of S.*

Proof. Assume that X is a subsemigroup of S, that is, $XX \subseteq X$. Then, we have:

$$\mathcal{S}_X \circ \mathcal{S}_X = \mathcal{S}_{XX} \widetilde{\subseteq} \mathcal{S}_X \ (\textit{by Theorem } 3.5 - (i) \textit{ and Theorem } 3.5 - (iii))$$

and so $\mathcal{S}_X$ is an SI-semigroup over U by Theorem 4.4.

Conversely, let $x \in XX$ and $\mathcal{S}_X$ be an SI-semigroup of S. Then, by Theorem 4.4,

$$\mathcal{S}_X(x) \supseteq (\mathcal{S}_X \circ \mathcal{S}_X)(x) = \mathcal{S}_{XX}(x) = U$$

implying that $\mathcal{S}_X(x) = U$, hence $x \in X$. Thus, $XX \subseteq X$ and so, X is a subsemigroup of S. □

Proposition 4.6. *Let f_S and f_T be SI-semigroup over U. Then, $f_S \wedge f_T$ is an SI-semigroup over U.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in S \times T$. Then,

$$\begin{aligned} f_{S\wedge T}((x_1,y_1)(x_2,y_2)) &= f_{S\wedge T}(x_1x_2, y_1y_2) \\ &= f_S(x_1x_2) \cap f_T(y_1y_2) \\ &\supseteq (f_S(x_1) \cap f_S(x_2)) \cap (f_T(y_1) \cap f_T(y_2)) \\ &= (f_S(x_1) \cap f_T(y_1)) \cap (f_S(x_2) \cap f_T(y_2)) \\ &= f_{S\wedge T}(x_1, y_1) \cap f_{S\wedge T}(x_2, y_2) \end{aligned}$$

Therefore, $f_S \wedge f_T$ is an SI-semigroup over U. □

Definition 4.7. *Let f_S, f_T be SI-semigroups over U. Then, the product of soft intersection semigroups f_S and f_T is defined as $f_S \times f_T = f_{S\times T}$, where $f_{S\times T}(x, y) = f_S(x) \times f_T(y)$ for all $(x, y) \in S \times T$.*

Proposition 4.8. *If f_S and f_T are SI-semigroups over U, then so is $f_S \times f_T$ over $U \times U$.*

Proof. By Definition 4.7, let $f_S \times f_T = f_{S\times T}$, where $f_{S\times T}(x, y) = f_S(x) \times f_T(y)$ for all $(x, y) \in S \times T$. Then, for all $(x_1, y_1), (x_2, y_2) \in S \times T$,

$$\begin{aligned} f_{S\times T}((x_1,y_1)(x_2,y_2)) &= f_{S\times T}(x_1x_2, y_1y_2) \\ &= f_S(x_1x_2) \times f_T(y_1y_2) \\ &\supseteq (f_S(x_1) \cap f_S(x_2)) \times (f_T(y_1) \cap f_T(y_2)) \\ &= (f_S(x_1) \times f_T(y_1)) \cap (f_S(x_2) \times f_T(y_2)) \\ &= f_{S\times T}(x_1, y_1) \cap f_{S\times T}(x_2, y_2) \end{aligned}$$

Hence, $f_S \times f_T = f_{S\times T}$ is an SI-semigroup over $U \times U$. □

Proposition 4.9. *If f_S and h_S are SI-semigroups over U, then so is $f_S \widetilde{\cap} h_S$ over U.*

Proof. Let $x, y \in S$, then

$$\begin{aligned} (f_S \widetilde{\cap} h_S)(xy) &= f_S(xy) \cap h_S(xy) \\ &\supseteq (f_S(x) \cap f_S(y)) \cap (h_S(x) \cap h_S(y)) \\ &= (f_S(x) \cap h_S(x)) \cap (f_S(y) \cap h_S(y)) \\ &= (f_S \widetilde{\cap} h_S)(x) \cap (f_S \widetilde{\cap} h_S)(y) \end{aligned}$$

Therefore, $f_S \widetilde{\cap} h_S$ is an SI-semigroup over U. □

Proposition 4.10. *Let f_S be a soft set over U and α be a subset of U such that $\alpha \in Im(f_S)$, where $Im(f_S) = \{\alpha \subseteq U : f_S(x) = \alpha,\ for\ x \in S\}$. If f_S is an SI-semigroup over U, then $\mathcal{U}(f_S; \alpha)$ is a subsemigroup of S.*

Proof. Since $f_S(x) = \alpha$ for some $x \in S$, then $\emptyset \neq \mathcal{U}(f_S; \alpha) \subseteq S$. Let $x, y \in \mathcal{U}(f_S; \alpha)$, then $f_S(x) \supseteq \alpha$ and $f_S(y) \supseteq \alpha$. We need to show that $xy \in \mathcal{U}(f_S; \alpha)$ for all $x, y \in \mathcal{U}(f_S; \alpha)$. Since f_S is an SI-semigroup over U, it follows that

$$f_S(xy) \supseteq f_S(x) \cap f_S(y) \supseteq \alpha \cap \alpha = \alpha$$

implying that $xy \in \mathcal{U}(f_S; \alpha)$. Thus, the proof is completed. □

Definition 4.11. *Let f_S be an SI-semigroup over U. Then, the subsemigroups $\mathcal{U}(f_S; \alpha)$ are called upper α-subsemigroups of f_S.*

Proposition 4.12. *Let f_S be a soft set over U, $\mathcal{U}(f_S; \alpha)$ be upper α-subsemigroups of f_S for each $\alpha \subseteq U$ and $Im(f_S)$ be an ordered set by inclusion. Then, f_S is an SI-semigroup over U.*

Proof. Let $x, y \in S$ and $f_S(x) = \alpha_1$ and $f_S(y) = \alpha_2$. Suppose that $\alpha_1 \subseteq \alpha_2$. It is obvious that $x \in \mathcal{U}(f_S;\alpha_1)$ and $y \in \mathcal{U}(f_S;\alpha_2)$. Since $\alpha_1 \subseteq \alpha_2$, $x, y \in \mathcal{U}(f_S;\alpha_1)$ and since $\mathcal{U}(f_S;\alpha)$ is a subsemigroup of S for all $\alpha \subseteq U$, it follows that $xy \in \mathcal{U}(f_S;\alpha_1)$. Hence, $f_S(xy) \supseteq \alpha_1 = \alpha_1 \cap \alpha_2 = f_S(x) \cap f_S(y)$. Thus, f_S is an *SI*-semigroup over U. □

Proposition 4.13. *Let f_S and f_T be soft sets over U and Ψ be a semigroup isomorphism from S to T. If f_S is an SI-semigroup over U, then so is $\Psi(f_S)$.*

Proof. Let $t_1, t_2 \in T$. Since Ψ is surjective, then there exist $s_1, s_2 \in S$ such that $\Psi(s_1) = t_1$ and $\Psi(s_2) = t_2$. Then,

$$\begin{aligned}
&(\Psi(f_S))(t_1t_2)\\
&= \bigcup\{f_S(s) : s \in S, \Psi(s) = t_1t_2\}\\
&= \bigcup\{f_S(s) : s \in S, s = \Psi^{-1}(t_1t_2)\}\\
&= \bigcup\{f_S(s) : s \in S, s = \Psi^{-1}(\Psi(s_1s_2)) = s_1s_2\}\\
&= \bigcup\{f_S(s_1s_2) : s_i \in S, \Psi(s_i) = t_i, i = 1,2\}\\
&\supseteq \bigcup\{f_S(s_1) \cap f_S(s_2) : s_i \in S, \Psi(s_i) = t_i, i = 1,2\}\\
&= (\bigcup\{f_S(s_1) : s_1 \in S, \Psi(s_1) = t_1\}) \cap (\bigcup\{f_S(s_2) : s_2 \in S, \Psi(s_2) = t_2\})\\
&= (\Psi(f_S))(t_1) \cap (\Psi(f_S))(t_2)
\end{aligned}$$

Hence, $\Psi(f_S)$ is an *SI*-semigroup over U. □

Proposition 4.14. *Let f_S and f_T be soft sets over U and Ψ be a semigroup homomorphism from S to T. If f_T is an SI-semigroup over U, then so is $\Psi^{-1}(f_T)$.*

Proof. Let $s_1, s_2 \in S$. Then,

$$\begin{aligned}
(\Psi^{-1}(f_T))(s_1s_2) &= f_T(\Psi(s_1s_2))\\
&= f_T(\Psi(s_1)\Psi(s_2))\\
&\supseteq f_T(\Psi(s_1)) \cap f_T(\Psi(s_2))\\
&= (\Psi^{-1}(f_T))(s_1) \cap (\Psi^{-1}(f_T))(s_2)
\end{aligned}$$

Hence, $\Psi^{-1}(f_T)$ is an *SI*-semigroup over U. □

5. Soft Intersection Left (Right, Two-Sided) Ideals of Semigroups

In this section, we define soft intersection left (right, two-sided) ideal of semigroups and obtain their basic properties related with soft set operations and soft int-product.

Definition 5.1. *A soft set over U is called a soft intersection left (right) ideal of S over U if*

$$f_S(ab) \supseteq f_S(b) \ (f_S(ab) \supseteq f_S(a))$$

for all $a, b \in S$. A soft set over U is called a soft intersection two-sided ideal (soft intersection ideal) of S if it is both soft intersection left and soft intersection right ideal of S over U.

For the sake of brevity, soft intersection left (right) ideal is abbreviated by *SI*-left (right) ideal in what follows.

Example 5.2. *Consider the semigroup $S = \{0, x, 1\}$ defined by the following table:*

.	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

Let f_S be a soft set over S such that $f_S(0) = \{0, x, 1\}$, $f_S(x) = \{0, x\}$, $f_S(1) = \{x\}$. Then, one can easily show that f_S is an SI-ideal of S over U. However if we define a soft set h_S over S such that $h_S(0) = \{1\}$, $h_S(x) = \{x, 1\}$, $h_S(1) = \{0, x, 1\}$, then, $h_S(x0) = h_S(0) \not\supseteq h_S(x)$ Thus, h_S is not an SI-left ideal over S.

It is known that a nonempty subset A of S is a left ideal of S if and only if $SA \subseteq A$. It is natural to extend this property to SI-semigroups with the following:

Theorem 5.3. *Let f_S be a soft set over U. Then, f_S is an SI-left ideal of S over U if and only if*

$$\widetilde{\mathbb{S}} \circ f_S \widetilde{\subseteq} f_S.$$

Proof. First assume that f_S is an SI-left ideal of S over U. Let $s \in S$. If

$$(\widetilde{\mathbb{S}} \circ f_S)(s) = \emptyset,$$

then it is clear that $\widetilde{\mathbb{S}} \circ f_S \widetilde{\subseteq} f_S$. Otherwise, there exist elements $x, y \in S$ such that $s = xy$. Then, since f_S is an SI-left ideal of S over U, we have:

$$\begin{aligned}
(\widetilde{\mathbb{S}} \circ f_S)(s) &= \bigcup_{s=xy} (\widetilde{\mathbb{S}}(x) \cap f_S(y)) \\
&\subseteq \bigcup_{s=xy} (U \cap f_S(xy)) \\
&= \bigcup_{s=xy} (U \cap f_S(s)) \\
&= f_S(s)
\end{aligned}$$

Thus, we have $\widetilde{\mathbb{S}} \circ f_S \widetilde{\subseteq} f_S$.

Conversely, assume that $\widetilde{\mathbb{S}} \circ f_S \widetilde{\subseteq} f_S$. Let $x, y \in S$ and $s = xy$. Then, we have:

$$\begin{aligned}
f_S(xy) &= f_S(s) \\
&\supseteq (\widetilde{\mathbb{S}} \circ f_S)(s) \\
&= \bigcup_{s=mn} (\widetilde{\mathbb{S}}(m) \cap f_S(n)) \\
&\supseteq \widetilde{\mathbb{S}}(x) \cap f_S(y) \\
&= U \cap f_S(y) \\
&= f_S(y)
\end{aligned}$$

Hence, f_S is an SI-left ideal over U. This completes the proof. □

It is known that a nonempty subset A of S is a right ideal of S if and only is $AS \subseteq A$. It is natural to extend this property to SI-semigroups with the following:

Theorem 5.4. *Let f_S be a soft set over U. Then, f_S is an SI-right ideal of S over U if and only if*

$$f_S \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} f_S$$

Proof. Similar to the proof of Theorem 5.3. □

Theorem 5.5. *Let f_S be a soft set over U. Then, f_S is an SI-ideal of S over U if and only if*

$$f_S \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} f_S \text{ and } \widetilde{\mathbb{S}} \circ f_S \widetilde{\subseteq} f_S$$

Corollary 5.6. *$\widetilde{\mathbb{S}}$ is both SI-right and SI-left ideal of S.*

Proof. Follows from Lemma 4.3-(i). □

Theorem 5.7. *Let X be a nonempty subset of a semigroup S. Then, X is a left (right, two-sided) ideal of S if and only if $\mathcal{S}_X$ is an SI-left (right, two-sided) ideal of S over U.*

Proof. We give the proof for the SI-left ideals. Assume that X is a left ideal of S, that is, $SX \subseteq X$. Then, we have:

$$\widetilde{\mathbb{S}} \circ \mathcal{S}_X = \mathcal{S}_S \circ \mathcal{S}_X = \mathcal{S}_{SX} \widetilde{\subseteq} \mathcal{S}_X$$

thus, $\mathcal{S}_X$ is an SI-left ideal of S over U by Theorem 5.3.

Conversely, let $x \in SX$ and $\mathcal{S}_X$ be an SI-left ideal of S over U. Then,

$$\mathcal{S}_X(x) \supseteq (\widetilde{\mathbb{S}} \circ \mathcal{S}_X)(x) = (\mathcal{S}_S \circ \mathcal{S}_X)(x) = \mathcal{S}_{SX}(x) = U$$

implying that $\mathcal{S}_X(x) = U$, hence $x \in X$. Thus, $SX \subseteq X$ and X is a left ideal of S. □

Proposition 5.8. *Let f_S be a soft set over U. Then, f_S is an SI-ideal of S over U if and only if*

$$f_S(xy) \supseteq f_S(x) \cup f_S(y)$$

for all $x, y \in S$.

Proof. Let f_S be an SI-ideal of S over U. Then,

$$f_S(xy) \supseteq f_S(x) \textit{ and } f_S(xy) \supseteq f_S(y)$$

for all $x, y \in S$. Thus, $f_S(xy) \supseteq f_S(x) \cup f_S(y)$ Conversely suppose that $f_S(xy) \supseteq f_S(x) \cup f_S(y)$ for all $x, y \in S$. It follows that

$$f_S(xy) \supseteq f_S(x) \cup f_S(y) \supseteq f_S(x) \textit{ and } f_S(xy) \supseteq f_S(x) \cup f_S(y) \supseteq f_S(y)$$

so f_S is an SI-ideal of S over U. □

It is obvious that every left (right, two-sided) ideal of S is a subsemigroup of S. Moreover, we have the following:

Theorem 5.9. *Let f_S be a soft set over U. Then, if f_S is an SI-left (right, two-sided) ideal of S over U, f_S is an SI-semigroup over U.*

Proof. We give the proof for SI-left ideals. Let f_S be an SI-left ideal of S over U. Then, $f_S(xy) \supseteq f_S(y)$ for all $x, y \in S$. Thus, $f_S(xy) \supseteq f_S(y) \supseteq f_S(x) \cap f_S(y)$, so f_S is an SI-semigroup over U. □

Proposition 5.10. *If f_S is an SI-right (left) ideal of S over U, then*

$$f_S \widetilde{\cup} (\widetilde{\mathbb{S}} \circ f_S) \ (f_S \widetilde{\cup} (f_S \circ \widetilde{\mathbb{S}})$$

is an SI-ideal of S over U.

Proof. Assume that f_S is an SI-right ideal of S. Then,

$$\begin{aligned}
\widetilde{\mathbb{S}} \circ (f_S \widetilde{\cup} (\widetilde{\mathbb{S}} \circ f_S)) &= (\widetilde{\mathbb{S}} \circ f_S) \widetilde{\cup} (\widetilde{\mathbb{S}} \circ (\widetilde{\mathbb{S}} \circ f_S)) \ (\textit{by Theorem } 3.3\ (iii)) \\
&= (\widetilde{\mathbb{S}} \circ f_S) \widetilde{\cup} ((\widetilde{\mathbb{S}} \circ \widetilde{\mathbb{S}}) \circ f_S) \ (\textit{by Theorem } 3.3\ (i)) \\
&\widetilde{\subseteq} (\widetilde{\mathbb{S}} \circ f_S) \widetilde{\cup} (\widetilde{\mathbb{S}} \circ f_S) \ (\textit{by Lemma } 4.3\ (i)) \\
&= \widetilde{\mathbb{S}} \circ f_S \\
&\widetilde{\subseteq} f_S \widetilde{\cup} (\widetilde{\mathbb{S}} \circ f_S)
\end{aligned}$$

Thus, $f_S\widetilde{\cup}(\widetilde{\mathbb{S}}\circ f_S)$ is an *SI*-left ideal of S over U. Also,

$$\begin{aligned}(f_S\widetilde{\cup}(\widetilde{\mathbb{S}}\circ f_S))\circ\widetilde{\mathbb{S}} &= (f_S\circ\widetilde{\mathbb{S}})\widetilde{\cup}((\widetilde{\mathbb{S}}\circ f_S)\circ\widetilde{\mathbb{S}})\\ &= (f_S\circ\widetilde{\mathbb{S}})\widetilde{\cup}(\widetilde{\mathbb{S}}\circ(f_S\circ\widetilde{\mathbb{S}}))\\ &\widetilde{\subseteq} (f_S\circ\widetilde{\mathbb{S}})\widetilde{\cup}(\widetilde{\mathbb{S}}\circ f_S)\ (since f_S\circ\widetilde{\mathbb{S}}\widetilde{\subseteq} f_S)\\ &\widetilde{\subseteq} f_S\widetilde{\cup}(\widetilde{\mathbb{S}}\circ f_S)\end{aligned}$$

Hence, $f_S\widetilde{\cup}(\widetilde{\mathbb{S}}\circ f_S)$ is an *SI*-right ideal of S over U. This completes the proof. □

It is known that if R is a right ideal of S and L left ideal o f S, then $RL\subseteq R\cap L$ holds. Moreover, we have the following:

Theorem 5.11. *Let f_S be an SI-right ideal of S over U and g_S be an SI-left ideal of S over U. Then*

$$f_S\circ g_S\widetilde{\subseteq} f_S\widetilde{\cap} g_S$$

Proof. Let f_S and g_S be *SI*-right and *SI*-left ideal of S over U, respectively. Then, since $f_S, g_S\widetilde{\subseteq}\widetilde{\mathbb{S}}$ always holds, we have:

$$f_S\circ g_S\widetilde{\subseteq} f_S\circ\widetilde{\mathbb{S}}\widetilde{\subseteq} f_S \textit{ and } f_S\circ g_S\widetilde{\subseteq}\widetilde{\mathbb{S}}\circ g_S\widetilde{\subseteq} g.$$

It follows that $f_S\circ g_S\widetilde{\subseteq} f_S\widetilde{\cap} g_S$. □

Now, we show that if f_S is an *SI*-right ideal of S over U and g_S is an *SI*-left ideal of S over U, then

$$f_S\circ g_S\widetilde{\not\supseteq} f_S\widetilde{\cup} g_S$$

with the following example:

Example 5.12. *Consider the semigroup S and SI-ideal f_S in Example 5.2. Let g_S be a soft set over S such that $g_S(0)=\{x,1\}$, $g_S(x)=\{x\}$, $g_S(1)=\{x\}$, One can easily show that g_S is an SI-ideal of S over U. However,*

$$(f_S\circ g_S)(x)=\bigcup_{x=ab}(f_S(a)\cap g_S(b))=\{x\}\not\supseteq (f_S\widetilde{\cup} g_S)(x)=\{0,x\}.$$

Proposition 5.13. *Let f_S and h_S be SI-left (right) ideals of S over U. Then, $f_S\circ h_S$ is an SI-left (right) ideal of S over U.*

Proof. Let f_S and h_S be *SI*-left ideal of S and $x, y\in S$. Then,

$$(f_S\circ h_S)(y)=\bigcup_{y=pq}(f_S(p)\cap h_S(q))$$

If $y=pq$, then $xy=x(pq)=(xp)q$. Since f_S is an *SI*-left ideal of S, $f_S(xp)\supseteq f_S(p)$. Thus,

$$\begin{aligned}(f_S\circ h_S)(y) &= \bigcup_{y=pq}(f_S(p)\cap h_S(q))\\ &\subseteq \bigcup_{xy=xpq}(f_S(xp)\cap h_S(q))\\ &= (f_S\circ h_S)(xy)\end{aligned}$$

So,

$$(f_S\circ h_S)(xy)\supseteq (f_S\circ h_S)(y)$$

If y is not expressible as $y=pq$, then $(f_S\circ h_S)(y)=\emptyset\subseteq(f_S\circ h_S)(xy)$. Thus, $f_S\circ h_S$ is an *SI*-left ideal of S. □

We give the following propositions without proof. The proofs are similar to those in Section 4.

Proposition 5.14. *Let f_S and f_T be SI-left (right) ideals of S over U. Then, $f_S \wedge f_T$ is an SI-left (right) ideal of $S \times T$ over U.*

Proposition 5.15. *If f_S and f_T are SI-left (right) ideals of S over U, then so is $f_S \times f_T$ of $S \times T$ over $U \times U$.*

Proposition 5.16. *If f_S and h_S are two SI-left (right) ideals of S over U, then so is $f_S \widetilde{\cap} h_S$ of S over U.*

Proposition 5.17. *Let f_S be a soft set over U and α be a subset of U such that $\alpha \in Im(f_S)$. If f_S is an SI-left (right) ideal of S over U, then $\mathcal{U}(f_S;\alpha)$ is a left (right) ideal of S.*

Definition 5.18. *Let f_S be an SI-left (right) ideal of S over U. Then, the left (right) ideals $\mathcal{U}(f_S;\alpha)$ are called upper α-left (right) ideals of f_S.*

Proposition 5.19. *Let f_S be a soft set over U, $\mathcal{U}(f_S;\alpha)$ be upper α-ideals of f_S for each $\alpha \subseteq U$ and $Im(f_S)$ be an ordered set by inclusion. Then, f_S is an SI-left (right) ideal of S over U.*

In order to show Proposition 5.17, we have the following example:

Example 5.20. *Consider the semigroup in Example 3.2. Define a soft set f_S over $U = D_2 = \{e, x, y, yx\}$ such that $f_S(a) = \{e, x, y, yx\}$, $f_S(b) = \{e, x, y\}$, $f_S(c) = \{e, x\}$, $f_S(d) = \{e, y\}$. Then, one can easily show that f_S is an SI-ideal of S over U. By taking into account $Im(f_S)$, we have: $\mathcal{U}(f_S;\{e, x, y, yx\}) = \{a\}, \mathcal{U}(f_S;\{e, x, y\}) = \{a, b\}, \mathcal{U}(f_S;\{e, x\}) = \{a, b, c\}, \mathcal{U}(f_S;\{e, y\}) = \{a, b, d\}$. One can easily show that $\{a\}, \{a, b\}, \{a, b, c\}$ and $\{a, b, d\}$ are two-sided ideals of S.*

In order to show Proposition 5.19, we have the following example:

Example 5.21. *Consider the semigroup in Example 3.2. Define a soft set f_S over $U = D_2 = \{e, x, y, yx\}$ such that $f_S(a) = \{e, x, y, yx\}$, $f_S(b) = \{e, x, yx\}$, $f_S(c) = \{e, x\}$, $f_S(d) = \{x\}$, By taking into account*

$$Im(f_S) = \{\{e, x, y, yx\}, \{e, x, yx\}, \{e, x\}, \{x\}\}$$

and considering that $Im(f_S)$ is ordered by inclusion, we have:

$$\mathcal{U}(f_S;\alpha) = \begin{cases} \{a, b, c, d\}, & \text{if } \alpha = \{x\} \\ \{a, b, c\}, & \text{if } \alpha = \{e, x\} \\ \{a, b\}, & \text{if } \alpha = \{e, x, yx\} \\ \{a\}, & \text{if } \alpha = \{e, x, y, yx\} \end{cases}$$

Since $\{a\}, \{a, b\}, \{a, b, c\}$ and $\{a, b, c, d\}$ are two-sided ideals of S, f_S is an SI-ideal of S over U.

Now we define a soft set h_S over $U = D_2$ such that $h_S(a) = \{e, x, y, yx\}, h_S(b) = \{e, x\}$, $h_S(c) = \{e\}$, $h_S(d) = \{e, x, yx\}$. By taking into account $Im(f_S) = \{\{e, x, y, yx\}, \{e, x\}, \{e\}, \{e, x, yx\}\}$ and considering that $Im(f_S)$ is ordered by inclusion, we have:

$$\mathcal{U}(f_S;\alpha) = \begin{cases} \{a, b, c, d\}, & \text{if } \alpha = \{e\} \\ \{a, b, d\}, & \text{if } \alpha = \{e, x\} \\ \{a, d\}, & \text{if } \alpha = \{e, x, yx\} \\ \{a\}, & \text{if } \alpha = \{e, x, y, yx\} \end{cases}$$

Since $\{a, d\}S \nsubseteq \{a, d\}$ and $S\{a, d\} \nsubseteq \{a, d\}$ $\{a, d\}$ is not a two-sided ideal of S. Moreover, since $h_S(dd) = h_S(b) \nsupseteq h_S(d)$ h_S is not an SI-ideal of S over U.

Proposition 5.22. *Let f_S and f_T be soft sets over U and Ψ be a semigroup isomorphism from S to T. If f_S is an SI-left (right) ideal of S over U, then so is $\Psi(f_S)$ of T over U.*

Proposition 5.23. *Let f_S and f_T be soft sets over U and Ψ be a semigroup homomorphism from S to T. If f_T is an SI-left (right) ideal of T over U, then so is $\Psi^{-1}(f_T)$ of S over U.*

6. Soft Intersection Bi-Ideals of Semigroups

In this section, we define soft intersection bi-ideals and study their properties as regards soft set operations and soft int-product.

Definition 6.1. *An SI-semigroup f_S over U is called a soft intersection bi-ideal of S over U if*

$$f_S(xyz) \supseteq f_S(x) \cap f_S(z)$$

for all $x, y, z \in S$.

For the sake of brevity, soft intersection bi-ideal is abbreviated by *SI*-bi-ideal in what follows.

Example 6.2. *Let $S = \{0, a, b, c\}$ be the semigroup with the operation table given below.*

+	0	a	b	c
0	0	0	0	0
a	0	a	b	0
b	0	0	0	0
c	0	c	0	0

Define the soft set f_S over $U = \mathbb{Z}_4$ such that $f_S(0) = \{\overline{0}, \overline{1}, \overline{2}\}$, $f_S(a) = \{\overline{0}, \overline{1}\}$, $f_S(b) = \{\overline{0}\}$, $f_S(c) = \{\overline{1}, \overline{2}\}$. Then, one can easily show that f_S is an SI bi-ideal of S over U.

It is known that a nonempty subset A of S is a bi-ideal of S if and only if $AA \subseteq A$ and $ASA \subseteq A$. It is natural to extend this property to *SI*-semigroups with the following:

Theorem 6.3. *Let f_S be a soft set over U. Then, f_S is an SI-bi-ideal of S over U if and only if*

$$f_S \circ f_S \widetilde{\subseteq} f_S \text{ and } f_S \circ \widetilde{\mathbb{S}} \circ f_S \widetilde{\subseteq} f_S$$

Proof. First assume that f_S is an *SI*-bi-ideal of S over U. Since f_S is an *SI*-semigroup over U, by Theorem 4.4, we have

$$f_S \circ f_S \widetilde{\subseteq} f_S.$$

Let $s \in S$. In the case, when $(f_S \circ \widetilde{\mathbb{S}} \circ f_S)(s) = \emptyset$, then it is clear that $f_S \circ \widetilde{\mathbb{S}} \circ f_S \widetilde{\subseteq} f_S$, Otherwise, there exist elements $x, y, p, q \in S$ such that

$$s = xy \text{ and } x = pq$$

Then, since f_S is an *SI*-bi-ideal of S over U, we have:

$$f_S(s) = f_S(xy) = f_S((pq)y) \supseteq f_S(p) \cap f_S(y)$$

Thus, we have

$$\begin{aligned}
(f_S \circ \widetilde{\mathbb{S}} \circ f_S)(s) &= [(f_S \circ \widetilde{\mathbb{S}}) \circ f_S](s) \\
&= \bigcup_{s=xy} [(f_S \circ \widetilde{\mathbb{S}})(x) \cap f_S(y)] \\
&= \bigcup_{s=xy} [(\bigcup_{x=pq} (f_S(p) \cap \widetilde{\mathbb{S}}(q)) \cap f_S(y)] \\
&= \bigcup_{s=xy} [(\bigcup_{x=pq} (f_S(p) \cap U) \cap f_S(y)] \\
&= \bigcup_{s=pqy} (f_S(p) \cap f_S(y)) \\
&\subseteq \bigcup_{s=pqy} f_S(pqy) \\
&= f_S(xy) \\
&= f_S(s)
\end{aligned}$$

Hence, $f_S \circ \widetilde{\mathbb{S}} \circ f_S \widetilde{\subseteq} f_S$. Here, note that if $x \neq pq$, then $(f_S \circ \widetilde{\mathbb{S}})(x) = \emptyset$, and so, $(f_S \circ \widetilde{\mathbb{S}} \circ f_S)(s) = \emptyset \subseteq f_S(s)$.

Conversely, assume that $f_S \circ f_S \widetilde{\subseteq} f_S$. By Theorem 4.4, f_S is an *SI*-semigroup of S. Let $x, y, z \in S$ and $s = xyz$. Then, since $f_S \circ \widetilde{\mathbb{S}} \circ f_S \widetilde{\subseteq} f_S$, we have

$$\begin{aligned} f_S(xyz) &= f_S(s) \\ &\supseteq (f_S \circ \widetilde{\mathbb{S}} \circ f_S)(s) \\ &= [(f_S \circ \widetilde{\mathbb{S}}) \circ f_S](s) \\ &= \bigcup_{s=mn} [(f_S \circ \widetilde{\mathbb{S}})(m) \cap f_S(n)] \\ &\supseteq (f_S \circ \widetilde{\mathbb{S}})(xy) \cap f_S(z) \\ &= [\bigcup_{xy=pq} (f_S(p) \cap \widetilde{\mathbb{S}}(q)] \cap f_S(z) \\ &\supseteq ((f_S(x) \cap \widetilde{\mathbb{S}}(y)) \cap f_S(z) \\ &= ((f_S(x) \cap U) \cap f_S(z) \\ &= f_S(x) \cap f_S(z) \end{aligned}$$

Thus, f_S is an *SI*-bi-ideal of S over U. This completes the proof. □

Theorem 6.4. *Let X be a nonempty subset of a semigroup S. Then, X is a bi-ideal of S if and only if $\mathcal{S}_X$ is an SI-bi-ideal of S over U.*

Proof. Assume that X is a bi-ideal of S, that is, $XX \subseteq X$ and $XSX \subseteq X$. Then, we have

$$\mathcal{S}_X \circ \mathcal{S}_X = \mathcal{S}_{XX} \widetilde{\subseteq} \mathcal{S}_X \text{ (since } XX \subseteq X).$$

Thus, $\mathcal{S}_X$ is an *SI*-semigroup over U. Moreover;

$$\mathcal{S}_X \circ \widetilde{\mathbb{S}} \circ \mathcal{S}_X = \mathcal{S}_X \circ \mathcal{S}_S \circ \mathcal{S}_X = \mathcal{S}_{XSX} \widetilde{\subseteq} \mathcal{S}_X \text{ (since } XSX \subseteq X)$$

This means that $\mathcal{S}_X$ is a bi-ideal of S.

Conversely, let $\mathcal{S}_X$ be an *SI*-bi-ideal of S over U. It means that $\mathcal{S}_X$ is an *SI*-semigroup over U. Let $x \in XX$. Then,

$$\mathcal{S}_X(x) \supseteq (\mathcal{S}_X \circ \mathcal{S}_X)(x) = \mathcal{S}_{XX}(x) = U$$

and so $x \in X$. Thus, $XX \subseteq X$ and X is a subsemigroup S. Next, let $y \in XSX$. Thus;

$$\mathcal{S}_X(y) \supseteq (\mathcal{S}_X \circ \widetilde{\mathbb{S}} \circ \mathcal{S}_X)(y) = (\mathcal{S}_X \circ \mathcal{S}_S \circ \mathcal{S}_X)(y) = \mathcal{S}_{XSX}(y) = U$$

and so $y \in X$. Thus, $XSX \subseteq X$ and X is a bi-ideal of S. □

It is known that every left (right, two sided) ideal of a semigroup S is a bi-ideal of S. Moreover, we have the following:

Theorem 6.5. *Every SI-left (right, two sided) ideal of a semigroup S over U is an SI-bi-ideal of S over U.*

Proof. Let f_S be an *SI*-left (right, two sided) ideal of S over U and $x, y, z \in S$. Then, f_S is as *SI*-semigroup by Theorem 5.9. Moreover,

$$f_S(xyz) = f_S((xy)z) \supseteq f_S(z) \supseteq f_S(x) \cap f_S(z)$$

Thus, f_S is an *SI*-bi-ideal of S. □

Theorem 6.6. *Let f_S be any soft subset of a semigroup S and g_S be any SI-bi-ideal of S over U. Then, the soft int-products $f_S \circ g_S$ and $g_S \circ f_S$ are SI-bi-ideals of S over U.*

Proof. We show the proof for $f_S \circ g_S$. To see that $f_S \circ g_S$ is an *SI*-bi-ideal of S over U, first we need to show that $f_S \circ g_S$ is an *SI*-semigroup over U. Thus,

$$\begin{aligned}(f_S \circ g_S) \circ (f_S \circ g_S) &= f_S \circ (g_S \circ (f_S \circ g_S)) \\ &\widetilde{\subseteq} f_S \circ (g_S \circ (\widetilde{\mathbb{S}} \circ g_S)) \text{ (since } f_S \widetilde{\subseteq} \widetilde{\mathbb{S}}) \\ &= f_S \circ (g_S \circ \widetilde{\mathbb{S}} \circ g_S) \\ &\widetilde{\subseteq} f_S \circ g_S \text{ (since } g_S \circ \widetilde{\mathbb{S}} \circ g_S \widetilde{\subseteq} g_S))\end{aligned}$$

Hence, by Theorem 4.4, $f_S \circ g_S$ is an *SI*-semigroup over U. Moreover we have:

$$\begin{aligned}(f_S \circ g_S) \circ \widetilde{\mathbb{S}} \circ (f_S \circ g_S) &= f_S \circ (g_S \circ (\widetilde{\mathbb{S}} \circ f_S) \circ g_S) \\ &\widetilde{\subseteq} f_S \circ (g_S \circ \widetilde{\mathbb{S}} \circ g_S) \text{ (since } \widetilde{\mathbb{S}} \circ f_S \widetilde{\subseteq} \widetilde{\mathbb{S}}) \\ &\widetilde{\subseteq} f_S \circ g_S\end{aligned}$$

Thus, it follows that $f_S \circ g_S$ is an *SI*-bi-ideal of S over U. It can be seen in a similar way that $g_S \circ f_S$ is an *SI*-bi-ideal of S over U. This completes the proof. □

Proposition 6.7. *Let f_S and f_T be SI-bi-ideals over U. Then, $f_S \wedge f_T$ is an SI-bi-ideal of $S \times T$ over U.*

Proposition 6.8. *If f_S and f_T are SI-bi-ideals of S over U, then so is $f_S \times f_T$ of $S \times T$ over $U \times U$.*

Proposition 6.9. *If f_S and h_S are two SI-bi-ideals of S over U, then so is $f_S \widetilde{\cap} h_S$ of S over U.*

Proposition 6.10. *Let f_S be a soft set over U and α be a subset of U such that $\alpha \in Im(f_S)$. If f_S is an SI-bi-ideal of S over U, then $\mathcal{U}(f_S; \alpha)$ is a bi-ideal of S.*

Definition 6.11. *If f_S is an SI-bi-ideal of S over U, then bi-ideals $\mathcal{U}(f_S; \alpha)$ are called upper α bi-ideals of f_S.*

Proposition 6.12. *Let f_S be a soft set over U, $\mathcal{U}(f_S; \alpha)$ be upper α bi-ideals of f_S for each $\alpha \subseteq U$ and $Im(f_S)$ be an ordered set by inclusion. Then, f_S is an SI-bi-ideal of S over U.*

Proposition 6.13. *Let f_S and f_T be soft sets over U and Ψ be a semigroup isomorphism from S to T. If f_S is an SI-bi-ideal of S over U, then so is $\Psi(f_S)$ of T over U.*

Proposition 6.14. *Let f_S and f_T be soft sets over U and Ψ be a semigroup homomorphism from S to T. If f_T is an SI-bi-ideal of T over U, then so is $\Psi^{-1}(f_T)$ of S over U.*

7. Regular semigroups

In this section, we characterize a regular semigroup in terms of *SI*-ideals.

A semigroup S is called *regular* if for every element a of S there exists an element x in S such that

$$a = axa$$

or equivalently $a \in aSa$. There is a characterization of a regular semigroup in [20] as follows:

Proposition 7.1. *[20] For a semigroup S, the following conditions are equivalent:*

1) S is regular.

2) $RL = R \cap L$ for every right ideal R and left ideal L of S.

It is natural to extend this property to *SI*-ideals of S with the following:

Theorem 7.2. *For a semigroup S, the following conditions are equivalent:*

1) S is regular.

2) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ for every SI-right ideal f_S of S over U and SI-left ideal g_S of S over U.

Proof. Let S be a regular semigroup and f_S be an SI-right ideal of S and g_S be an SI-left ideal of S over U. In Theorem 5.11, we show that

$$f_S \circ g_S \widetilde{\subseteq} f_S \widetilde{\cap} g_S$$

for every SI-right ideal f_S of S and SI-left ideal g_S of S over U. Therefore, it suffices to show that $f_S \widetilde{\cap} g_S \widetilde{\subseteq} f_S \circ g_S$. Let s be any element of S. Then, since S is regular, there exists an element x in S such that $s = sxs$. Thus, we have

$$\begin{aligned}(f_S \circ g_S)(s) &= \bigcup_{s=ab}(f_S(a) \cap g_S(b)) \\ &\supseteq f_S(sx) \cap g_S(s) \\ &\supseteq f_S(s) \cap g_S(s) \\ &= (f_S \widetilde{\cap} g_S)(s)\end{aligned}$$

Thus, $f_S \circ g_S = f_S \widetilde{\cap} g_S$.

Conversely, assume that (2) holds. In order to show that S is regular, we need to illustrate that $RL = R \cap L$ for every for every right ideal R of S and left ideal L of S over U. Let R and L be any right ideal and left ideal of S, respectively. It is known that $RL \subseteq R \cap L$ always holds. So it is enough to show that $R \cap L \subseteq RL$. Let a be any element of $R \cap L$. Then, by Theorem 5.7, the soft characteristic functions $\mathcal{S}_R$ and $\mathcal{S}_L$ of R and L are SI-right ideal and SI-left ideal of S, respectively. Thus, we have:

$$\mathcal{S}_{RL}(a) = (\mathcal{S}_R \circ \mathcal{S}_L)(a) = (\mathcal{S}_R \widetilde{\cap} \mathcal{S}_L)(a) = \mathcal{S}_{R\cap L}(a) = U$$

which implies that $a \in RL$. Thus, $R \cap L \subseteq RL$. It follows by Proposition 7.1 that S is regular. Hence (2) implies (1). □

Corollary 7.3. *For a semigroup S, the following conditions are equivalent:*

1) S is regular.

2) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ for every SI-ideals f_S and g_S of S over U.

Proposition 7.4. *Every SI-left (right) ideal of a regular semigroup is idempotent.*

Proof. Let h_S be an SI-right ideal of S. Then,

$$h_S \circ h_S \widetilde{\subseteq} h_S \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} h_S.$$

Now, we show that $h_S \widetilde{\supseteq} h_S \circ h_S$. Since S is regular, there exists an element $x \in S$ such that $a = axa$ for all $a \in S$. So, we have;

$$\begin{aligned}(h_S \circ h_S)(a) &= \bigcup_{a=axa}(h_S(ax) \cap h_S(a)) \\ &\supseteq h_S(a) \cap h_S(a) \\ &= h_S(a)\end{aligned}$$

Hence, $h_S \widetilde{\subseteq} h_S \circ h_S$ and so $(h_S)^2 = h_S \circ h_S = h_S$.

Now, let k_S be any SI-left ideal of S. Then,

$$k_S \circ k_S \widetilde{\subseteq} \widetilde{\mathbb{S}} \circ k_S \widetilde{\subseteq} k_S.$$

Thus, we show that $k_S \widetilde{\subseteq} k_S \circ k_S$. Since S is regular, there exists an element $x \in S$ such that $a = axa$ for all $a \in S$. Thus, we have;

$$\begin{aligned}(k_S \circ k_S)(a) &= \bigcup_{a=axa} (k_S(a) \cap k_S(xa)) \\ &\supseteq (k_S(a) \cap k_S(a)) \\ &= k_S(a)\end{aligned}$$

Hence, $k_S \widetilde{\subseteq} k_S \circ k_S$ and so $(k_S)^2 = k_S \circ k_S = k_S$. □

Corollary 7.5. *Every SI-ideal of a regular semigroup is idempotent.*

Corollary 7.6. *The set of all SI-ideals of a regular semigroup S forms a semilattice under the soft int-product.*

Proof. Let S be a regular semigroup and f_S, g_S and h_S be SI-ideals of S over U. Then, it follows from Theorem 3.3 (i) that

$$(f_S \circ g_S) \circ h_S = f_S \circ (g_S \circ h_S).$$

By Corollary 7.5, f_S is idempotent. Moreover, since $f_S \widetilde{\cap} g_S = g_S \widetilde{\cap} f_S$, by Corollary 7.3, $f_S \circ g_S = g_S \circ f_S$. Hence, the soft int-product is commutative. This completes the proof. □

Proposition 7.7. *Let the set of all SI-ideals of S be a regular semigroup of S under the soft int-product. Then, every SI-ideal of S has the form $f_S = f_S \circ \widetilde{\mathbb{S}} \circ f_S$.*

Proof. Let f_S be an SI-ideal of S. Then, by assumption, there exists an SI-ideal g_S of S such that

$$f_S = f_S \circ g_S \circ f_S.$$

Thus, we have

$$f_S = f_S \circ g_S \circ f_S \widetilde{\subseteq} f_S \circ \widetilde{\mathbb{S}} \circ f_S \widetilde{\subseteq} (f_S \circ \widetilde{\mathbb{S}}) \widetilde{\cap} (\widetilde{\mathbb{S}} \circ f_S) \widetilde{\subseteq} f_S \widetilde{\cap} f_S = f_S,$$

since

$$f_S \circ \widetilde{\mathbb{S}} \circ f_S \widetilde{\subseteq} f_S \circ \widetilde{\mathbb{S}} \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} f_S \circ \widetilde{\mathbb{S}}$$

and

$$f_S \circ \widetilde{\mathbb{S}} \circ f_S \widetilde{\subseteq} \widetilde{\mathbb{S}} \circ \widetilde{\mathbb{S}} \circ f_S \widetilde{\subseteq} \widetilde{\mathbb{S}} \circ f_S.$$

Hence, $f_S = f_S \circ \widetilde{\mathbb{S}} \circ f_S$. □

Definition 7.8. *An SI-ideal f_S of a semigroup S is said to be soft strongly irreducible if and only if for every SI- ideals g_S and h_S of S, $g_S \widetilde{\cap} h_S \widetilde{\subseteq} f_S$ implies that $g_S \widetilde{\subseteq} f_S$ or $h_S \widetilde{\subseteq} f_S$.*

Definition 7.9. *An SI-ideal h_S of a semigroup S is said to be soft prime ideal if for any SI-ideals f_S and g_S of S, $f_S \circ g_S \widetilde{\subseteq} h_S$ implies that $f_S \widetilde{\subseteq} h_S$ or $g_S \widetilde{\subseteq} h_S$.*

Definition 7.10. *The set of SI-ideals of a semigroup is called totally ordered under inclusion if for any SI-ideals f_S and g_S of S, either $f_S \widetilde{\subseteq} g_S$ or $g_S \widetilde{\subseteq} f_S$.*

Proposition 7.11. *In a regular semigroup S, an SI-ideal is soft strongly irreducible if and only if it is soft prime.*

Proof. It follows from Corollary 7.3, Definition 7.8 and Definition 7.9. □

Proposition 7.12. *Every SI-ideal of a regular semigroup S is soft prime if and only if the set of SI-ideals of S is totally ordered under inclusion.*

Proof. It follows from Corollary 7.3, Definition 7.9 and Definition 7.10. □

As is known a semigroup S is regular if and only if $B = BSB$ for all bi-ideals B of S. Now, we shall give a characterization of a regular semigroup by SI-bi-ideals.

Theorem 7.13. *For a semigroup S, the following conditions are equivalent:*

1) S is regular.

2) $f_S = f_S \circ \widetilde{\mathbb{S}} \circ f_S$ for every SI-bi-ideal f_S of S over U.

Proof. First assume that (1) holds. Let f_S be any SI-bi-ideal f_S of S over U and s be any element of S. Then, since S is regular, there exists an element $x \in S$ such that $s = sxs$. Thus, we have;

$$\begin{aligned}(f_S \circ \widetilde{\mathbb{S}} \circ f_S)(s) &= [(f_S \circ \widetilde{\mathbb{S}}) \circ f_S](s)\\ &= \bigcup_{s=ab}[(f_S \circ \widetilde{\mathbb{S}})(a) \cap f_S(b)]\\ &\supseteq (f_S \circ \widetilde{\mathbb{S}})(sx) \cap f_S(s)\\ &= \bigcup_{sx=mn}\{(f_S(m) \cap \widetilde{\mathbb{S}}(n)\} \cap f_S(s)\\ &\supseteq (f_S(s) \cap \widetilde{\mathbb{S}}(x)) \cap f_S(s)\\ &= (f_S(s) \cap U) \cap f_S(s)\\ &= f_S(s)\end{aligned}$$

and so, we have $f_S \circ \widetilde{\mathbb{S}} \circ f_S \widetilde{\supseteq} f_S$. Since f_S is an SI-bi-ideal of S, $f_S \circ \widetilde{\mathbb{S}} \circ f_S \widetilde{\subseteq} f_S$. Thus, $f_S \circ \widetilde{\mathbb{S}} \circ f_S = f_S$ which means that (1) implies (2).

Conversely assume that (2) holds. In order to show that S is regular, we need to illustrate that $B = BSB$ for every bi-ideal B of S. It is obvious that $BSB \subseteq B$. Therefore, it is enough to show that $B \subseteq BSB$. Let $b \in B$. Then, by Theorem 6.4, the soft characteristic function $\mathcal{S}_B$ of B is an SI-bi-ideal of S. Thus, we have;

$$(\mathcal{S}_{BSB})(b) = (\mathcal{S}_B \circ \mathcal{S}_S \circ \mathcal{S}_B)(b) = (\mathcal{S}_B \circ \widetilde{\mathbb{S}} \circ \mathcal{S}_B)(b) = (\mathcal{S}_B)(b) = U$$

which means that $b \in BSB$. Thus, $B \subseteq BSB$ and so $B = BSB$. It follows that S is regular, so (2) implies (1). □

Theorem 7.14. *Let f_S be a soft set of a regular semigroup S. Then, the following conditions are equivalent:*

1) f_S is an SI-bi-ideal of S.

2) f_S may be presented in the form $f_S = g_S \circ h_S$, where g_S is an SI-right ideal and h_S is an SI-left ideal of S over U.

Proof. First assume that (1) holds. Since S is regular, it follows from Theorem 7.13 that $f_S = f_S \circ \widetilde{\mathbb{S}} \circ f_S$. Thus, we have

$$\begin{aligned}f_S &= f_S \circ \widetilde{\mathbb{S}} \circ f_S\\ &= f_S \circ \widetilde{\mathbb{S}} \circ (f_S \circ \widetilde{\mathbb{S}} \circ f_S)\\ &= [f_S \circ (\widetilde{\mathbb{S}} \circ f_S)] \circ (\widetilde{\mathbb{S}} \circ f_S)\\ &\widetilde{\subseteq} (f_S \circ \widetilde{\mathbb{S}}) \circ (\widetilde{\mathbb{S}} \circ f_S) \ (\textit{since } \widetilde{\mathbb{S}} \circ f_S \widetilde{\subseteq} \widetilde{\mathbb{S}})\end{aligned}$$

Similarly,

$$\begin{aligned}(f_S \circ \widetilde{\mathbb{S}}) \circ (\widetilde{\mathbb{S}} \circ f_S) &= f_S \circ (\widetilde{\mathbb{S}} \circ \widetilde{\mathbb{S}}) \circ f_S)\\ &\widetilde{\subseteq} f_S \circ \widetilde{\mathbb{S}} \circ f_S \ (\textit{since } \widetilde{\mathbb{S}} \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} \widetilde{\mathbb{S}})\\ &= f_S\end{aligned}$$

Namely, $f_S = (f_S \circ \widetilde{\mathbb{S}}) \circ (\widetilde{\mathbb{S}} \circ f_S)$. Here, we can easily show that $f_S \circ \widetilde{\mathbb{S}}$ is an SI-right ideal of S and $\widetilde{\mathbb{S}} \circ f_S$ is an SI-left ideal of S. In fact

$$(f_S \circ \widetilde{\mathbb{S}}) \circ \widetilde{\mathbb{S}} = f_S \circ (\widetilde{\mathbb{S}} \circ \widetilde{\mathbb{S}}) \widetilde{\subseteq} f_S \circ \widetilde{\mathbb{S}}$$

Similarly

$$\widetilde{\mathbb{S}} \circ (\widetilde{\mathbb{S}} \circ f_S) = (\widetilde{\mathbb{S}} \circ \widetilde{\mathbb{S}}) \circ f_S \widetilde{\subseteq} \widetilde{\mathbb{S}} \circ f_S$$

implying that $\widetilde{\mathbb{S}} \circ f_S$ is an SI-left ideal of S.

Conversely assume that (2) holds. It means that there exists an SI-right ideal g_S and SI-left ideal h_S of S such that $f_S = g_S \circ h_S$. By Theorem 6.5, every SI-left (right) ideal of S is an SI-bi-ideal of S. Thus, g_S and h_S are SI-bi-ideals of S. Moreover, $g_S \circ h_S = f_S$ is an SI-bi-ideal of S by Theorem 6.6. Therefore, we obtain that (2) implies (1). This completes the proof. □

Theorem 7.15. *For a semigroup S, the following conditions are equivalent:*

1) S is regular.

2) $f_S \widetilde{\cap} g_S = f_S \circ g_S \circ f_S$ for every SI-bi-ideal f_S of S and SI-ideal g_S of S over U.

Proof. First assume that (1) holds. Let f_S be any SI-bi-ideal and g_S be SI-ideal of S over U. Then,

$$f_S \circ g_S \circ f_S \widetilde{\subseteq} f_S \circ \widetilde{\mathbb{S}} \circ f_S \widetilde{\subseteq} f_S$$

and

$$f_S \circ g_S \circ f_S \widetilde{\subseteq} \widetilde{\mathbb{S}} \circ (g_S \circ \widetilde{\mathbb{S}}) \widetilde{\subseteq} \widetilde{\mathbb{S}} \circ g_S \widetilde{\subseteq} g_S$$

so $f_S \circ g_S \circ f_S \widetilde{\subseteq} f_S \widetilde{\cap} g_S$. To show that $f_S \widetilde{\cap} g_S \widetilde{\subseteq} f_S \circ g_S \circ f_S$ holds, let s be any element of S. Since S is regular, there exists an element x in S such that

$$s = sxs \ (s = sx(sxs))$$

Since g_S is an SI-ideal of S, we have

$$g_s(xsx) = g_S(x(sx)) \supseteq g_S(sx) \supseteq g_S(s)$$

Therefore, we have

$$\begin{aligned}
(f_S \circ g_S \circ f_S)(s) &= [f_S \circ (g_S \circ f_S)](s) \\
&= \bigcup_{s=mn} [f_S(m) \cap (g_S \circ f_S)(n)] \\
&\supseteq f_S(s) \cap (g_S \circ f_S)(xsxs) \\
&= f_S(s) \cap \{ \bigcup_{xsxs=yz} [g_S(y) \cap f_S(z)]\} \\
&= f_S(s) \cap (g_S(xsx) \cap f_S(s)) \\
&\supseteq (f_S(s) \cap g_S(s) \cap f_S(s) \\
&\supseteq f_S(s) \cap g_S(s) \\
&= (f_S \widetilde{\cap} g_S)(s)
\end{aligned}$$

so we have $f_S \widetilde{\cap} g_S \widetilde{\subseteq} f_S \circ g_S \circ f_S$. Thus we obtain that $f_S \widetilde{\cap} g_S = f_S \circ g_S \circ f_S$, hence (1) implies (2).

Conversely assume that (2) holds. In order to show that S is regular, it is enough to show that $f_S = f_S \circ \widetilde{\mathbb{S}} \circ f_S$ for all SI-bi-ideals of S over U by Theorem 7.13. Since $\widetilde{\mathbb{S}}$ is an SI-ideal of S, we have $f_S = f_S \widetilde{\cap} \widetilde{\mathbb{S}} = f_S \circ \widetilde{\mathbb{S}} \circ f_S$ Thus, (2) implies (1). This completes the proof. □

Theorem 7.16. *For a semigroup S, the following conditions are equivalent:*

1) S is regular.

2) $h_S \widetilde{\cap} f_S \widetilde{\cap} g_S \widetilde{\subseteq} h_S \circ f_S \circ g_S$ *for every SI-right ideal* h_S*, every SI-bi-ideal* f_S *and every SI-left ideal* g_S *of S.*

Proof. Assume that (1) holds. Let h_S, f_S and g_S be *SI*-right, *SI*-bi-ideal and *SI*-left ideal of S, respectively. Let a be any element of S. Since S is regular, there exists an element x in S such that $a = axa$. Hence, we have:

$$\begin{aligned}
(h_S \circ f_S \circ g_S)(a) &= [h_S \circ (f_S \circ g_S)](a)\\
&= \bigcup_{a=yz}[h_S(y) \cap (f_S \circ g_S)(z)]\\
&\supseteq h_S(ax) \cap (f_S \circ g_S)(a)\\
&= h_S(ax) \cap \{\bigcup_{a=pq}[f_S(p) \cap g_S(q)]\}\\
&\supseteq h_S(a) \cap (f_S(a) \cap g_S(xa))\\
&\supseteq h_S(a) \cap (f_S(a) \cap g_S(a))\\
&= (h_S \widetilde{\cap} f_S \widetilde{\cap} g_S)(a)
\end{aligned}$$

so we have $h_S \circ f_S \circ g_S \widetilde{\subseteq} h_S \cap f_S \cap g_S$. Thus, (1) implies (2).

Conversely assume that (2) holds. Let h_S and g_S be any *SI*-right ideal and *SI*-left ideal of S, respectively. It is obvious that

$$h_S \circ g_S \widetilde{\subseteq} h_S \widetilde{\cup} g_S.$$

Since $\widetilde{\mathbb{S}}$ itself is an *SI*-bi-ideal of S by Theorem 6.3, by assumption we have:

$$h_S \widetilde{\cap} g_S = h_S \widetilde{\cap} \widetilde{\mathbb{S}} \widetilde{\cap} g_S \widetilde{\subseteq} h_S \circ \widetilde{\mathbb{S}} \circ g_S = h_S \circ (\widetilde{\mathbb{S}} \circ g_S) \widetilde{\subseteq} h_S \circ g_S$$

It follows that $h_S \widetilde{\cap} g_S \widetilde{\subseteq} h_S \circ g_S$ for every *SI*-right ideal h_S and *SI*-left ideal g_S of S. It follows by Theorem 7.2 that S is regular. Hence, (2) implies (1). This completes the proof. □

Theorem 7.17. *For a regular semigroup S, the following conditions are equivalent:*

1) Every bi-ideal of S is a right (left, two-sided) ideal of S.

2) Every SI-bi-ideal of S is an SI-right (left, two-sided) ideal of S.

Proof. We give the proof for the *SI*-right ideals. First assume that (1) holds. Let f_S any *SI* bi-ideal of S and a, b any elements in S. One easily show that aSa is a bi-ideal of S. By assumption, aSa is a right ideal of S. Since S is regular,

$$ab \in (aSa)S = a((Sa)S) \subseteq aSa$$

This implies that there exists an element $x \in S$ such that

$$ab = axa.$$

Then, since f_S is an *SI* bi-ideal of S, we have

$$f_S(ab) = f_S(axa) \supseteq f_S(a) \cap f_S(a) = f_S(a).$$

This means that f_S is an *SI*-right ideal of S and that (1) implies (2).

Conversely, assume that (2) holds. Let B be any bi-ideal of S. Then, by Theorem 6.4, the soft characteristic function $\mathcal{S}_B$ of B is an *SI* bi-ideal of S. Thus, by assumption, $\mathcal{S}_B$ is an *SI*-right ideal of S. Again, by Theorem 6.4, B is a right ideal of S. Therefore, (2) implies (1). This completes the proof. □

8. Intra-Regular Semigroups

In this section, we characterize an intra-regular semigroup in terms of SI-ideals. A semigroup S is called *intra-regular* if for every element a of S there exist elements x and y in S such that

$$a = xa^2y$$

Proposition 8.1. *[25] For a semigroup S, the following conditions are equivalent:*

1) S is intra-regular.

2) $L \cap R \subseteq LR$ for every left ideal L and every right ideal R of S.

It is natural to extend this property to SI-ideals of S with the following:

Theorem 8.2. *For a semigroup S, the following conditions are equivalent:*

1) S is intra-regular.

2) $g_S \widetilde{\cap} f_S \widetilde{\subseteq} g_S \circ f_S$ for every SI-right ideal f_S of S and SI-left ideal g_S of S over U.

Proof. First assume that (1) holds. Let f_S be any SI-right ideal and g_S be SI-left ideal of S over U and a be any element of S. Then, since S is intra-regular, there exist elements x and y in S such that $a = xa^2y$. Thus,

$$\begin{aligned}(g_S \circ f_S)(a) &= \bigcup_{a=bc}(g_S(b) \cap f_S(c)) \\ &\supseteq (g_S(xa) \cap f_S(ay)) \\ &\supseteq (g_S(a) \cap f_S(a)) \\ &= (g_S \widetilde{\cap} f_S)(a)\end{aligned}$$

Thus, $g_S \widetilde{\cap} f_S \widetilde{\subseteq} g_S \circ f_S$, which means that (1) implies (2).

Conversely assume that $g_S \widetilde{\cap} f_S \widetilde{\subseteq} g_S \circ f_S$ for every SI-right ideal f_S and SI-left ideal g_S of S over U. In order to show that S in intra-regular, it suffices to illustrate $L \cap R \subseteq LR$ for every left ideal L and for every right ideal R of S. Let L be a left ideal and R be a right ideal of S and $a \in L \cap R$. Then, $a \in L$ and $a \in R$. Thus, the soft characteristic functions $\mathcal{S}_L$ of L and $\mathcal{S}_R$ of R is an SI-left ideal and SI-right ideal of S, respectively. Thus, we have;

$$\mathcal{S}_{LR}(a) = (\mathcal{S}_L \circ \mathcal{S}_R)(a) \widetilde{\supseteq} (\mathcal{S}_L \widetilde{\cap} \mathcal{S}_R(a) = \mathcal{S}_L(a) \cap \mathcal{S}_R(a) = U$$

which means that $a \in LR$. Thus, $L \cap R \subseteq LR$. It follows that S is intra-regular, so (2) implies (1). □

The following characterization of a semigroup is both regular and intra-regular.

Proposition 8.3. *[25] For a semigroup S, the following conditions are equivalent:*

1) S is both regular and intra-regular.

2) $B^2 = B$ for every bi-ideal B of S. (That is, every bi-ideal of S is idempotent).

Theorem 8.4. *For a semigroup S, the following conditions are equivalent:*

1) S is both regular and intra-regular.

2) $f_S \circ f_S = f_S$ for every SI-bi-ideal f_S of S. (That is, every SI-bi-ideal of S is idempotent).

3) $f_S \widetilde{\cap} g_S \widetilde{\subseteq} (f_S \circ g_S) \widetilde{\cap} (g_S \circ f_S)$ for every SI-bi-ideals f_S and g_S of S.

4) $f_S \widetilde{\cap} g_S \widetilde{\subseteq} (f_S \circ g_S) \widetilde{\cap} (g_S \circ f_S)$ for every SI bi-ideal f_S and for every SI-left ideal g_S of S.

5) $f_S \widetilde{\cap} g_S \widetilde{\subseteq} (f_S \circ g_S) \widetilde{\cap} (g_S \circ f_S)$ *for every SI bi-ideal* f_S *and for every SI-right ideal* g_S *of* S*.*

6) $f_S \widetilde{\cap} g_S \widetilde{\subseteq} (f_S \circ g_S) \widetilde{\cap} (g_S \circ f_S)$ *for every SI-right ideal* f_S *and for every SI-left ideal* g_S *of* S*.*

Proof. First assume that (1) holds. In order to show that (3) holds, let f_S and g_S be SI-bi-ideals of S and $a \in S$. Since S is intra-regular, there exist elements y and z in S such that $a = ya^2z$ for every element a of S. Thus,

$$a = axa = (axa)xa = ax(ya^2z)xa = (axya)(azxa)$$

Since f_S and g_S be SI-bi-ideals of S, we have;

$$f_S(a(xy)a) \supseteq f_S(a) \cap f_S(a) = f_S(a)$$

$$g_S(a(zx)a) \supseteq g_S(a) \cap g_S(a) = g_S(a)$$

Then, we have:

$$\begin{aligned}(f_S \circ g_S)(a) &= \bigcup_{a=bc} (f_S(b) \cap g_S(c)) \\ &\supseteq (f_S(axya) \cap g_S(azxa)) \\ &\supseteq f_S(a) \cap g_S(a) \\ &= (f_S \widetilde{\cap} g_S)(a)\end{aligned}$$

and so we have $f_S \circ g_S \widetilde{\supseteq} f_S \widetilde{\cap} g_S$. One can similarly show that $g_S \circ f_S \widetilde{\supseteq} g_S \widetilde{\cap} f_S$, which means that $f_S \widetilde{\cap} g_S \widetilde{\subseteq} (f_S \circ g_S) \widetilde{\cap} (g_S \circ f_S)$. This shows that (1) implies (3).

It is obvious that (3) implies (4), (4) implies (6) , (3) implies (5) and (5) implies (6).

Assume that (6) holds. Let f_S and g_S be any SI-right ideal and SI-left ideal of S, respectively. Then, we have

$$f_S \widetilde{\cap} g_S = g_S \widetilde{\cap} f_S \widetilde{\subseteq} (f_S \circ g_S) \widetilde{\cap} (g_S \circ f_S) \widetilde{\subseteq} g_S \circ f_S$$

It follows by Theorem 8.2 that S is intra-regular. On the other hand,

$$f_S \widetilde{\cap} g_S \widetilde{\subseteq} (f_S \circ g_S) \widetilde{\cap} (g_S \circ f_S) \widetilde{\subseteq} f_S \circ g_S$$

Since, the inclusion $f_S \circ g_S \widetilde{\subseteq} f_S \widetilde{\cap} g_S$ always hold, we have $f_S \widetilde{\cap} g_S = f_S \circ g_S$. It follows that S is regular. Hence, (6) implies (1).

It is clear that (3) implies (2). In fact, by taking g_S as f_S in (3), we get

$$f_S \widetilde{\cap} f_S = f_S = (f_S \circ f_S) \widetilde{\cap} (f_S \circ f_S) = f_S \circ f_S$$

Finally assume that (2) holds. In order to show that (1) holds, it is enough to show that $B^2 = B$ for every bi-ideal B of S. Let B be any bi-ideal of S. Then, $BB \subseteq B$ always holds. We show that $B \subseteq BB$. Let $b \in B$. Since B is a bi-ideal of S, the soft characteristic function $\mathcal{S}_B$ is an SI-bi-ideal of S. So we have;

$$(\mathcal{S}_{BB})(b) = (\mathcal{S}_B \circ \mathcal{S}_B)(b) = \mathcal{S}_B(b) = U$$

which means that $b \in BB$. Thus, $B \subseteq BB$ and so $B = BB = B^2$. It follows that S is both regular and intra-regular, so (2) implies (1). □

Theorem 8.5. *For a semigroup* S*, the following conditions are equivalent:*

1) S *is both regular and intra-regular.*

2) $f_S \widetilde{\cap} g_S \widetilde{\cap} h_S \widetilde{\subseteq} f_S \circ g_S \circ h_S$ *for every SI-bi-ideals* f_S*,* g_S *and* h_S *of* S*.*

3) $f_S \widetilde{\cap} g_S \widetilde{\cap} h_S \widetilde{\subseteq} f_S \circ g_S \circ h_S$ *for every SI bi-ideals* f_S *and* h_S *of* S *and for every SI-right ideal* g_S *of* S.

4) $f_S \widetilde{\cap} g_S \widetilde{\cap} h_S \widetilde{\subseteq} f_S \circ g_S \circ h_S$ *for every SI-left ideals* f_S *and* h_S *of* S *and for every SI-right ideal* g_S *of* S.

Proof. First assume that (1) holds. In order to show that (4) holds, let f_S and h_S be any SI-left ideals of S and g_S be any SI-right ideal of S and a be any element in S. Since S is regular, there exists element x in S such that $a = axa$. Since S is intra-regular, there exist elements y, z in S such that $a = ya^2z$. Thus, we have

$$a = axa = (axa)x(axa) = (ax(yaaz))x((yaaz)xa) = (axya)(azxya)(azxa)$$

Therefore, we have

$$\begin{aligned}
(f_S \circ g_S \circ h_S)(a) &= [f_S \circ (g_S \circ h_S)](a) \\
&= \bigcup_{a=pq} [f_S(p) \cap (g_S \circ h_S)(q)] \\
&\supseteq f_S(axya) \cap (g_S \circ h_S)(azxyaazxa) \\
&= f_S(a) \cap \{ \bigcup_{azxyaazxa=uv} (g_S(u) \cap h_S(v)) \} \\
&\supseteq f_S(a) \cap (g_S(azxya) \cap h_S(azxa)) \\
&\supseteq f_S(a) \cap g_S(a) \cap h_S(a) \\
&= (f_S \widetilde{\cap} g_S \widetilde{\cap} h_S)(a)
\end{aligned}$$

so we have $f_S \widetilde{\cap} g_S \widetilde{\cap} h_S \widetilde{\subseteq} f_S \circ g_S \circ h_S$. Thus, (1) implies (4). Assume that (4) holds. Let f_S and g_S be SI-left and SI-right ideal of S, respectively. Since $\widetilde{\mathbb{S}}$, itself is an SI-left ideal of S,

$$g_S \widetilde{\cap} f_S = g_S \widetilde{\cap} \widetilde{\mathbb{S}} \widetilde{\cap} f_S \widetilde{\subseteq} g_S \circ \widetilde{\mathbb{S}} \circ f_S \widetilde{\subseteq} g_S \circ f_S$$

Since the inclusion $g_S \circ f_S \widetilde{\subseteq} g_S \widetilde{\cap} f_S$ always hold, $g_S \widetilde{\cap} f_S = g_S \circ f_S$. Hence, it follows that S is regular. Now, let f_S and g_S be any SI-left ideal and SI-right ideal of S, respectively. Since $\widetilde{\mathbb{S}}$ itself is an SI-left ideal of S, by assumption we have:

$$f_S \widetilde{\cap} g_S = f_S \widetilde{\cap} g_S \widetilde{\cap} \widetilde{\mathbb{S}} \widetilde{\subseteq} f_S \circ g_S \circ \widetilde{\mathbb{S}} = f_S \circ (g_S \circ \widetilde{\mathbb{S}}) \widetilde{\subseteq} f_S \circ g_S$$

Thus, it follows by Theorem 8.2 that S is intra-regular. So, (4) implies (1). It is obvious that (2) implies (3) and (3) implies (4). Thus, the proof is completed. □

Now we give a new characterization for an intra-regular semigroup: First, we have the following definition:

Definition 8.6. *A soft set* f_S *over* U *is called soft semiprime if for all* $a \in S$,

$$f_S(a) \supseteq f_S(a^2).$$

Theorem 8.7. *For a nonempty subset* A *of* S*, the following conditions are equivalent:*

1) A *is semiprime.*

2) The soft characteristic function $\mathcal{S}_A$ *of* A *is soft semiprime.*

Proof. First assume that (1) holds. Let a be any element of S. We need to show that $\mathcal{S}_A(a) \supseteq \mathcal{S}_A(a^2)$ for all $a \in S$. If $a^2 \in A$, then since A is semiprime, $a \in A$. Thus,

$$\mathcal{S}_A(a) = U = \mathcal{S}_A(a^2)$$

If $a^2 \notin A$, then

$$\mathcal{S}_A(a) \supseteq \emptyset = \mathcal{S}_A(a^2)$$

In any case, $\mathcal{S}_A(a) \supseteq \mathcal{S}_A(a^2)$ for all $a \in S$. Thus, $\mathcal{S}_A$ is soft semiprime. Hence (1) implies (2).

Conversely assume that (2) holds. Let $a^2 \in A$. Since $\mathcal{S}_A$ is soft semiprime, we have

$$\mathcal{S}_A(a) \supseteq \mathcal{S}_A(a^2) = U$$

implying that $\mathcal{S}_A(a) = U$ and that $a \in A$. Hence, A is semiprime. Thus, (2) implies (1). □

Theorem 8.8. *For any SI-semigroup f_S, the following conditions are equivalent:*

1) f_S is soft semiprime.

2) $f_S(a) = f_S(a^2)$ for all $a \in S$.

Proof. (2) implies (1) is clear. Assume that (1) holds. Let a be any element of S. Since f_S is an SI-semigroup, we have;

$$f_S(a) \supseteq f_S(a^2) = f_S(aa) \supseteq f_S(a) \cap f_S(a) = f_S(a)$$

So, $f_S(a^2) = f_S(a)$ and (1) implies (2). This completes the proof. □

Theorem 8.9. *For a semigroup S, the following conditions are equivalent:*

1) S is intra-regular.

2) Every SI-ideal of S is soft semiprime.

3) $f_S(a) = f_S(a^2)$ for all SI-ideal of S and for all $a \in S$.

Proof. First assume that (1) holds. Let f_S be any SI-ideal of S and a any element of S. Since S is intra-regular, there exist elements x and y in S such that $a = xa^2y$. Thus,

$$f_S(a) = f_S(xa^2y) \supseteq f_S(xa^2) \supseteq f_S(a^2) = f_S(aa) \supseteq f_S(a)$$

so, we have $f_S(a) = f_S(a^2)$. Hence, (1) implies (3).

Conversely, assume that (3) holds. It is known that $J[a^2]$ is an ideal of S. Thus, the soft characteristic function $\mathcal{S}_{J[a^2]}$ is an SI-ideal of S. Since $a^2 \in J[a^2]$, we have;

$$\mathcal{S}_{J[a^2]}(a) = \mathcal{S}_{J[a^2]}(a^2) = U$$

Thus, $a \in J[a^2] = \{a^2\} \cup Sa^2 \cup a^2S \cup Sa^2S \subseteq Sa^2S$. Here, one can easily show that S is intra-regular. Hence (3) implies (1).

It is obvious that (3) implies (2). Now, assume that (2) holds. Let f_S be an SI-ideal of S. Since f_S is a soft semiprime ideal of S,

$$f_S(a) \supseteq f_S(a^2) = f_S(aa) \supseteq f_S(a)$$

Thus, $f_S(a) = f_S(a^2)$. Hence (2) implies (3). This completes the proof. □

Theorem 8.10. *Let S be an intra-regular semigroup. Then, for every SI-ideal f_S of S,*

$$f_S(ab) = f_S(ba)$$

for all $a, b \in S$.

Proof. Let f_S be an SI-ideal of an intra-regular semigroup S. Then, by Theorem 8.9, we have;

$$f_S(ab) = f_S((ab)^2) = f_S(a(ba)b) \supseteq f_S(ba) = f_S((ba)^2) = f_S(b(ab)a) \supseteq f_S(ab)$$

so, we have $f_S(ab) = f_S(ba)$. This completes the proof. □

9. Completely Regular Semigroups

In this section, we characterize a completely regular semigroups in terms of SI-ideals. An element a of S is called a *completely regular* if there exists an element $x \in S$ such that

$$a = axa \text{ and } ax = xa$$

A semigroup S is called *completely regular* if every element of S is completely regular. A semigroup is called *left (right) regular* if for each element a of S, there exists an element $x \in S$ such that

$$a = xa^2 \ (a = a^2x).$$

Proposition 9.1. *[29] For a semigroup S, the following conditions are equivalent:*

1) S is completely regular.

2) S is left and right regular, that is, $a \in Sa^2$ and $a \in a^2S$ for all $a \in S$.

3) $a \in a^2Sa^2$ for all $a \in S$.

Theorem 9.2. *For a left regular semigroup S, the following conditions are equivalent:*

1) Every left ideal of S is a two-sided ideal of S.

2) Every SI-left ideal of S is an SI-ideal of S.

Proof. Assume that (1) holds. Let f_S be any SI-left ideal of S and a and b be any elements of S. Then, since the left ideal Sa is a two-sided ideal by assumption and since S is left regular, we have

$$ab \in (Sa^2)b \subseteq (Sa)bS \subseteq Sa$$

This implies that there exists an element $x \in S$ such that $ab = xa$. Thus, since fS is an SI-left ideal of S, we have

$$f_S(ab) = f_S(xa) \supseteq f_S(a).$$

Hence, f_S is an SI-right ideal of S and so f_S is an SI-ideal of S. Thus (1) implies (2).

Assume that (2) holds. Let A be any left ideal of S. Then, the soft characteristic function $\mathcal{S}_A$ is an SI-left ideal of S. Then, by assumption, $\mathcal{S}_A$ is an SI-right ideal of S and so A is a right ideal of S and so A is a two-sided ideal of S. Hence (2) implies (1). □

Theorem 9.3. *For a semigroup S, the following conditions are equivalent:*

1) S is left regular.

2) For every SI-left ideal f_S of S, $f_S(a) = f_S(a^2)$ for all $a \in S$.

Proof. First assume that (1) holds. Let f_S be any SI-left ideal of S and a be any element of S. Since S is left regular, there exists an element x in S such that $a = xa^2$. Thus, we have

$$f_S(a) = f_S(xa^2) \supseteq f_S(a^2) \supseteq f_S(a)$$

implying that $f_S(a) = f_S(a^2)$. Hence (1) implies (2).

Conversely, assume that (2) holds. Let a be any element of S. Since $L[a^2]$ is a left ideal of S, the soft characteristic function $\mathcal{S}_{L[a^2]}$ is an SI-left ideal of S. Since $a^2 \in L[a^2]$, we have

$$\mathcal{S}_{L[a^2]}(a) = \mathcal{S}_{L[a^2]}(a^2) = U$$

implying that $a \in L[a^2] = \{a^2\} \cup Sa^2$. This obviously means that S is left regular. So (2) implies (1). This completes the proof. □

Theorem 9.4. *For a semigroup S, the following conditions are equivalent:*

1) S is right regular.

2) For every SI-right ideal f_S of S, $f_S(a) = f_S(a^2)$ for all $a \in S$.

Theorem 9.5. *For a semigroup S, the following conditions are equivalent:*

1) S is completely regular.

2) Every bi-ideal of S is semiprime.

3) Every SI-bi-ideal of S is soft semiprime.

4) $f_S(a) = f_S(a^2)$ for every SI-bi-ideal f_S of S and for all $a \in S$.

Proof. First assume that (1) holds. Let f_S be any SI-bi-ideal of S. Since S is completely regular, there exists an element $x \in S$ such that $a = a^2xa^2$. Thus, we have

$$f_S(a) = f_S(a^2xa^2) \supseteq f_S(a^2) \cap f_S(a^2) = f_S(a^2) = f_S(aa) = f_S(a(a^2xa^2) =$$

$$f_S(a(a^2xa)a) \supseteq f_S(a) \cap f_S(a) = f_S(a)$$

and so, $f_S(a) = f_S(a^2)$. Thus (1) implies (4). (4) implies (3) is clear by Theorem 8.9. Assume that (3) holds. Let B be any bi-ideal of S and $a^2 \in B$. Since the soft characteristic function $\mathcal{S}_B$ of B is an SI-bi-ideal of S, it is soft semiprime by hypothesis. Thus,

$$\mathcal{S}_B(a) \supseteq \mathcal{S}_B(a^2) = U$$

Hence, $a \in B$ and so B is semiprime. Thus (3) implies (2).

Finally assume that (2) holds. Let a be any element of S. Then, since the principal ideal $B[a^2]$ generated by a^2 is a bi-ideal and so by assumption semiprime and since $a^2 \in B[a^2]$,

$$\mathcal{S}_{B[a^2]}(a) = \mathcal{S}_{B[a^2]}(a^2) = U$$

implying that

$$a \in B[a^2] = \{a^2\} \cup \{a^4\} \cup a^2Sa^2 \subseteq a^2Sa^2.$$

This implies that S is completely regular. Thus (2) implies (1). This completes the proof. □

10. Weakly Regular Semigroups

In this section, we characterize a weakly regular semigroup in terms of SI-ideals. A semigroup S is called weakly-regular if for every $x \in S$, $x \in (xS)^2$.

Proposition 10.1. *[29] A monoid is weakly regular if and only if $I \cap J = IJ$ for all right ideal I and all two-sided ideal J of S.*

Theorem 10.2. *For a monoid S, the following conditions are equivalent:*

1) S is weakly regular.

2) $f_S \widetilde{\cap} g_S \widetilde{\subseteq} f_S \circ g_S$ for every SI-right ideal f_S of S and for every SI-ideal g_S of S.

Proof. First assume that (1) holds. Let f_S be an SI-right ideal of S, g_S be an SI-left ideal of S and $x \in S$. Then, since S is weakly regular, $x \in (xS)^2$. Thus, $x = xsxt$ for some $s, t \in S$. Hence,

$$\begin{aligned}(f_S \circ g_S)(x) &= \bigcup_{x=xsxt} (f_S(xs) \cap g_S(xt)) \\ &\supseteq f_S(x) \cap g_S(x) \\ &= (f_S \widetilde{\cap} g_S)(x)\end{aligned}$$

Since $f_S \widetilde{\cap} g_S \widetilde{\supseteq} f_S \circ g_S$ always holds for every SI-right ideal f_S and SI-left ideal g_S of S, $f_S \widetilde{\cap} g_S = f_S \circ g_S$. Thus, (1) implies (2).

Conversely assume that (2) holds. In order to show that S is weakly regular, we show that $R \cap L = RL$ for every right ideal R and left ideal L of S. It is obvious that $RL \subseteq R \cap L$ always holds. In order to see that $R \cap L \subseteq RL$, let a be any element in $R \cap L$. Then $a \in R$ and $a \in L$. Thus, the soft characteristic functions $\mathcal{S}_R$ of R and $\mathcal{S}_L$ of L is SI-right and SI-left ideal of S, respectively. Thus, we have:

$$\mathcal{S}_{RL}(a) = (\mathcal{S}_R \circ \mathcal{S}_L)(a) = (\mathcal{S}_R \widetilde{\cap} \mathcal{S}_L)(a) = (\mathcal{S}_{R \cap L})(a) = U$$

so, $a \in RL$. Thus, $R \cap L \subseteq RL$ and $R \cap L = RL$. It follows that S is weakly-regular. Hence (2) implies (1). □

Theorem 10.3. *For a monoid S, the following conditions are equivalent:*

1) S is weakly regular.

2) $f_S \widetilde{\cap} g_S \widetilde{\cap} h_S \widetilde{\subseteq} f_S \circ g_S \circ h_S$ for every SI-bi-ideal f_S of S, for every SI-ideal g_S of S and for every SI-right ideal h_S of S.

Proof. First assume that (1) holds. Let $x \in S$. Then, $x \in (xS)^2$. Thus, $x = xsxt$ for some $s, t \in S$. Hence,

$$\begin{aligned}(f_S \circ g_S \circ h_S)(x) &= [f_S \circ (g_S \circ h_S)](x) \\ &= \bigcup_{x=xsxt} [f_S(x) \cap (g_S \circ h_S)(sxt)] \\ &\supseteq f_S(x) \cap \{\bigcup_{sxt=pv} (g_S(p) \cap h_S(v))\} \\ &\supseteq f_S(x) \cap g_S(sxs) \cap h_S(xt^2) \\ &\supseteq f_S(x) \cap g_S(x) \cap h_S(x) \\ &= (f_S \widetilde{\cap} g_S \widetilde{\cap} h_S)(x)\end{aligned}$$

since $sxt = s(xsxt)t = (sxs)(xt^2)$. Thus, (1) implies (2).

Now, assume that (2) holds. Let f_S be an SI-right ideal of S, g_S be an SI-ideal of S and let $h_S = \widetilde{\mathbb{S}}$. Then, we have

$$f_S \widetilde{\cap} g_S \widetilde{\cap} h_S = f_S \widetilde{\cap} g_S \widetilde{\cap} \widetilde{\mathbb{S}} = f_S \widetilde{\cap} g_S$$

and

$$f_S \circ g_S \circ h_S = f_S \circ g_S \circ \widetilde{\mathbb{S}} = f_S \circ (g_S \circ \widetilde{\mathbb{S}}) \widetilde{\subseteq} f_S \circ g_S$$

Then, $f_S \widetilde{\cap} g_S = f_S \widetilde{\cap} g_S \widetilde{\cap} h_S \widetilde{\subseteq} f_S \circ g_S \circ h_S \widetilde{\subseteq} f_S \circ g_S$ that is, $f_S \widetilde{\cap} g_S \widetilde{\subseteq} f_S \circ g_S$ for every SI-right ideal f_S of S and SI-ideal g_S of S. Thus, S is weakly regular. Hence (2) implies (1). This completes the proof. □

Theorem 10.4. *For a monoid S, the following conditions are equivalent:*

1) S is weakly regular.

2) $f_S \widetilde{\cap} g_S \widetilde{\subseteq} f_S \circ g_S$ for every SI-bi-ideal f_S of S and for every SI-ideal g_S of S.

Proof. Similar to the the proof of Theorem 10.3. □

11. Quasi-Regular Semigroups

In this section, we study a semigroup whose *SI*-left (right, two-sided) ideals are all idempotent. A semigroup S is called *left (right) quasi-regular* if every left (right) ideal of S is idempotent, and is called *quasi-regular* if every left ideal and right ideal of S is idempotent ([9]). It is easy to prove that S is left (right) quasi-regular if and only if $a \in SaSa$ ($a \in aSaS$), this implies that there exist elements $x, y \in S$ such that $a = xaya$ ($a = axay$).

Theorem 11.1. *A semigroup S is left (right) quasi-regular if and only if every SI-left (right) ideal is idempotent.*

Proof. Assume that f_S is an *SI*-left ideal. Then, there exist $x, y \in S$ such that $a = xaya$. So, we have;

$$\begin{aligned}(f_S \circ f_S)(a) &= \bigcup_{a=xaya} (f_S(xa) \cap f_S(ya)) \\ &\supseteq f_S(xa) \cap f_S(ya) \\ &\supseteq f_S(a) \cap f_S(a) \\ &= f_S(a)\end{aligned}$$

and so, $f_S \circ f_S \tilde{\supseteq} f_S$. Thus, $f_S \circ f_S = f_S$ and f_S is idempotent.

Conversely, assume that every *SI*-left ideal of S is idempotent. Let $a \in S$. Then, since $L[a]$ is a principal left ideal of S, the soft characteristic function $\mathcal{S}_{L[a]}$ is an *SI*-left ideal of S. Thus, by assumption

$$\mathcal{S}_{L[a]L[a]}(a) = (\mathcal{S}_{L[a]} \circ \mathcal{S}_{L[a]})(a) = \mathcal{S}_{L[a]}(a) = U$$

and so,

$$a \in L[a]L[a] = (\{a\} \cup Sa)(\{a\} \cup Sa) = \{a^2\} \cup aSa \cup Sa^2 \cup SaSa \subseteq SaSa$$

Hence, S is left quasi-regular. The case when S is right quasi-regular can be similarly proved. □

Theorem 11.2. *Let S be a semigroup. If $f_S = (f_S \circ \widetilde{\mathbb{S}})^2 \tilde{\cap} (\widetilde{\mathbb{S}} \circ f_S)^2$ for every SI-ideal f_S of S, then S is quasi-regular.*

Proof. Let f_S be any *SI*-right ideal of S. Thus, we have

$$f_S = (f_S \circ \widetilde{\mathbb{S}})^2 \tilde{\cap} (\widetilde{\mathbb{S}} \circ f_S)^2 \tilde{\subseteq} (f_S \circ \widetilde{\mathbb{S}})^2 \tilde{\subseteq} f_S \circ f_S \tilde{\subseteq} f_S \circ \widetilde{\mathbb{S}} \tilde{\subseteq} f_S$$

and so $f_S = (f_S)^2$. It follows that S is right quasi-regular by Theorem 11.1. One can similarly show that S is left quasi-regular. □

Theorem 11.3. *For a semigroup S, the following conditions are equivalent:*

1) S is both intra-regular and left quasi-regular.

2) $g_S \tilde{\cap} h_S \tilde{\cap} f_S = g_S \circ h_S \circ f_S$ for every SI-bi-ideal f_S, for every SI-left ideal g_S and every SI-right ideal h_S of S.

Proof. Assume that (1) holds. Let f_S be any *SI*-bi-ideal, g_S be any *SI*-left ideal and h_S be any *SI*-right ideal of S. Let a be any element of S. Since S is intra-regular, there exist elements $x, y \in S$ such that $a = xa^2y$. Since S is left quasi-regular, there exist elements $u, v \in S$ such that $a = uava$. Hence

$$a = uava = u(xaay)va = ((ux)a)((a(yv)a)$$

Thus,

$$\begin{aligned}
(g_S \circ h_S \circ f_S)(a) &= [g_S \circ (h_S \circ f_S)](a) \\
&= \bigcup_{a=((ux)a)((a(yv)a)} [g_S((ux)a)) \cap (h_S \circ f_S)(a(yv)a))] \\
&\supseteq g_S((ux)a)) \cap (h_S \circ f_S)(a(yv)a)) \\
&\supseteq g_S(a) \cap (\bigcup_{(a(yv))a=mn} h_S(m) \cap f_S(n)) \\
&\supseteq g_S(a) \cap (h_S(a(yv)) \cap f_S(a)) \\
&\supseteq g_S(a) \cap h_S(a) \cap f_S(a) \\
&= (g_S \widetilde{\cap} h_S \widetilde{\cap} f_S)(a)
\end{aligned}$$

and so $g_S \circ h_S \circ f_S \supseteq g_S \widetilde{\cap} h_S \widetilde{\cap} f_S$. Thus, (1) implies (2). Assume that (2) holds. Let g_S be any SI-left ideal and f_S be any SI-right ideal of S. Then, since SI-left ideal g_S is a bi-ideal of S, and since $\widetilde{\mathbb{S}}$ itself is an SI-right ideal of S, we have

$$g_S = g_S \widetilde{\cap} \widetilde{\mathbb{S}} \widetilde{\cap} g_S = g_S \circ \widetilde{\mathbb{S}} \circ g_S = g_S \circ (\widetilde{\mathbb{S}} \circ g_S) \widetilde{\subseteq} g_S \circ g_S \widetilde{\subseteq} \widetilde{\mathbb{S}} \circ g_S \widetilde{\subseteq} g_S$$

Hence $g_S = g_S \circ g_S$. Thus, by Theorem 11.1, S is left quasi-regular.

Now, since SI-right ideal f_S is an SI-bi-ideal of S, and since $\widetilde{\mathbb{S}}$ itself is an SI-right ideal of S, we have:

$$g_S \widetilde{\cap} f_S = g_S \widetilde{\cap} \widetilde{\mathbb{S}} \widetilde{\cap} f_S = g_S \circ \widetilde{\mathbb{S}} \circ f_S = g_S \circ (\widetilde{\mathbb{S}} \circ f_S) \widetilde{\subseteq} g_S \circ f_S$$

Thus, by Theorem 8.2, S is intra-regular. Hence (2) implies (1). This completes the proof. □

12. Conclusion

Throughout this paper, soft intersection semigroup, soft intersection left (right, two-sided) ideals, soft intersection bi-ideals and soft semiprime ideals are studied and regular, intra-regular, completely regular, weakly regular and quasi-regular semigroups are characterized by the properties of these ideals. Based on these results, some further work can be done on the properties of other soft intersection ideals of semigroups, which may be useful to characterize the classical semigroups.

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