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Existence of Solution and Iterative Approximation of a System of Generalized Variational-like Inclusion Problems in Semi-inner Product Spaces

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Abstract. In this paper, we consider the system of generalized variational-like inclusion problems in semi-inner product spaces. We define a class of (H, φ)-η-monotone operators and its associated class of generalized resolvent operators. Further, using generalized resolvent operator technique, we give the existence of solution of the generalized variational-like inclusion problems. Furthermore, we suggest an iterative algorithm and give the convergence analysis of the sequences generated by the iterative algorithm. The results presented in this paper extend and unify the related known results in the literature.

1. Introduction

Variational inclusions, as the generalization of variational inequalities, have been widely studied by many authors in recent years. One of the most interesting and important problems in the theory of variational inclusions is the development of an efficient and implementable iterative algorithm. Various kinds of iterative algorithms have been suggested to find solutions for variational inclusions. Among these methods, the resolvent operator techniques for solving variational inclusions have been widely used by many authors. For further study in this direction, we refer to [3,5,6,8,12-16,18,21,23,25,28,33,34] and the related references therein.

In 2014, Sahu et al. [23] proved the existence of solutions for a class of nonlinear implicit variational inclusion problems in semi-inner product spaces, which is more general than the results studied in [24]. Moreover, they constructed an iterative algorithm for approximating the solution for the class of implicit variational inclusion problems involving A-monotone and H-monotone operators by using the generalized resolvent operator technique. It is remarked that they discussed the existence and convergence analysis by relaxing the condition of monotonicity on the set-valued map considered.

Very recently Luo and Huang [19], introduced and studied a class of (H, ϕ)-η-monotone operators in Banach spaces which provides a unifying framework for classes of maximal monotone operators, maximal η-monotone operators, m-η-accretive operators, H-monotone operators and (H, η)-monotone operators. Using proximal-point operator technique, they studied the convergence analysis of the iterative algorithms for

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some classes of variational inclusions. For further study of these types of operators and their applications in variational inequalities and variational inclusions, see for example [3,6-9,14-16,18,19,21,25-27,30-34]. Motivated and inspired by the research work mentioned above, in this paper, we define the generalized resolvent operator associated with (H,φ)-η-monotone operator in semi-inner product spaces. Using the properties of graph convergence of (H,φ)-η-monotone operator, we construct a new class of iterative algorithms to solve the system of generalized variational-like inclusions involving (H,φ)-η-monotone operator in semi-inner product spaces. Using the technique in this paper, one may generalize the results for symmetric generalized quasi-variational inclusion problems considered in [1,11]. The methods and results presented in this paper improve and generalize many known results in the literature.

2. Preliminaries and Basic Results

The following definitions and results are needed in the sequel.

Definition 2.1. *(Lumer [20], Sahu et al. [23]) Let X be a vector space over the field F of real or complex numbers. A functional $[\cdot,\cdot] : X \times X \to F$ is called a semi-inner product if it satisfies the following conditions:*

(i) $\left[x+y,z\right] = \left[x,z\right] + \left[y,z\right],\ \forall x,y,z \in X,$

(ii) $\left[\lambda x,y\right] = \lambda\left[x,y\right],\ \forall \lambda \in F,\ x,y \in X,$

(iii) $\left[x,x\right] > 0,\ \text{ for } x \neq 0,$

(iv) $\left|\left[x,y\right]\right|^2 \leq \left[x,x\right]\left[y,y\right].$

The pair $\left(X,[\cdot,\cdot]\right)$ is called a semi-inner product space. We note that $\|x\| = \left[x,x\right]^{\frac{1}{2}}$ is a norm on X, hence every semi-inner product space is a normed linear space. On the other hand, in a normed linear space, one can generate semi-inner product in infinitely many different ways. Giles [10] had proved that if the underlying space X is a uniformly convex smooth Banach space then it is possible to define a semi-inner product, uniquely. Also the unique semi-inner product has the following properties:

(i) $\left[x,y\right] = 0$ *iff y is orthogonal to x, that is iff $\|y\| \leq \|y+\lambda x\|$, for all scalars λ.*

(ii) Generalized Riesz representation theorem: If f is a continuous linear functional on X then there is a unique vector $y \in X$ such that $f(x) = [x,y]$, for all $x \in X$.

(iii) The semi-inner product is continuous, that is for each $x,y \in X$, we have $Re[y,x+\lambda y] \to Re[y,x]$ as $\lambda \to 0$.

The sequence space $l^p, p > 1$ and the function space $L^p, p > 1$ are uniformly convex smooth Banach spaces. So one can define semi-inner product on these spaces, uniquely.

Example 2.2. *(Sahu et al. [23]) The real sequence space l^p for $1 < p < \infty$ is a semi-inner product space with the semi-inner product defined by*

$$\left[x,y\right] = \frac{1}{\|y\|_p^{p-2}} \sum_i x_i y_i |y_i|^{p-2},\ x,y \in l^p.$$

Example 2.3. *(Giles [10], Sahu et al. [23]) The real Banach space $L^p(X,\mu)$ for $1 < p < \infty$ is a semi-inner product space with the semi-inner product defined by*

$$\left[f,g\right] = \frac{1}{\|g\|_p^{p-2}} \int_X f(x)|g(x)|^{p-1}\, sgn(g(x))d\mu,\ f,g \in L^p.$$

Definition 2.4. *(Sahu et al. [23], Xu [29]) Let X be a real Banach space. The modulus of smoothness of X is the function* $\rho_X : [0,+\infty) \longrightarrow [0,+\infty)$ *defined by*

$$\rho_X(t) = \sup\Big\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| = 1, \|y\| = t,\ t > 0\Big\}.$$

A Banach space X is said to be uniformly smooth, if

$$\lim_{t\to 0} \frac{\rho_X(t)}{t} = 0.$$

X is said to be q-uniformly smooth, if there exists a constant $c > 0$*, such that*

$$\rho_X(t) \le c\, t^q, q > 1.$$

X is said to be 2*-uniformly smooth, if there exists a constant* $c > 0$*, such that*

$$\rho_X(t) \le c\, t^2.$$

Lemma 2.5. *(Sahu et al. [23], Xu [29]) Let* $p > 1$ *be a real number and X be a smooth Banach space. Then the following statements are equivalent:*

(i) X is 2*-uniformly smooth.*

(ii) there is a constant $c > 0$*, such that for every* $x, y \in X$*, the following inequality holds*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, f_x\rangle + c\|y\|^2, \tag{2.1}$$

where $f_x \in J(x)$ *and* $J(x) = \Big\{x^\star \in X^\star : \langle x, x^\star\rangle = \|x\|^2, \|x\|^\star = \|x\|\Big\}$ *is the normalized duality mapping.*

Remark 2.6. *(Sahu et al. [23]) Every normed linear space is a semi-inner product space (see, [20]). Infact by Hahn Banach theorem, for each* $x \in X$*, there exists at least one functional* $f_x \in X^\star$ *such that* $\langle x, f_x\rangle = \|x\|^2$*. Given any such mapping f from X into* $X^\star$*, we can verify that* $\Big[y, x\Big] = \langle y, f_x\rangle$ *defines a semi-inner product. Hence we can write the inequality (2.1) as*

$$\|x+y\|^2 \le \|x\|^2 + 2\Big[y, x\Big] + c\|y\|^2,\ \forall x, y \in X, \tag{2.2}$$

where c is the constant of smoothness of X and is chosen with best possible minimum value.

Example 2.7. *(Sahu et al. [23]) The functions space* L^p *is* 2*-uniformly smooth for* $p \ge 2$ *and it is p-uniformly smooth for* $1 < p < 2$*. If* $2 \le p < \infty$*, then we have for all* $x, y \in L^p$*,*

$$\|x+y\|^2 \le \|x\|^2 + 2\Big[y, x\Big] + (p-1)\|y\|^2.$$

Here the constant of smoothness is $p - 1$*.*

Definition 2.8. *(Luo and Huang [19], Sahu et al. [23]) Let X be a real* 2*-uniformly smooth Banach space,* $\eta : X \times X \to X$ *and* $T : X \to X$ *be single-valued mappings. Then T is said to be*

(i) monotone, if

$$\Big[T(x) - T(y), x - y\Big] \ge 0,\ \ \forall x, y \in X,$$

(ii) strictly monotone, if

$$\Big[T(x) - T(y), x - y\Big] \ge 0,\ \ \forall x, y \in X,$$

and equality holds if and only if $x = y$*,* *(iii)* *γ-strongly monotone, if there exists a constant $\gamma > 0$ such that*

$$\Big[T(x) - T(y), x - y\Big] \geq \gamma \|x - y\|^2, \ \ \forall x, y \in X,$$

(iv) *δ-Lipschitz continuous, if there exists a constant $\delta > 0$ such that*

$$\|T(x) - T(y)\| \leq \delta \|x - y\|, \ \ \forall x, y \in X,$$

(v) *η-monotone, if*

$$\Big[T(x) - T(y), \eta(x, y)\Big] \geq 0, \ \ \forall x, y \in X,$$

(vi) *strictly-η-monotone, if*

$$\Big[T(x) - T(y), \eta(x, y)\Big] \geq 0, \ \ \forall x, y \in X,$$

and equality holds if and only if $x = y$,

(vii) *γ-strongly η-monotone, if there exists a constant $\gamma > 0$ such that*

$$\Big[T(x) - T(y), \eta(x, y)\Big] \geq \gamma \|x - y\|^2, \ \ \forall x, y \in X,$$

(viii) *λ-cocoercive if there exists a constant $\lambda > 0$ such that*

$$\Big[T(x) - T(y), x - y\Big] \geq \lambda \|T(x) - T(y)\|^2, \ \ \forall x, y \in X.$$

Let $M : X \to 2^X$ be a set-valued mapping. We denote its graph by graph(M), that is, graph(M) $= \Big\{(x, y) : y \in M(x)\Big\}$. The domain of M is defined by

$$Dom(M) = \Big\{x \in X : \exists y \in X : (x, y) \in M\Big\}.$$

The range of M is defined by

$$Range(M) = \Big\{y \in X : \exists x \in X : (x, y) \in M\Big\}.$$

The inverse M^{-1} of M is $\Big\{(y, x) : (x, y) \in M\Big\}$.

For any two set-valued mappings N and M, and any real number ρ, we define

$$N + M = \Big\{(x, y + z) : (x, y) \in N, \ (x, z) \in M\Big\},$$

$$\rho M = \Big\{(x, \rho y) : (x, y) \in M\Big\}.$$

For a mapping $A : X \to X$ and a set-valued map $M : X \to 2^X$, we define

$$A + M = \Big\{(x, y + z) : Ax = y \text{ and } (x, z) \in M\Big\}.$$

Definition 2.9. *(Luo and Huang [19], Sahu et al. [23]) Let X be a real 2-uniformly smooth Banach space. The mapping $M : X \to 2^X$ is said to be*

(i) *monotone if*

$$\Big[u - v, x - y\Big] \geq 0, \ \ \forall \, x, y \in X, \ u \in M(x), v \in M(y),$$

(ii) *γ-strongly monotone if there exists a constant $\gamma > 0$, such that*

$$\Big[u - v, x - y\Big] \geq \gamma \|x - y\|^2, \ \ \forall \, x, y \in X, \ u \in M(x), v \in M(y),$$

(iii) *η-monotone if*

$$\left[u - v, \eta(x, y)\right] \geq 0, \ \ \forall\, x, y \in X,\ u \in M(x), v \in M(y),$$

(iv) *γ-strongly η-monotone if there exists a constant $\gamma > 0$, such that*

$$\left[u - v, \eta(x, y)\right] \geq \gamma\|x - y\|^2, \ \ \forall\, x, y \in X,\ u \in M(x), v \in M(y).$$

Definition 2.10. *(Sahu et al. [23]) Let $H : X \to X$ be an r-strongly monotone operator. The mapping $M : X \to 2^X$ is said to be H-monotone if*

(i) *M is monotone;*

(ii) *$(H + \rho M)(X) = X$, where ρ is a positive real number.*

Definition 2.11. *(Sahu et al. [23]) The generalized resolvent operator $J^H_{M,\rho} : X \to X$ is defined by $J^H_{M,\rho}(u) = (H + \rho M)^{-1}(u)$ for all $u \in X$.*

(H, φ)-η-Monotone Operator:

Definition 2.12. *Let X be a real 2-uniformly smooth Banach space. Let $H : X \to X$, $\varphi : X \to X$, $\eta : X \times X \to X$ be single-valued mappings and $M : X \to 2^X$ be a multi-valued mapping. The mapping M is said to be (H, φ)-η-monotone if $\varphi \circ M$ is η-monotone and $(H + \varphi \circ M)(X) = X$.*

Definition 2.13. *(Luo and Huang [19]) Let X be a real 2-uniformly smooth Banach space. Let $H : X \to X$, $\varphi : X \to X$, $\eta : X \times X \to X$ be single-valued mappings and $M : X \to 2^X$ be a multi-valued mapping. The mapping M is said to be*

(i) *(H, φ)-monotone if $(\varphi \circ M)$ is monotone and $(H + \varphi \circ M)(X) = X$.*

(ii) *maximal φ-monotone if $(\varphi \circ M)$ is monotone and $(J + \varphi \circ M)(X) = X$,*

(iii) *maximal φ-η-monotone if $(\varphi \circ M)$ is η-monotone and $(J + \varphi \circ M)(X) = X$.*

where J is the normalized duality mapping.

Now, we define the generalized resolvent operator associated with (H, φ)-η- monotone operator.

Definition 2.14. *Let X be a real 2-uniformly smooth Banach space. Let $\varphi : X \to X$, $\eta : X \times X \to X$ be single-valued mappings, $H : X \to X$ be a γ-strongly η-monotone and δ-Lipschitz continuous mapping and $M : X \times X \to 2^X$ be a (H, φ)-η-monotone mapping. Then the generalized resolvent operator $J^{H,\eta}_{M(\cdot,x),\varphi} : X \to X$ associated with (H, φ)-η-monotone operator is defined by $J^{H,\eta}_{M(\cdot,x),\varphi}(u) = (H + \rho\varphi \circ M(\cdot, x))^{-1}(u)$, $\forall x, u \in X$.*

Graph convergence plays a crucial role in variational problems, optimization problems and approximation theory. For details on graph convergence, see Aubin and Frankowska [2], Rockafellar [22] and Sahu *et.al.*, [24].

Definition 2.15. *(Sahu et al. [23]) Let $H : X \to X$ be an r-strongly monotone and s-Lipschitz continuous operator. Let $\{M^n\}, M^n : X \to 2^X$ be a sequence of H-monotone set-valued mappings for $n = 0, 1, 2, ...$ Then the sequence $\{M^n\}$ is graph convergent to M, denoted by $M^n \xrightarrow{HG} M$, if for every $(x, y) \in$ graph (M), there exists a sequence $\{(x_n, y_n)\} \subset$ graph (M^n) such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.*

Definition 2.16. *(Sahu et al. [24]) Let X be a real 2-uniformly smooth Banach space. Let $H : X \to X$ be an s-Lipschitz continuous and r-strongly monotone. Let $\{M^n\}, M^n : X \to 2^X$ be a sequence of H-monotone set-valued mappings. Then the sequence $M^n \xrightarrow{HG} M$ if and only if $J^H_{M^n,\rho}(u) \to J^H_{M,\rho}(u)$ for all $u \in X$ and $\rho > 0$, where $J^H_{M,\rho} = (H + \rho M)^{-1}$.*

Now we prove the following important lemma:

Lemma 2.17. *Let X be a real 2-uniformly smooth Banach space. Let $\varphi : X \to X$ be a single-valued mapping, $\eta : X \times X \to X$ be a τ-Lipschitz continuous mapping, $H : X \to X$ be a γ-strongly η-monotone and δ-Lipschitz continuous mapping; and $\{M^n\}, M^n : X \times X \to 2^X$ be a sequence of (H, φ)-η-monotone mappings for $n = 0, 1, 2, ...$ Then the sequence $\varphi \circ M^n(\cdot, x_n) \xrightarrow{HG} \varphi \circ M(\cdot, x)$ if and only if $J^{H,\eta}_{M^n(\cdot,x_n),\varphi}(u) \to J^{H,\eta}_{M(\cdot,x),\varphi}(u)$ for all $x, u \in X$ and $\rho > 0$, where $J^{H,\eta}_{M(\cdot,x),\varphi} = (H + \rho\varphi \circ M(\cdot, x))^{-1}$.*

Proof. For any $u \in X$, let

$$J^{H,\eta}_{M(\cdot,x),\varphi}(u) = v \tag{2.3}$$

$$\Rightarrow (H + \rho\varphi \circ M(\cdot, x))^{-1}(u) = v$$

$$\Rightarrow u \in H(v) + \rho\varphi \circ M(v, x)$$

$$\Rightarrow \frac{1}{\rho}(u - H(v)) \in \varphi \circ M(v, x)$$

Hence $(v, \frac{1}{\rho}(u - H(v))) \in \text{graph}(\varphi \circ M(\cdot, x))$. Since $\varphi \circ M^n(\cdot, x_n) \xrightarrow{HG} \varphi \circ M(\cdot, x)$, there exists a sequence $(v_n, \frac{1}{\rho}(u - H(v_n))) \in \text{graph}(\varphi \circ M^n(\cdot, x_n))$ such that

$$v_n \to v,\ \frac{1}{\rho}(u - H(v_n)) \to \frac{1}{\rho}(u - H(v)). \tag{2.4}$$

Again we have $\frac{1}{\rho}(u - H(v_n)) \in \varphi \circ M^n(v_n, x_n)$. This implies that

$$u \in (H + \rho\varphi \circ M^n(\cdot, x_n))(v_n) \Rightarrow (H + \rho\varphi \circ M^n(\cdot, x_n))^{-1}(u) = v_n. \tag{2.5}$$

Therefore from (2.3)-(2.5), we have $J^{H,\eta}_{M^n(\cdot,x_n),\varphi}(u) \to J^{H,\eta}_{M(\cdot,x),\varphi}(u)$.
Conversely, let $J^{H,\eta}_{M^n(\cdot,x_n),\varphi}(u) \to J^{H,\eta}_{M(\cdot,x),\varphi}(u)$, $\forall u \in X$,
i.e., $\left(H + \rho\varphi \circ M^n(\cdot, x_n)\right)^{-1}(u) \to \left(H + \rho\varphi \circ M(\cdot, x)\right)^{-1}(u)$.
Suppose that $\left(H + \rho\varphi \circ M^n(\cdot, x_n)\right)^{-1}(u) = v_n$ and $\left(H + \rho\varphi \circ M(\cdot, x)\right)^{-1}(u) = v$. This implies that $\frac{1}{\rho}(u - H(v_n)) \in \varphi \circ M^n(v_n, x_n)$ and $\frac{1}{\rho}(u - H(v)) \in \varphi \circ M(v, x)$.
Since H is continuous, we have

$$v_n \to v \Rightarrow \frac{1}{\rho}(u - H(v_n)) \to \frac{1}{\rho}(u - H(v)).$$

Hence, for each $(v, \frac{1}{\rho}(u-H(v))) \in \text{graph}\,\varphi\circ M(\cdot, x)$, there exists a sequence $(v_n, \frac{1}{\rho}(u-H(v_n))) \in \text{graph}\,\varphi\circ M^n(\cdot, x_n)$ such that $v_n \longrightarrow v$ and $\frac{1}{\rho}(u - H(v_n)) \longrightarrow \frac{1}{\rho}(u - H(v))$. □

Lemma 2.18. *Let X be a real 2-uniformly smooth Banach space. Let $\varphi : X \to X$ be a single-valued mapping, $\eta : X \times X \to X$ be a τ-Lipschitz continuous mapping, $H : X \to X$ be a γ-strongly η-monotone and δ-Lipschitz continuous mapping; and $M : X \times X \to 2^X$ be a (H, φ)-η-monotone mapping. Then, the resolvent operator $J^{H,\eta}_{M(\cdot,x),\varphi} : X \longrightarrow X$ is $\frac{\tau}{\gamma}$-Lipschitz continuous, i.e.,*

$$\|J^{H,\eta}_{M(\cdot,x),\varphi}(x^\star) - J^{H,\eta}_{M(\cdot,x),\varphi}(y^\star)\| \leq \frac{\tau}{\gamma}\|x^\star - y^\star\|,\ \ \forall x^\star, y^\star \in X.$$

Proof. Let $x^\star, y^\star \in X$. It follows that

$$\begin{cases} J^{H,\eta}_{M(\cdot,x),\varphi}(x^\star) = \Big(H + \rho\varphi \circ M(\cdot, x)\Big)^{-1}(x^\star), \\ J^{H,\eta}_{M(\cdot,x),\varphi}(y^\star) = \Big(H + \rho\varphi \circ M(\cdot, x)\Big)^{-1}(y^\star) \end{cases}$$

and hence

$$\begin{cases} \dfrac{1}{\rho}\Big\{x^\star - H\Big(J^{H,\eta}_{M(\cdot,x),\varphi}(x^\star)\Big)\Big\} \in \Big(\varphi \circ M(\cdot, x)\Big)\Big(J^{H,\eta}_{M(\cdot,x),\varphi}(x^\star)\Big), \\ \\ \dfrac{1}{\rho}\Big\{y^\star - H\Big(J^{H,\eta}_{M(\cdot,x),\varphi}(y^\star)\Big)\Big\} \in \Big(\varphi \circ M(\cdot, x)\Big)\Big(J^{H,\eta}_{M(\cdot,x),\varphi}(y^\star)\Big). \end{cases}$$

Since $(\varphi \circ M)$ is η-monotone, we have

$$\frac{1}{\rho}\Big[x^\star - H\Big(J^{H,\eta}_{M(\cdot,x),\varphi}(x^\star)\Big) - \Big(y^\star - H\Big(J^{H,\eta}_{M(\cdot,x),\varphi}(y^\star)\Big)\Big), \eta\Big(J^{H,\eta}_{M(\cdot,x),\varphi}(x^\star), J^{H,\eta}_{M(\cdot,x),\varphi}(y^\star)\Big)\Big] \geq 0.$$

It follows that

$$\begin{aligned} \tau\|x^\star - y^\star\| &\Big\|J^{H,\eta}_{M(\cdot,x),\varphi}(x^\star) - J^{H,\eta}_{M(\cdot,x),\varphi}(y^\star)\Big\| \\ &\geq \|x^\star - y^\star\|\Big\|\eta\Big(J^{H,\eta}_{M(\cdot,x),\varphi}(x^\star), J^{H,\eta}_{M(\cdot,x),\varphi}(y^\star)\Big)\Big\| \\ &\geq \Big[x^\star - y^\star, \eta\Big(J^{H,\eta}_{M(\cdot,x),\varphi}(x^\star), J^{H,\eta}_{M(\cdot,x),\varphi}(y^\star)\Big)\Big] \\ &\geq \Big[H\Big(J^{H,\eta}_{M(\cdot,x),\varphi}(x^\star)\Big) - H\Big(J^{H,\eta}_{M(\cdot,x),\varphi}(y^\star)\Big), \eta\Big(J^{H,\eta}_{M(\cdot,x),\varphi}(x^\star), J^{H,\eta}_{M(\cdot,x),\varphi}(y^\star)\Big)\Big] \\ &\geq \gamma\Big\|J^{H,\eta}_{M(\cdot,x),\varphi}(x^\star) - J^{H,\eta}_{M(\cdot,x),\varphi}(y^\star)\Big\|^2. \end{aligned}$$

Thus, $\Big\|J^{H,\eta}_{M(\cdot,x),\varphi}(x^\star) - J^{H,\eta}_{M(\cdot,x),\varphi}(y^\star)\Big\| \leq \dfrac{\tau}{\gamma}\|x^\star - y^\star\|, \ \ \forall x^\star, y^\star \in X.$ □

Lemma 2.19. *(Liu [17]) Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of non-negative real numbers that satisfy: there exists a positive integer n_0 such that $n \geq n_0$,*

$$a_{n+1} \leq (1 - t_n)a_n + b_n t_n + c_n,$$

where $t_n \in [0, 1]$, $\sum_{n=0}^{\infty} t_n = +\infty$, $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\sum_{n=0}^{\infty} a_n = 0$.

3. Formulation of the Problem and Existence of Solution

Let X be a real 2-uniformly smooth Banach space. For $i \in \{1, 2\}$, let $N_i : X \to X$, $F_i, \eta_i : X \times X \to X$, $g_i, H_i : X \to X$ be single-valued mappings, $M_i : X \times X \to 2^X$ be (H_i, φ_i)-η_i-monotone mappings, respectively. Then the system of generalized variational-like inclusions (in short, SGVLI) is: Find $(x, y) \in X \times X$ such that

$$\begin{cases} 0 \in N_1(g_1(x) - F_1(x, y)) + M_1(g_1(x), x), & (3.1) \\ \\ 0 \in N_2(g_2(y) - F_2(x, y)) + M_2(g_2(y), y). & (3.2) \end{cases}$$

Some Special Cases:

Case I: If $N_1 = N_2 \equiv I$, the Identity mapping, then SGVLI (3.1)-(3.2) reduces to the following system of variational inclusions: Find $(x, y) \in X \times X$ such that

$$\begin{cases} 0 \in g_1(x) - F_1(x, y) + M_1(g_1(x), x), & (3.3) \\ 0 \in g_2(y) - F_2(x, y) + M_2(g_2(y), y), & (3.4) \end{cases}$$

which is an important generalization of the problems considered by Chang et al. [4], Sahu et al. [23] and Tang and Wang [25].

Case II: If $g_1 = g_2 \equiv I$, the Identity mapping, then SGVLI (3.3)-(3.4) reduces to the following system of variational inclusions: Find $(x, y) \in X \times X$ such that

$$\begin{cases} 0 \in x - F_1(x, y) + M_1(x, x), \\ 0 \in y - F_2(x, y) + M_2(y, y), \end{cases}$$

which is an important generalization of the problems considered by Luo and Huang [19].

Case III: If $N_1 = N_2 \equiv I$, the Identity mapping, $g_1 = g_2 = g$, $F_1(x, y) = g(y)$, $F_2(x, y) = g(x)$, $M_1(g_1(x), x) = M(g(x))$, $M_2(g_2(y), y) = M(g(y))$, where $g : X \to X$ and $M : X \to 2^X$, then SGVLI (3.1)-(3.2) reduces to the following system of variational inclusions: Find $(x, y) \in X \times X$ such that

$$\begin{cases} 0 \in g(x) - g(y) + M(g(x)), \\ 0 \in g(y) - g(x) + M(g(y)), \end{cases}$$

which is an important generalization of the problem considered in [13-16,31].

Case IV: If $N_2 \equiv 0$, $N_1(g_1(x) - F_1(x, y)) = S(u) - T(u)$ and $M_1(g_1(x), x) = M(g(u))$, $\forall u \in X$ where $M : X \to 2^X$ is a set-valued mapping, $S, T, g : X \to X$ are single-valued mappings, then SGVLI (3.1)-(3.2) reduces to the following problem: Find an element $u \in X$ such that

$$0 \in S(u) - T(u) + M(g(u)),$$

which is the generalization of variational inclusion problem considered by Sahu et al. [24].

Now we prove the following technical lemma.

Lemma 3.1. *Let X be a real 2-uniformly smooth Banach space. For $i \in \{1, 2\}$, suppose $N_i : X \to X$, $F_i, \eta_i : X \times X \to X$, $g_i : X \to X$ and $H_i : X \to X$ be single-valued mappings. Let $\varphi_i : X \to X$ be single-valued mappings satisfying $\varphi_i(u + v) = \varphi_i(u) + \varphi_i(v), \forall u, v \in X$ and $Ker(\varphi_i) = \{0\}$ (i.e., $Ker(\varphi_i) = \{u \in X : \varphi_i(u) = 0\}$), $M_i : X \times X \to 2^X$ be (H_i, φ_i)-η_i-monotone mappings, respectively. Then $(x, y) \in X \times X$ is the solution of SGVLI (3.1)-(3.2) if and only if it satisfies:*

$$g_1(x) = J^{H_1,\eta_1}_{M_1(\cdot,x),\varphi_1}\Big\{H_1(g_1(x)) - \rho_1(\varphi_1 \circ N_1)\Big(g_1(x) - F_1(x, y)\Big)\Big\}; \ \ \rho_1 > 0 \tag{3.5}$$

and

$$g_2(y) = J^{H_2,\eta_2}_{M_2(\cdot,y),\varphi_2}\Big\{H_2(g_2(y)) - \rho_2(\varphi_2 \circ N_2)\Big(g_2(y) - F_2(x, y)\Big)\Big\}; \ \ \rho_2 > 0, \tag{3.6}$$

where $J^{H_1,\eta_1}_{M_1(\cdot,x),\varphi_1} := (H_1 + \rho_1\varphi_1 \circ M_1(\cdot, x))^{-1}$; $J^{H_2,\eta_2}_{M_2(\cdot,y),\varphi_2} := (H_2 + \rho_2\varphi_2 \circ M_2(\cdot, y))^{-1}$ are the generalized resolvent operators.

Proof. From the definition of $J^{H_1,\eta_1}_{M_1(\cdot,x),\varphi_1}$, we have

$$\begin{aligned} & H_1(g_1(x)) - \rho_1(\varphi_1 \circ N_1)\Big(g_1(x) - F_1(x, y)\Big) \in \Big\{H_1 + \rho_1\varphi_1 \circ M_1(\cdot, x)\Big\}g_1(x) \\ \Longleftrightarrow \quad & H_1(g_1(x)) - \rho_1(\varphi_1 \circ N_1)\Big(g_1(x) - F_1(x, y)\Big) \in H_1(g_1(x)) + \rho_1\varphi_1 \circ M_1(g_1(x), x\Big) \\ \Longleftrightarrow \quad & 0 \in \rho_1(\varphi_1 \circ N_1)\Big(g_1(x) - F_1(x, y)\Big) + \rho_1\varphi_1 \circ M_1(g_1(x), x) \\ \Longleftrightarrow \quad & 0 \in \varphi_1 \circ \Big\{N_1\Big(g_1(x) - F_1(x, y)\Big) + M_1(g_1(x), x)\Big\} \\ \Longleftrightarrow \quad & 0 \in N_1\Big(g_1(x) - F_1(x, y)\Big) + M_1(g_1(x), x), \end{aligned}$$

since $\varphi_i(u + v) = \varphi_i(u) + \varphi_i(v)$ and $Ker(\varphi_i) = \{0\}$.
Similarly, $0 \in N_2\Big(g_2(y) - F_2(x, y)\Big) + M_2(g_2(y), y)$, $\forall (x, y) \in X \times X$. Thus $(x, y) \in X \times X$ is the solution of SGVLI (3.1)-(3.2). □

Theorem 3.2. *Let X be a real 2-uniformly smooth Banach space. For $i \in \{1,2\}$, let $N_i : X \to X$ be r_i-Lipschitz continuous, $g_i : X \to X$ be β_i-Lipschitz continuous and q_i-strongly monotone and $\eta_i : X \times X \to X$ be τ_i-Lipschitz continuous, $H_i : X \to X$ be a δ_i-Lipschitz continuous and γ_i-strongly-η_i-monotone mappings, respectively. Suppose $\varphi_i : X \to X$ be single-valued mappings satisfying $\varphi_i(u+v) = \varphi_i(u) + \varphi_i(v)$ and $Ker(\varphi_i) = \{0\}$, and let φ_i be θ_i-Lipschitz continuous, $M_i : X \times X \to 2^X$ be (H_i, φ_i)-η_i-monotone mappings, respectively such that*

$$\|J^{H_1,\eta_1}_{M_1(\cdot,x_1),\varphi_1}(x) - J^{H_1,\eta_1}_{M_1(\cdot,x_2),\varphi_1}(x)\| \leq s_1\|x_1 - x_2\|, \quad \forall x_1, x_2 \in X,\ s_1 > 0, \tag{3.7}$$

$$\|J^{H_2,\eta_2}_{M_2(\cdot,y_1),\varphi_2}(x) - J^{H_2,\eta_2}_{M_2(\cdot,y_2),\varphi_2}(x)\| \leq s_2\|y_1 - y_2\|, \quad \forall y_1, y_2 \in X,\ s_2 > 0. \tag{3.8}$$

Suppose that $F_1 : X\times X \to X$ be a ζ_1-Lipschitz continuous in the first argument and $F_2 : X\times X \to X$ be a ζ_2-Lipschitz continuous in the second argument. In addition, if

$$(1 - 2q_1 + c\beta_1^2) > 0,\ \ (1 - 2q_2 + c\beta_2^2) > 0, \tag{3.9}$$

and

$$\left.\begin{aligned} 0 &< \sqrt{1 - 2q_1 + c\beta_1^2} + \frac{\tau_1}{\gamma_1}\Big\{\rho_1\theta_1 r_1(\beta_1 + \zeta_1) + \delta_1\beta_1\Big\} + s_1 < 1, \\ 0 &< \sqrt{1 - 2q_2 + c\beta_2^2} + \frac{\tau_2}{\gamma_2}\Big\{\rho_2\theta_2 r_2(\beta_2 + \zeta_2) + \delta_2\beta_2\Big\} + s_2 < 1, \end{aligned}\right\} \tag{3.10}$$

where c is constant of smoothness of Banach space X, then SGVLI (3.1)-(3.2) has a solution.

Proof. Define the mappings $S_1, S_2 : X \to X$ by

$$S_1(x_1) = x_1 - g_1(x_1) + J^{H_1,\eta_1}_{M_1(\cdot,x_1),\varphi_1}\Big\{H_1(g_1(x_1)) - \rho_1(\varphi_1 \circ N_1)\Big(g_1(x_1) - F_1(x_1, y)\Big)\Big\}$$

$$S_2(y_1) = y_1 - g_2(y_1) + J^{H_2,\eta_2}_{M_2(\cdot,y_1),\varphi_2}\Big\{H_2(g_2(y_1)) - \rho_2(\varphi_2 \circ N_2)\Big(g_2(y_1) - F_2(x, y_1)\Big)\Big\}.$$

Then for the elements $x_1, x_2 \in X$, we have

$$\begin{aligned} \|S_1(x_1) - S_1(x_2)\| \leq\ & \Big\|x_1 - x_2 - \Big(g_1(x_1) - g_1(x_2)\Big)\Big\| \\ & + \Big\|J^{H_1,\eta_1}_{M_1(\cdot,x_1),\varphi_1}\Big\{H_1(g_1(x_1)) - \rho_1(\varphi_1 \circ N_1)\Big(g_1(x_1) - F_1(x_1, y)\Big)\Big\} \\ & - J^{H_1,\eta_1}_{M_1(\cdot,x_2),\varphi_1}\Big\{H_1(g_1(x_2)) - \rho_1(\varphi_1 \circ N_1)\Big(g_1(x_2) - F_1(x_2, y)\Big)\Big\}\Big\|. \end{aligned} \tag{3.11}$$

Using (3.7) and Lemma 2.18, we have the following estimate

$$\begin{aligned} & \Big\|J^{H_1,\eta_1}_{M_1(\cdot,x_1),\varphi_1}\Big\{H_1(g_1(x_1)) - \rho_1(\varphi_1 \circ N_1)\Big(g_1(x_1) - F_1(x_1, y)\Big)\Big\} \\ & \quad - J^{H_1,\eta_1}_{M_1(\cdot,x_2),\varphi_1}\Big\{H_1(g_1(x_2)) - \rho_1(\varphi_1 \circ N_1)\Big(g_1(x_2) - F_1(x_2, y)\Big)\Big\}\Big\| \\ \leq\ & \Big\|J^{H_1,\eta_1}_{M_1(\cdot,x_1),\varphi_1}\Big\{H_1(g_1(x_1)) - \rho_1(\varphi_1 \circ N_1)\Big(g_1(x_1) - F_1(x_1, y)\Big)\Big\} \\ & - J^{H_1,\eta_1}_{M_1(\cdot,x_2),\varphi_1}\Big\{H_1(g_1(x_1)) - \rho_1(\varphi_1 \circ N_1)\Big(g_1(x_1) - F_1(x_1, y)\Big)\Big\}\Big\| \end{aligned}$$

$$
\begin{aligned}
&+\Big\|J^{H_1,\eta_1}_{M_1(\cdot,x_2),\varphi_1}\Big\{H_1(g_1(x_1))-\rho_1(\varphi_1\circ N_1)\Big(g_1(x_1)-F_1(x_1,y)\Big)\Big\}\\
&-J^{H_1,\eta_1}_{M_1(\cdot,x_2),\varphi_1}\Big\{H_1(g_1(x_2))-\rho_1(\varphi_1\circ N_1)\Big(g_1(x_2)-F_1(x_2,y)\Big)\Big\}\Big\|\\
\leq\ & s_1\|x_1-x_2\|+\frac{\tau_1}{\gamma_1}\Big\|H_1(g_1(x_1))-\rho_1(\varphi_1\circ N_1)\Big(g_1(x_1)-F_1(x_1,y)\Big)\\
&-\Big\{H_1(g_1(x_2))-\rho_1(\varphi_1\circ N_1)\Big(g_1(x_2)-F_1(x_2,y)\Big)\Big\}\Big\|\\
\leq\ & s_1\|x_1-x_2\|+\frac{\tau_1}{\gamma_1}\Big\|H_1(g_1(x_1))-H_1(g_1(x_2))\Big\|\\
&+\frac{\tau_1\rho_1}{\gamma_1}\Big\|(\varphi_1\circ N_1)\Big(g_1(x_1)-F_1(x_1,y)\Big)-(\varphi_1\circ N_1)\Big(g_1(x_2)-F_1(x_2,y)\Big)\Big\|. \qquad (3.12)
\end{aligned}
$$

Since H_1 is a δ_1-Lipschitz continuous, g_1 is a β_1-Lipschitz continuous, φ_1 is a θ_1-Lipschitz continuous, N_1 is an r_1-Lipschitz continuous and F_1 is a ζ_1-Lipschitz continuous in the first argument,we have

$$
\|H_1(g_1(x_1))-H_1(g_1(x_2))\Big\|\leq\delta_1\beta_1\|x_1-x_2\|, \qquad (3.13)
$$

and

$$
\begin{aligned}
&\Big\|(\varphi_1\circ N_1)\Big(g_1(x_1)-F_1(x_1,y)\Big)-(\varphi_1\circ N_1)\Big(g_1(x_2)-F_1(x_2,y)\Big)\Big\|\\
\leq\ & \theta_1r_1\Big\|g_1(x_1)-g_1(x_2)-\Big(F_1(x_1,y)-F_1(x_2,y)\Big)\Big\|\\
\leq\ & \theta_1r_1\Big\|g_1(x_1)-g_1(x_2)\Big\|+\theta_1r_1\Big\|F_1(x_1,y)-F_1(x_2,y)\Big\|\\
\leq\ & \theta_1r_1\beta_1\Big\|x_1-x_2\Big\|+\theta_1r_1\zeta_1\Big\|x_1-x_2\Big\|. \qquad (3.14)
\end{aligned}
$$

Using (3.13) and (3.14) in (3.12), we have

$$
\begin{aligned}
&\Big\|J^{H_1,\eta_1}_{M_1(\cdot,x_1),\varphi_1}\Big\{H_1(g_1(x_1))-\rho_1(\varphi_1\circ N_1)\Big(g_1(x_1)-F_1(x_1,y)\Big)\Big\}\\
&-J^{H_1,\eta_1}_{M_1(\cdot,x_2),\varphi_1}\Big\{H_1(g_1(x_2))-\rho_1(\varphi_1\circ N_1)\Big(g_1(x_2)-F_1(x_2,y)\Big)\Big\}\Big\|\\
\leq\ & s_1\Big\|x_1-x_2\Big\|+\frac{\tau_1\delta_1\beta_1}{\gamma_1}\Big\|x_1-x_2\Big\|+\frac{\tau_1\rho_1\theta_1r_1\beta_1}{\gamma_1}\Big\|x_1-x_2\Big\|+\frac{\tau_1\rho_1\theta_1r_1\zeta_1}{\gamma_1}\Big\|x_1-x_2\Big\|\\
=\ & \left\{s_1+\frac{\tau_1}{\gamma_1}\Big(\rho_1\theta_1r_1(\beta_1+\zeta_1)+\delta_1\beta_1\Big)\right\}\Big\|x_1-x_2\Big\|.
\end{aligned}
$$

Since X is a 2-uniformly smooth Banach space, we have

$$
\begin{aligned}
\Big\|x_1-x_2-\Big(g_1(x_1)-g_1(x_2)\Big)\Big\|^2 &\leq \|x_1-x_2\|^2-2\Big[g_1(x_1)-g_1(x_2),x_1-x_2\Big]\\
&\quad +c\|g_1(x_1)-g_1(x_2)\|^2\\
&\leq \|x_1-x_2\|^2-2q_1\|x_1-x_2\|^2+c\beta_1^2\|x_1-x_2\|^2.
\end{aligned}
$$

Thus,

$$
\Big\|x_1-x_2-\Big(g_1(x_1)-g_1(x_2)\Big)\Big\|\leq\sqrt{1-2q_1+c\beta_1^2}\,\|x_1-x_2\|.
$$

Therefore,

$$\begin{aligned}\Big\|S_1(x_1) - S_1(x_2)\Big\| &\leq \left\{\sqrt{1-2q_1+c\beta_1^2} + \frac{\tau_1}{\gamma_1}\Big(\rho_1\theta_1 r_1(\beta_1+\zeta_1) + \delta_1\beta_1\Big) + s_1\right\}\|x_1 - x_2\| \\ &\leq \Big\|x_1 - x_2\Big\|,\end{aligned}$$

since by (3.10), $0 < \sqrt{1-2q_1+c\beta_1^2} + \dfrac{\tau_1}{\gamma_1}\Bigg(\rho_1\theta_1 r_1(\beta_1+\zeta_1) + \delta_1\beta_1\Bigg) + s_1 < 1$.

Thus it follows S_1 is a contraction mapping. Hence by Banach contraction principle, S_1 admits a fixed point (say) $x \in X$. Thus, we have

$$g_1(x) = J^{H_1,\eta_1}_{M_1(\cdot,x),\varphi_1}\Big\{H_1(g_1(x)) - \rho_1(\varphi_1 \circ N_1)\Big(g_1(x) - F_1(x,y)\Big)\Big\};\ \ \rho_1 > 0.$$

Similarly, for the elements $y_1, y_2 \in X$, we have

$$\begin{aligned}\Big\|S_2(y_1) - S_2(y_2)\Big\| &\leq \Big\|y_1 - y_2 - \Big(g_2(y_1) - g_2(y_2)\Big)\Big\| \\ &+\Big\|J^{H_2,\eta_2}_{M_2(\cdot,y_1),\varphi_2}\Big\{H_2(g_2(y_1)) - \rho_2(\varphi_2 \circ N_2)\Big(g_2(y_1) - F_2(x,y_1)\Big)\Big\} \\ &-J^{H_2,\eta_2}_{M_2(\cdot,y_2),\varphi_2}\Big\{H_2(g_2(y_2)) - \rho_2(\varphi_2 \circ N_2)\Big(g_2(y_2) - F_2(x,y_2)\Big)\Big\}\Big\|. \qquad (3.15)\end{aligned}$$

Again using (3.8) and Lemma 2.18, we have the following estimate

$$\begin{aligned}&\Big\|J^{H_2,\eta_2}_{M_2(\cdot,y_1),\varphi_2}\Big\{H_2(g_2(y_1)) - \rho_2(\varphi_2 \circ N_2)\Big(g_2(y_1) - F_2(x,y_1)\Big)\Big\} \\ &\quad -J^{H_2,\eta_2}_{M_2(\cdot,y_2),\varphi_2}\Big\{H_2(g_2(y_2)) - \rho_2(\varphi_2 \circ N_2)\Big(g_2(y_2) - F_2(x,y_2)\Big)\Big\}\Big\| \\ &\leq \Big\|J^{H_2,\eta_2}_{M_2(\cdot,y_1),\varphi_2}\Big\{H_2(g_2(y_1)) - \rho_2(\varphi_2 \circ N_2)\Big(g_2(y_1) - F_2(x,y_1)\Big)\Big\} \\ &\quad -J^{H_2,\eta_2}_{M_2(\cdot,y_2),\varphi_2}\Big\{H_2(g_2(y_1)) - \rho_2(\varphi_2 \circ N_2)\Big(g_2(y_1) - F_2(x,y_1)\Big)\Big\}\Big\| \\ &\quad +\Big\|J^{H_2,\eta_2}_{M_2(\cdot,y_2),\varphi_2}\Big\{H_2(g_2(y_1)) - \rho_2(\varphi_2 \circ N_2)\Big(g_2(y_1) - F_2(x,y_1)\Big)\Big\} \\ &\quad -J^{H_2,\eta_2}_{M_2(\cdot,y_2),\varphi_2}\Big\{H_2(g_2(y_2)) - \rho_2(\varphi_2 \circ N_2)\Big(g_2(y_2) - F_2(x,y_2)\Big)\Big\}\Big\| \\ &\leq s_2\|y_1 - y_2\| + \frac{\tau_2}{\gamma_2}\Big\|H_2(g_2(y_1)) - \rho_2(\varphi_2 \circ N_2)\Big(g_2(y_1) - F_2(x,y_1)\Big) \\ &\quad -\Big\{H_2(g_2(y_2)) - \rho_2(\varphi_2 \circ N_2)\Big(g_2(y_2) - F_2(x,y_2)\Big)\Big\}\Big\| \\ &\leq s_2\|y_1 - y_2\| + \frac{\tau_2}{\gamma_2}\Big\|H_2(g_2(y_1)) - H_2(g_2(y_2))\Big\| \\ &\quad +\frac{\tau_2\rho_2}{\gamma_2}\Big\|(\varphi_2 \circ N_2)\Big(g_2(y_1) - F_2(x,y_1)\Big) - (\varphi_2 \circ N_2)\Big(g_2(y_2) - F_2(x,y_2)\Big)\Big\|. \qquad (3.16)\end{aligned}$$

Since H_2 is a δ_2-Lipschitz continuous, g_2 is a β_2-Lipschitz continuous, φ_2 is a θ_2-Lipschitz continuous, N_2 is an r_2-Lipschitz continuous and F_2 is a ζ_2-Lipschitz continuous in the second argument, we have

$$\Big\| H_2(g_2(y_1)) - H_2(g_2(y_2)) \Big\| \leq \delta_2\beta_2\|y_1 - y_2\|. \tag{3.17}$$

and

$$\begin{aligned} \Big\|(\varphi_2 \circ N_2)\big(g_2(y_1) - F_2(x, y_1)\big) &- (\varphi_2 \circ N_2)\big(g_2(y_2) - F_2(x, y_2)\big)\Big\| \\ &\leq \theta_2 r_2 \Big\| g_2(y_1) - g_2(y_2) - \big(F_2(x, y_1) - F_2(x, y_2)\big)\Big\| \\ &\leq \theta_2 r_2 \Big\| g_2(y_1) - g_2(y_2)\Big\| + \theta_2 r_2 \Big\| F_2(x, y_1) - F_2(x, y_2)\Big\| \\ &\leq \theta_2 r_2 \beta_2 \Big\| y_1 - y_2 \Big\| + \theta_2 r_2 \zeta_2 \Big\| y_1 - y_2 \Big\|. \end{aligned} \tag{3.18}$$

Using (3.17) and (3.18) in (3.16), we have

$$\begin{aligned} \Big\| J^{H_2,\eta_2}_{M_2(\cdot,y_1),\varphi_2}&\big\{H_2(g_2(y_1)) - \rho_2(\varphi_2 \circ N_2)\big(g_2(y_1) - F_2(x, y_1)\big)\big\} \\ &- J^{H_2,\eta_2}_{M_2(\cdot,y_2),\varphi_2}\big\{H_2(g_2(y_2)) - \rho_2(\varphi_2 \circ N_2)\big(g_2(y_2) - F_2(x, y_2)\big)\big\}\Big\| \\ &\leq \left\{ s_2 + \frac{\tau_2}{\gamma_2}\big(\rho_2\theta_2 r_2(\beta_2 + \zeta_2) + \delta_2\beta_2\big)\right\}\Big\| y_1 - y_2 \Big\|. \end{aligned}$$

Since X is a 2-uniformly smooth Banach space, we have

$$\begin{aligned} \Big\| y_1 - y_2 - \big(g_2(y_1) - g_2(y_2)\big)\Big\|^2 &\leq \|y_1 - y_2\|^2 - 2\big[g_2(y_1) - g_2(y_2), y_1 - y_2\big] \\ &\quad + c\|g_2(y_1) - g_2(y_2)\|^2 \\ &\leq \|y_1 - y_2\|^2 - 2q_2\|y_1 - y_2\|^2 + c\beta_2^2\|y_1 - y_2\|^2. \end{aligned}$$

Thus,

$$\Big\| y_1 - y_2 - \big(g_2(y_1) - g_2(y_2)\big)\Big\| \leq \sqrt{1 - 2q_2 + c\beta_2^2}\ \|y_1 - y_2\|.$$

Therefore,

$$\begin{aligned} \Big\| S_2(y_1) - S_2(y_2)\Big\| &\leq \left\{ \sqrt{1 - 2q_2 + c\beta_2^2} + \frac{\tau_2}{\gamma_2}\big(\rho_2\theta_2 r_2(\beta_2 + \zeta_2) + \delta_2\beta_2\big) + s_2\right\}\|y_1 - y_2\| \\ &\leq \Big\| y_1 - y_2 \Big\|, \end{aligned}$$

since by (3.10), $0 < \sqrt{1 - 2q_2 + c\beta_2^2} + \frac{\tau_2}{\gamma_2}\Big(\rho_2\theta_2 r_2(\beta_2 + \zeta_2) + \delta_2\beta_2\Big) + s_2 < 1$.

Thus it follows that the mapping $S_2 : X \to X$ is a contraction mapping. Hence, by Banach contraction principle, S_2 has a fixed point (say) $y \in X$. Thus, we have $g_2(y) = J^{H_2,\eta_2}_{M_2(\cdot,y),\varphi_2}\big\{H_2(g_2(y)) - \rho_2(\varphi_2 \circ N_2)\big(g_2(y) - F_2(x, y)\big)\big\};\ \rho_2 > 0$. □

When $X = L^p(\boldsymbol{R}),\ 2 \leq p < \infty$, we have the following corollary:

Corollary 3.3. *For $i \in \{1,2\}$, let $N_i : L^p \to L^p$ be r_i-Lipschitz continuous, $g_i : L^p \to L^p$ be β_i-Lipschitz continuous and q_i-strongly monotone and $\eta_i : L^p \times L^p \to L^p$ be τ_i-Lipschitz continuous, $H_i : L^p \to L^p$ be a δ_i-Lipschitz continuous and γ_i-strongly-η_i-monotone mappings, respectively. Suppose $\varphi_i : L^p \to L^p$ be single-valued mappings satisfying $\varphi_i(u+v) = \varphi_i(u) + \varphi_i(v)$ and $Ker(\varphi_i) = \{0\}$, and let φ_i be θ_i-Lipschitz continuous, $M_i : L^p \times L^p \to 2^{L^p}$ be (H_i, φ_i)-η_i-monotone mappings, respectively, such that (3.7)-(3.8) holds. Furthermore, let $F_1 : L^p \times L^p \to L^p$ be a ζ_1-Lipschitz continuous in the first argument and $F_2 : L^p \times L^p \to L^p$ be a ζ_2-Lipschitz continuous in the second argument. In addition, if*

$$(1 - 2q_1 + (p-1)\beta_1^2) > 0, \;\; (1 - 2q_2 + (p-1)\beta_2^2) > 0,$$

and

$$\left.\begin{aligned} 0 < \sqrt{1 - 2q_1 + (p-1)\beta_1^2} + \frac{\tau_1}{\gamma_1}\Big\{\rho_1\theta_1 r_1(\beta_1 + \zeta_1) + \delta_1\beta_1\Big\} + s_1 < 1, \\ 0 < \sqrt{1 - 2q_2 + (p-1)\beta_2^2} + \frac{\tau_2}{\gamma_2}\Big\{\rho_2\theta_2 r_2(\beta_2 + \zeta_2) + \delta_2\beta_2\Big\} + s_2 < 1, \end{aligned}\right\}$$

where $(p-1)$ is constant of smoothness, then SGVLI (3.1)-(3.2) has a solution.

4. Iterative Algorithm and Convergence Criteria

Lemma 3.1 is important from the numerical point of view. It allows us to suggest the following iterative algorithm for finding the approximate solution of SGVLI (3.1)-(3.2):

Iterative Algorithm 4.1. *For arbitrary point $(x_0, y_0) \in X \times X$, compute the sequences $\{x_n\}$, $\{y_n\}$ by the iterative scheme:*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n\Big\{x_n - g_1(x_n) \\ &\quad + J^{H_1,\eta_1}_{M_1^n(\cdot,x_n),\varphi_1}\Big\{H_1(g_1(x_n)) - \rho_1(\varphi_1 \circ N_1)\Big(g_1(x_n) - F_1(x_n, y_n)\Big)\Big\}\Big\}; \;\; \rho_1 > 0 \end{aligned}$$

and

$$g_2(y_n) = J^{H_2,\eta_2}_{M_2^n(\cdot,y_n),\varphi_2}\Big\{H_2(g_2(y_n)) - \rho_2(\varphi_2 \circ N_2)\Big(g_2(y_n) - F_2(x_n, y_n)\Big)\Big\}; \;\; \rho_2 > 0$$

where $M_i^n : X \times X \to X$ are (H_i, φ_i)-η_i-monotone mappings, respectively, for $i \in \{1,2\}$, $n = 0, 1, 2, \ldots$ and

$$J^{H_1,\eta_1}_{M_1^n(\cdot,x_n),\varphi_1} \equiv (H_1 + \rho_1\varphi_1 \circ M_1^n(\cdot, x_n))^{-1}; J^{H_2,\eta_2}_{M_2^n(\cdot,y_n),\varphi_2} \equiv (H_2 + \rho_2\varphi_2 \circ M_2^n(\cdot, y_n))^{-1};$$

$\forall x_n, y_n \in X$ and α_n be the sequence of real numbers with $\alpha_n \in [0,1]$ and $\sum_{n=0}^{\infty} \alpha_n = +\infty$.

Now, we prove the following theorem, which ensures the convergence of the sequences generated by the Iterative Algorithm 4.1 for SGVLI (3.1)-(3.2).

Theorem 4.2. *For $i \in \{1,2\}$, let X be a real 2-uniformly smooth Banach space, $N_i : X \to X$ be r_i-Lipschitz continuous, $g_i : X \to X$ be β_i-Lipschitz continuous and q_i-strongly monotone, $\eta_i : X \times X \to X$ be τ_i-Lipschitz continuous, $H_i : X \to X$ be a δ_i-Lipschitz continuous and γ_i-strongly-η_i-monotone mappings, repectively. Suppose $F_1 : X \times X \to X$ be a ζ_1'-Lipschitz continuous with respect to second argument, $F_2 : X \times X \to X$ be a ζ_2'-Lipschitz continuous with respect to first argument and let $\varphi_i : X \to X$ be single-valued mappings satisfying $\varphi_i(u+v) = \varphi_i(u) + \varphi_i(v)$ and $Ker(\varphi_i) = \{0\}$ such that φ_i be θ_i-Lipschitz continuous, $M_i^n : X \times X \to 2^X$ be (H_i, φ_i)-η_i-monotone mappings such that $M_i^n \xrightarrow{G} M_i$ as $n \to \infty$, respectively. Also φ_i, N_i, F_i, g_i and H_i are single-valued mappings such that $\Big\{H_1(g_1)(\cdot) - \rho_1(\varphi_1 \circ N_1)\Big(g_1(\cdot) - F_1(\cdot, y_n)\Big)\Big\}$ is λ_1-cocoercive and $\Big\{H_2(g_2)(\cdot) - \rho_2(\varphi_2 \circ N_2)\Big(g_2(\cdot) - F_2(x_n, \cdot)\Big)\Big\}$ is λ_2-cocoercive. In addition, if*

$$\left.\begin{array}{l} 1-2q_1+c\beta_1^2>0;\quad q_2\gamma_2\lambda_2>\tau_2;\quad \left(\dfrac{1}{\lambda_1}+\dfrac{\rho_1\rho_2\tau_2\theta_1\theta_2 r_1 r_2\lambda_2\zeta_1'\zeta_2'}{q_2\gamma_2\lambda_2-\tau_2}\right)>0;\\ 0<\sqrt{1-2q_1+c\beta_1^2}+\dfrac{\tau_1}{\gamma_1}\left(\dfrac{1}{\lambda_1}+\dfrac{\rho_1\rho_2\tau_2\theta_1\theta_2 r_1 r_2\lambda_2\zeta_1'\zeta_2'}{q_2\gamma_2\lambda_2-\tau_2}\right)<1, \end{array}\right\} \tag{4.1}$$

where c is constant of smoothness of Banach space X. Then the iterative sequences $\{x_n\}$, $\{y_n\}$ *generated by Iterative Algorithm 4.1 converges strongly to a solution* $(x,y)\in X\times X$ *of SGVLI (3.1)-(3.2).*

Proof. Let $(x,y)\in X\times X$ be the solution of SGVLI (3.1)-(3.2). By Lemma 3.1, we have

$$x=(1-\alpha_n)x+\alpha_n\Big\{x-g_1(x)+J^{H_1,\eta_1}_{M_1(\cdot,x),\varphi_1}\Big\{H_1(g_1(x))-\rho_1(\varphi_1\circ N_1)\Big(g_1(x)-F_1(x,y)\Big)\Big\}\Big\}$$

Now from Iterative Algorithm 4.1 and above condition, we have

$$\begin{aligned}\|x_{n+1}-x\| &= \Big\|(1-\alpha_n)x_n+\alpha_n\Big\{x_n-g_1(x_n)+J^{H_1,\eta_1}_{M_1^n(\cdot,x_n),\varphi_1}\Big\{H_1(g_1(x_n))\\ &\quad -\rho_1(\varphi_1\circ N_1)\Big(g_1(x_n)-F_1(x_n,y_n)\Big)\Big\}\Big\}-(1-\alpha_n)x-\alpha_n\Big\{x-g_1(x)\\ &\quad +J^{H_1,\eta_1}_{M_1(\cdot,x),\varphi_1}\Big\{H_1(g_1(x))-\rho_1(\varphi_1\circ N_1)\Big(g_1(x)-F_1(x,y)\Big)\Big\}\Big\}\Big\|\\ &\le (1-\alpha_n)\Big\|x_n-x\Big\|\\ &\quad +\alpha_n\Big\|x_n-x-\Big(g_1(x_n)-g_1(x)\Big)\Big\|\\ &\quad +\alpha_n\Big\|J^{H_1,\eta_1}_{M_1^n(\cdot,x_n),\varphi_1}\Big\{H_1(g_1(x_n))-\rho_1(\varphi_1\circ N_1)\Big(g_1(x_n)-F_1(x_n,y_n)\Big)\Big\}\\ &\quad -J^{H_1,\eta_1}_{M_1^n(\cdot,x_n),\varphi_1}\Big\{H_1(g_1(x))-\rho_1(\varphi_1\circ N_1)\Big(g_1(x)-F_1(x,y)\Big)\Big\}\Big\|\\ &\quad +\alpha_n\Big\|J^{H_1,\eta_1}_{M_1^n(\cdot,x_n),\varphi_1}\Big\{H_1(g_1(x))-\rho_1(\varphi_1\circ N_1)\Big(g_1(x)-F_1(x,y)\Big)\Big\}\\ &\quad -J^{H_1,\eta_1}_{M_1(\cdot,x),\varphi_1}\Big\{H_1(g_1(x))-\rho_1(\varphi_1\circ N_1)\Big(g_1(x)-F_1(x,y)\Big)\Big\}\Big\|.\end{aligned} \tag{4.2}$$

Using Lemma 2.18, we have the following estimate:

$$\Big\|J^{H_1,\eta_1}_{M_1^n(\cdot,x_n),\varphi_1}\Big\{H_1(g_1(x_n))-\rho_1(\varphi_1\circ N_1)\Big(g_1(x_n)-F_1(x_n,y_n)\Big)\Big\}-J^{H_1,\eta_1}_{M_1^n(\cdot,x_n),\varphi_1}\Big\{H_1(g_1(x))-\rho_1(\varphi_1\circ N_1)\Big(g_1(x)-F_1(x,y)\Big)\Big\}\Big\|$$

$$\begin{aligned}&\le \frac{\tau_1}{\gamma_1}\Big\|H_1(g_1(x_n))-H_1(g_1(x))-\rho_1(\varphi_1\circ N_1)\Big(g_1(x_n)-F_1(x_n,y_n)\Big)\\ &\quad +\rho_1(\varphi_1\circ N_1)\Big(g_1(x))-F_1(x,y)\Big)\Big\|\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\tau_1}{\gamma_1}\Big\|H_1(g_1(x_n)) - \rho_1(\varphi_1 \circ N_1)\big(g_1(x_n) - F_1(x_n, y_n)\big) \\
&-\Big\{H_1(g_1(x)) - \rho_1(\varphi_1 \circ N_1)\big(g_1(x) - F_1(x, y_n)\big)\Big\} \\
&-\rho_1(\varphi_1 \circ N_1)\big(g_1(x) - F_1(x, y_n)\big) + \rho_1(\varphi_1 \circ N_1)\big(g_1(x) - F_1(x, y)\big)\Big\| \\
&\leq \frac{\tau_1}{\gamma_1}\Big\|H_1(g_1(x_n)) - \rho_1(\varphi_1 \circ N_1)\big(g_1(x_n) - F_1(x_n, y_n)\big) \\
&-\Big\{H_1(g_1(x)) - \rho_1(\varphi_1 \circ N_1)\big(g_1(x) - F_1(x, y_n)\big)\Big\}\Big\| \\
&+\frac{\tau_1\rho_1}{\gamma_1}\Big\|(\varphi_1 \circ N_1)\big(g_1(x) - F_1(x, y_n)\big) - (\varphi_1 \circ N_1)\big(g_1(x) - F_1(x, y)\big)\Big\|. \qquad (4.3)
\end{aligned}$$

Since $\Big\{H_1g_1(\cdot) - \rho_1(\varphi_1 \circ N_1)\big(g_1(\cdot) - F_1(\cdot, y_n)\big)\Big\}$ is λ_1-cocoercive, then

$$\begin{aligned}
&\Big\|H_1(g_1(x_n)) - \rho_1(\varphi_1 \circ N_1)\big(g_1(x_n) - F_1(x_n, y_n)\big) \\
&\qquad -\Big\{H_1(g_1(x)) - \rho_1(\varphi_1 \circ N_1)\big(g_1(x) - F_1(x, y_n)\big)\Big\}\Big\| \, \|x_n - x\| \\
&\geq \Big[H_1(g_1(x_n)) - \rho_1(\varphi_1 \circ N_1)\big(g_1(x_n) - F_1(x_n, y_n)\big) \\
&\qquad -\Big\{H_1(g_1(x)) - \rho_1(\varphi_1 \circ N_1)\big(g_1(x) - F_1(x, y_n)\big)\Big\}, x_n - x\Big] \\
&\geq \lambda_1\Big\|H_1(g_1(x_n)) - \rho_1(\varphi_1 \circ N_1)\big(g_1(x_n) - F_1(x_n, y_n)\big) \\
&\qquad -\Big\{H_1(g_1(x)) - \rho_1(\varphi_1 \circ N_1)\big(g_1(x) - F_1(x, y_n)\big)\Big\}\Big\|^2
\end{aligned}$$

Thus,

$$\begin{aligned}
&\Big\|H_1(g_1(x_n)) - \rho_1(\varphi_1 \circ N_1)\big(g_1(x_n) - F_1(x_n, y_n)\big) - \Big\{H_1(g_1(x)) - \rho_1(\varphi_1 \circ N_1)\big(g_1(x) - F_1(x, y_n)\big)\Big\}\Big\| \\
&\qquad \leq \frac{1}{\lambda_1}\, \|x_n - x\| \qquad (4.4)
\end{aligned}$$

Since φ_1 is a θ_1-Lipschitz continuous, N_1 is an r_1-Lipschitz continuous and F_1 is a ζ_1'-Lipschitz continuous with respect to second argument, then we have

$$\begin{aligned}
\Big\|(\varphi_1 \circ N_1)\big(g_1(x) - F_1(x, y_n)\big) - (\varphi_1 \circ N_1)\big(g_1(x) - F_1(x, y)\big)\Big\| &\leq \theta_1\Big\|N_1\big(g_1(x) - F_1(x, y_n)\big) - N_1\big(g_1(x) - F_1(x, y)\big)\Big\| \\
&\leq \theta_1 r_1\Big\|F_1(x, y_n) - F_1(x, y)\Big\| \\
&\leq \theta_1 r_1 \zeta_1' \|y_n - y\|. \qquad (4.5)
\end{aligned}$$

Since g_1 is a q_1-strongly monotone and β_1-Lipschitz continuous, we have

$$\Big\|x_n - x - \big(g_1(x_n) - g_1(x)\big)\Big\| \leq \sqrt{1 - 2q_1 + c\beta_1^2}\, \|x_n - x\|. \qquad (4.6)$$

Using (4.3)-(4.6) in (4.2), we get

$$\begin{aligned}\|x_{n+1}-x\| &\leq (1-\alpha_n)\|x_n-x\| + \alpha_n\sqrt{1-2q_1+c\beta_1^2}\,\|x_n-x\| \\ &\quad +\alpha_n\left\{\frac{\tau_1}{\gamma_1\lambda_1}\|x_n-x\| + \frac{\tau_1\rho_1\theta_1 r_1\zeta_1'}{\gamma_1}\|y_n-y\|\right\} + \alpha_n f_n \\ &= \left\{(1-\alpha_n)+\alpha_n\Big(\sqrt{1-2q_1+c\beta_1^2}+\frac{\tau_1}{\gamma_1\lambda_1}\Big)\right\}\|x_n-x\| \\ &\quad +\frac{\alpha_n\tau_1\rho_1\theta_1 r_1\zeta_1'}{\gamma_1}\|y_n-y\| + \alpha_n f_n\end{aligned}$$

where

$$\begin{aligned}f_n &= \Big\| J^{H_1,\eta_1}_{M_1^n(\cdot,x_n),\varphi_1}\Big\{H_1(g_1(x)) - \rho_1(\varphi_1\circ N_1)\Big(g_1(x)-F_1(x,y)\Big)\Big\} \\ &\quad - J^{H_1,\eta_1}_{M_1(\cdot,x),\varphi_1}\Big\{H_1(g_1(x)) - \rho_1(\varphi_1\circ N_1)\Big(g_1(x)-F_1(x,y)\Big)\Big\}\Big\|,\end{aligned}$$

and $f_n \to 0$ as $n\to\infty$. Thus, we have

$$\begin{aligned}\|x_{n+1}-x\| &\leq \left\{(1-\alpha_n)+\alpha_n\Big(\sqrt{1-2q_1+c\beta_1^2}+\frac{\tau_1}{\gamma_1\lambda_1}\Big)\right\}\|x_n-x\| \\ &\quad +\frac{\alpha_n\tau_1\rho_1\theta_1 r_1\zeta_1'}{\gamma_1}\|y_n-y\|.\end{aligned} \tag{4.7}$$

Since $g_2 : X\to X$ is a q_2-strongly monotone, then we have

$$\|g_2(y_n)-g_2(y)\|\,\|y_n-y\| \geq \Big[g_2(y_n)-g_2(y),\, y_n-y\Big] \geq q_2\|y_n-y\|^2.$$

Thus,

$$\begin{aligned}\|y_n-y\| &\leq \frac{1}{q_2}\|g_2(y_n)-g_2(y)\| \\ &= \frac{1}{q_2}\Big\|J^{H_2,\eta_2}_{M_2^n(\cdot,y_n),\varphi_2}\Big\{H_2(g_2(y_n)) - \rho_2(\varphi_2\circ N_2)\Big(g_2(y_n)-F_2(x_n,y_n)\Big)\Big\} \\ &\quad -J^{H_2,\eta_2}_{M_2(\cdot,y),\varphi_2}\Big\{H_2(g_2(y)) - \rho_2(\varphi_2\circ N_2)\Big(g_2(y)-F_2(x,y)\Big)\Big\}\Big\| \\ &\leq \frac{1}{q_2}\Big\|J^{H_2,\eta_2}_{M_2^n(\cdot,y_n),\varphi_2}\Big\{H_2(g_2(y_n)) - \rho_2(\varphi_2\circ N_2)\Big(g_2(y_n)-F_2(x_n,y_n)\Big)\Big\} \\ &\quad -J^{H_2,\eta_2}_{M_2^n(\cdot,y_n),\varphi_2}\Big\{H_2(g_2(y)) - \rho_2(\varphi_2\circ N_2)\Big(g_2(y)-F_2(x,y)\Big)\Big\}\Big\| \\ &\quad +\frac{1}{q_2}\Big\|J^{H_2,\eta_2}_{M_2^n(\cdot,y_n),\varphi_2}\Big\{H_2(g_2(y)) - \rho_2(\varphi_2\circ N_2)\Big(g_2(y)-F_2(x,y)\Big)\Big\} \\ &\quad -J^{H_2,\eta_2}_{M_2(\cdot,y),\varphi_2}\Big\{H_2(g_2(y)) - \rho_2(\varphi_2\circ N_2)\Big(g_2(y)-F_2(x,y)\Big)\Big\}\Big\|.\end{aligned} \tag{4.8}$$

Using Lemma 2.18, we have the following estimate:

$$\Big\|J^{H_2,\eta_2}_{M_2^n(\cdot,y_n),\varphi_2}\Big\{H_2(g_2(y_n)) - \rho_2(\varphi_2\circ N_2)\Big(g_2(y_n)-F_2(x_n,y_n)\Big)\Big\} - J^{H_2,\eta_2}_{M_2^n(\cdot,y_n),\varphi_2}\Big\{H_2(g_2(y)) - \rho_2(\varphi_2\circ N_2)\Big(g_2(y)-F_2(x,y)\Big)\Big\}\Big\|$$

$$\leq \frac{\tau_2}{\gamma_2}\Big\|H_2(g_2(y_n)) - \rho_2(\varphi_2 \circ N_2)\big(g_2(y_n) - F_2(x_n, y_n)\big)$$

$$-\Big\{H_2(g_2(y)) - \rho_2(\varphi_2 \circ N_2)\big(g_2(y) - F_2(x, y)\big)\Big\}\Big\|$$

$$\leq \frac{\tau_2}{\gamma_2}\Big\|H_2(g_2(y_n)) - \rho_2(\varphi_2 \circ N_2)\big(g_2(y_n) - F_2(x_n, y_n)\big)$$

$$-\Big\{H_2(g_2(y)) - \rho_2(\varphi_2 \circ N_2)\big(g_2(y) - F_2(x_n, y)\big)\Big\}\Big\|$$

$$+\frac{\tau_2\rho_2}{\gamma_2}\Big\|(\varphi_2 \circ N_2)\big(g_2(y) - F_2(x_n, y)\big) - (\varphi_2 \circ N_2)(g_2(y) - F_2(x, y))\Big\|. \tag{4.9}$$

Since $\Big\{H_2(g_2(\cdot)) - \rho_2(\varphi_2 \circ N_2)\big(g_2(\cdot) - F_2(x_n, \cdot)\big)\Big\}$ is λ_2-cocoercive, then

$$\Big\|H_2(g_2(y_n)) - \rho_2(\varphi_2 \circ N_2)\big(g_2(y_n) - F_2(x_n, y_n)\big) - \Big\{H_2(g_2(y)) - \rho_2(\varphi_2 \circ N_2)\big(g_2(y) - F_2(x_n, y)\big)\Big\}\Big\| \, \|y_n - y\|$$

$$\geq \Big[H_2(g_2(y_n)) - \rho_2(\varphi_2 \circ N_2)\big(g_2(y_n) - F_2(x_n, y_n)\big)$$

$$-\Big\{H_2(g_2(y)) - \rho_2(\varphi_2 \circ N_2)\big(g_2(y) - F_2(x_n, y)\big)\Big\}, y_n - y\Big]$$

$$\geq \lambda_2\Big\|H_2(g_2(y_n)) - \rho_2(\varphi_2 \circ N_2)\big(g_2(y_n) - F_2(x_n, y_n)\big)$$

$$-\Big\{H_2(g_2(y)) - \rho_2(\varphi_2 \circ N_2)\big(g_2(y) - F_2(x_n, y)\big)\Big\}\Big\|^2.$$

Thus,

$$\Big\|H_2(g_2(y_n)) - \rho_2(\varphi_2 \circ N_2)\big(g_2(y_n) - F_2(x_n, y_n)\big) - \Big\{H_2(g_2(y)) - \rho_2(\varphi_2 \circ N_2)\big(g_2(y) - F_2(x_n, y)\big)\Big\}\Big\|$$

$$\leq \frac{1}{\lambda_2}\Big\|y_n - y\Big\|. \tag{4.10}$$

Since φ_2 is a θ_2-Lipschitz continuous, N_2 is an r_2-Lipschitz continuous and $F_2(\cdot, \cdot)$ is a ζ_2'-Lipschitz continuous with respect to first argument, then we have

$$\Big\|(\varphi_2 \circ N_2)\big(g_2(y) - F_2(x_n, y)\big) - (\varphi_2 \circ N_2)\big(g_2(y) - F_2(x, y)\big)\Big\|$$

$$\leq \theta_2\Big\|N_2\big(g_2(y) - F_2(x_n, y)\big) - N_2\big(g_2(y) - F_2(x, y)\big)\Big\|$$

$$\leq \theta_2 r_2 \Big\|F_2(x_n, y) - F_2(x, y)\Big\|$$

$$\leq \theta_2 r_2 \zeta_2' \|x_n - x\|. \tag{4.11}$$

Using (4.9)-(4.11) in (4.8), we have

$$\|y_n - y\| \leq \frac{1}{q_2}\left\{\frac{\tau_2}{\gamma_2\lambda_2}\|y_n - y\| + \frac{\tau_2\rho_2\theta_2 r_2\zeta_2'}{\gamma_2}\|x_n - x\| + h_n\right\},$$

where

$$h_n = \Big\|J^{H_2,\eta_2}_{M_2^n(\cdot, y_n),\varphi_2}\Big\{H_2(g_2(y)) - \rho_2(\varphi_2 \circ N_2)\big(g_2(y) - F_2(x, y)\big)\Big\} - J^{H_2,\eta_2}_{M_2(\cdot, y),\varphi_2}\Big\{H_2(g_2(y)) - \rho_2(\varphi_2 \circ N_2)\big(g_2(y) - F_2(x, y)\big)\Big\}\Big\|$$

and $h_n \to 0$ as $n \to \infty$.
Thus, we have

$$\|y_n - y\| \leq \left(\frac{\tau_2\rho_2\lambda_2\theta_2 r_2\zeta_2'}{q_2\gamma_2\lambda_2 - \tau_2}\right)\|x_n - x\| + \left(\frac{\gamma_2\lambda_2}{q_2\gamma_2\lambda_2 - \tau_2}\right)h_n.$$

Thus, from (4.7), we have

$$\begin{aligned}
\|x_{n+1} - x\| &\leq \Big\{(1-\alpha_n) + \alpha_n\Big(\sqrt{1-2q_1+c\beta_1^2} + \frac{\tau_1}{\gamma_1\lambda_1}\Big) \\
&\quad + \frac{\alpha_n\lambda_2\tau_1\tau_2\rho_1\rho_2\theta_1\theta_2 r_1 r_2\zeta_1'\zeta_2'}{\gamma_1(q_2\gamma_2\lambda_2-\tau_2)}\Big\}\|x_n - x\| + \frac{\alpha_n\gamma_2\lambda_2\tau_1\rho_1\theta_1 r_1\zeta_1'}{\gamma_1(q_2\gamma_2\lambda_2-\tau_2)}\,h_n \\
&\leq \left\{1 - \alpha_n\left(1 - \sqrt{1-2q_1+c\beta_1^2} - \frac{\tau_1}{\gamma_1}\left(\frac{1}{\lambda_1} + \frac{\tau_2\lambda_2\rho_1\rho_2\theta_1\theta_2 r_1 r_2\zeta_1'\zeta_2'}{(q_2\gamma_2\lambda_2-\tau_2)}\right)\right)\right\} \\
&\quad \times \|x_n - x\| + \alpha_n\left(\frac{\rho_1\tau_1\gamma_2\lambda_2\theta_1 r_1\zeta_1'}{\gamma_1(q_2\gamma_2\lambda_2-\tau_2)}\right)h_n \\
&\leq \Big\{1 - \alpha_n(1-k_1)\Big\}\|x_n - x\| + \alpha_n d_n,
\end{aligned}$$

where

$$k_1 = \sqrt{1-2q_1+c\beta_1^2} + \frac{\tau_1}{\gamma_1}\left(\frac{1}{\lambda_1} + \frac{\rho_1\rho_2\theta_1\theta_2 r_1 r_2\zeta_1'\zeta_2'\tau_2\lambda_2}{(q_2\gamma_2\lambda_2-\tau_2)}\right)$$

and $k_1 < 1$ because of the assumption (4.1).
Hence we can write

$$\|x_{n+1} - x\| \leq \Big(1 - \alpha_n(1-k_1)\Big)\|x_n - x\| + \alpha_n(1-k_1)\frac{d_n}{(1-k_1)}. \tag{4.12}$$

We suppose that $a_n = \|x_n - x\|$, $b_n = \frac{d_n}{(1-k_1)}$, $c_n = \alpha_n(1-k_1)$, then we can express (4.12) in the form $a_{n+1} = (1-c_n)a_n + c_n b_n$. Hence by Lemma 2.19, we get $a_n \to 0$ as $n \to \infty$. As a result $x_n \to x$ as $n \to \infty$ which infact implies that $y_n \to y$ as $n \to \infty$. Hence the sequences $\{x_n\}$, $\{y_n\}$ converges strongly to the solution $(x, y) \in X \times X$ of SGVLI (3.1)-(3.2). □

When $X = L^p(\mathbf{R})$, $2 \leq p < \infty$, we have the following corollary.

Corollary 4.3. *For $i \in \{1, 2\}$, let $N_i : L^p \to L^p$ be r_i-Lipschitz continuous, $g_i : L^p \to L^p$ be β_i-Lipschitz continuous and q_i-strongly monotone, $\eta_i : L^p \times L^p \to L^p$ be τ_i-Lipschitz continuous, $H_i : L^p \to L^p$ be a δ_i-Lipschitz continuous and γ_i-strongly-η_i-monotone mappings, respectively. Suppose $F_1 : L^p \times L^p \to L^p$ be a ζ_1'-Lipschitz continuous with respect to second argument, $F_2 : L^p \times L^p \to L^p$ be a ζ_2'-Lipschitz continuous with respect to first argument and let $\varphi_i : L^p \to L^p$ be single-valued mappings satisfying $\varphi_i(u + v) = \varphi_i(u) + \varphi_i(v)$ and $Ker(\varphi_i) = \{0\}$ such that φ_i be θ_i-Lipschitz continuous, $M_i^n : L^p \times L^p \to 2^{L^p}$ be (H_i, φ_i)-η_i-monotone mappings such that $M_i^n \xrightarrow{G} M_i$ as $n \to \infty$, respectively. Also φ_i, N_i, F_i, g_i and H_i are single-valued mappings such that $\Big\{H_1(g_1)(\cdot) - \rho_1(\varphi_1 \circ N_1)\Big(g_1(\cdot) - F_1(\cdot, y_n)\Big)\Big\}$ is λ_1-cocoercive and $\Big\{H_2(g_2)(\cdot) - \rho_2(\varphi_2 \circ N_2)\Big(g_2(\cdot) - F_2(x_n, \cdot)\Big)\Big\}$ is λ_2-cocoercive. In addition, if*

$$1 - 2q_1 + (p-1)\beta_1^2 > 0; \quad q_2\gamma_2\lambda_2 > \tau_2; \quad \left(\frac{1}{\lambda_1} + \frac{\rho_1\rho_2\tau_2\theta_1\theta_2 r_1 r_2\lambda_2\zeta_1'\zeta_2'}{q_2\gamma_2\lambda_2 - \tau_2}\right) > 0;$$

$$0 < \sqrt{1 - 2q_1 + (p-1)\beta_1^2} + \frac{\tau_1}{\gamma_1}\left(\frac{1}{\lambda_1} + \frac{\rho_1\rho_2\tau_2\theta_1\theta_2 r_1 r_2\lambda_2\zeta_1'\zeta_2'}{q_2\gamma_2\lambda_2 - \tau_2}\right) < 1,$$

where $(p-1)$ *is constant of smoothness. Then the iterative sequences* $\{x_n\}$, $\{y_n\}$ *generated by Iterative Algorithm 4.1 converges strongly to a solution* $(x, y) \in L^p \times L^p$ *of SGVLI (3.1)-(3.2).*

Remark 4.4. *Using the technique in this paper one can extend the results of various authors, see for example [3,6,7-9,18,19,21,25-27,30-34] and the related references cited therein in this direction.*

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