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Characterization of Two-sided Order Preserving of Convex Majorization on $\ell^p(I)$

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Abstract. In this paper, we consider an equivalence relation $\sim_c$ on $\ell^p(I)$, which is said to be "convex equivalent" for $p \in [1, +\infty)$ and a nonempty set I. We characterize the structure of all bounded linear operators $T : \ell^p(I) \to \ell^p(I)$ that strongly preserve the convex equivalence relation. We prove that the rows of the operator which preserve convex equivalent, belong to $\ell^1(I)$. Also, we show that any bounded linear operators $T : \ell^p(I) \longrightarrow \ell^p(I)$ which preserve convex equivalent, also preserve convex majorization.

1. Introduction and Preliminaries

Majorization theory plays an important role in various areas and gives a lot of applications in the operator theory and linear algebra. For an account of the majorization theory we refer the reader to [1, 2, 4–9].

Throughout this work, I is a nonempty set, $p \in [1, +\infty)$, and $\ell^p(I)$ is the Banach space of all functions $f : I \to \mathbb{R}$ with the finite norm defined by

$$\|f\|_p = \left(\sum_{i \in I} |f(i)|^p \right)^{\frac{1}{p}}.$$

For $f \in \ell^p(I)$, to shorten notation, we will write $\operatorname{co}(f)$, instead of the convex combination of the set $\operatorname{Im}(f) = \{f(i); i \in I\}$.

Definition 1.1. *[3] For any given $f, g \in \ell^p(I)$, f is said to be convex majorized by g, and denoted by $f \prec_c g$, if $\operatorname{co}(f) \subseteq \operatorname{co}(g)$. Also, f is said to be convex equivalent to g, denoted by $f \sim_c g$, whenever $f \prec_c g \prec_c f$, i.e., $\operatorname{co}(f) = \operatorname{co}(g)$.*

It is easy to see that the relation $\sim_c$ is an equivalence relation on $\ell^p(I)$.

Definition 1.2. *[4] Let X be a linear space and $\mathcal{R}$ be a relation on X. The linear operator $T : X \to X$ is said to preserve $\mathcal{R}$ if for each $x, y \in X$*

$$\mathcal{R}(x, y) \ \textit{implies} \ \ \mathcal{R}(Tx, Ty).$$

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Moreover, T is called two-sided(or strongly) preserve $\mathcal{R}$ if

$$\mathcal{R}(x,y) \text{ if and only if } \mathcal{R}(Tx,Ty).$$

Let $\mathcal{E}$ denote the set of all bounded linear operators $T : \ell^p(I) \to \ell^p(I)$ which satisfy $\mathrm{co}(Tf) = \mathrm{co}(f)$, for all $f \in \ell^p(I)$. The set of all bounded linear operators $T : \ell^p(I) \to \ell^p(I)$ which preserve convex majorization, convex equivalent, strongly preserve convex majorization, and strongly preserve convex equivalent will be denoted by $\mathcal{P}_c$, $\mathcal{P}_e$, $\mathcal{P}_{sc}$, and $\mathcal{P}_{se}$, respectively. It is obvious that $\mathcal{P}_c \subseteq \mathcal{P}_e$.

An element $f \in \ell^p(I)$ can be represented by $\sum_{i\in I} f(i)e_i$, where $e_i : I \to \mathbb{R}$ is defined by $e_i(j) = \delta_{ij}$, the Kronecker delta. Let $T : \ell^p(I) \to \ell^p(I)$ be a bounded linear operator. Then an easy computation shows that, T is represented by a (finite or infinite) matrix $(t_{ij})_{i,j\in I}$ in the sense that

$$(Tf)(i) = \sum_{j\in I} t_{ij} f(j) \qquad (f \in \ell^p(I),\ i \in I),$$

where $t_{ij} = (Te_j)(i)$. To simplify notation, we can incorporate T to its matrix form $(t_{ij})_{i,j\in I}$. Both the values of $\inf_{i\in I}\{Te_j(i)\}$ and $\sup_{i\in I}\{Te_j(i)\}$ are independent of the choice of $j \in I$, and we denote them by a and b, respectively [3].

Theorem 1.3. *[3] Let $T : \ell^p(I) \longrightarrow \ell^p(I)$ be a linear operator. Then $T \in \mathcal{P}_c$ if and only if*

(i) *For any $j \in I$, the value of $\min_{i\in I}\{Te_j(i)\}$ exists and independent of j is equal to a.*

(ii) *For any $j \in I$, the value of $\max_{i\in I}\{Te_j(i)\}$ exists and independent of j is equal to b.*

(iii) *If $a < 0 < b$, we have $\frac{1}{a}\sum_{j\in I^-} Te_j(i) + \frac{1}{b}\sum_{j\in I^+} Te_j(i) \le 1$; if $a < 0 = b$, then we have $\sum_{j\in I} Te_j(i) \ge a$, and if $a = 0 < b$, then it implies $\sum_{j\in I} Te_j(i) \le b$, where $(Te_j(i))_{j\in I}$ is an arbitrary row of T and $I^+ = \{j \in I;\ Te_j(i) > 0\}$, $I^- = \{j \in I;\ Te_j(i) < 0\}$.*

Theorem 1.4. *[3] Let $I \neq \emptyset$ and $T : \ell^p(I) \longrightarrow \ell^p(I)$ be a bounded linear operator. Then*

(i) *for finite set I, $T \in \mathcal{E}$ if and only if T is a permutation.*

(ii) *for infinite set I, $T \in \mathcal{E}$ if and only if for all $j \in I$, we have $\min_{i\in I}\{Te_j(i)\} = 0$, $\max_{i\in I}\{Te_j(i)\} = 1$, and for each $i \in I$ we have $0 < \sum_{j\in I} Te_j(i) \le 1$, when I is countable, and $0 \le \sum_{j\in I} Te_j(i) \le 1$, when I is uncountable.*

We prepare this work as follows. In the next section, we consider some properties of the operators in $\mathcal{P}_e$ when I is an infinite set. We prove that the rows of $T \in \mathcal{P}_e$ belong to $\ell^1(I)$. Also, we show that any operators $T : \ell^p(I) \longrightarrow \ell^p(I)$ which preserve convex equivalent, also preserve convex majorization. In the third section we proceed with the study of the structure of operators that preserve convex equivalent when I is a finite set. Section 4 is devoted to characterize strong preservers of convex majorization. For an infinite set I, we prove that any elements in $\mathcal{P}_{se}$ are nonzero constant coefficient of an element in $\mathcal{E}$, and $\mathcal{P}_{sc} = \mathcal{P}_{se}$.

2. Operators in $\mathcal{P}_e$ when I is an Infinite Set

For each $T \in \mathcal{P}_e$ the values $a := \inf Te_j$ and $b := \sup Te_j$ are constants, independent of the choice of $j \in I$ [3].

In Theorem 1.3, Bayati et al. characterized the operators in $\mathcal{P}_c$. In this section, we consider all the operators in $\mathcal{P}_e$, in the case that I is an infinite set.

Theorem 2.1. *Let I be an infinite set and $T \in \mathcal{P}_e$. Then*

(i) *The values* $\|Te_j\|_\infty$ *and* $\|Te_{j_1} - Te_{j_2}\|_\infty$ *are constants and equal, independent of the choice of* $j, j_1, j_2 \in I$ *with* $j_1 \neq j_2$.

(ii) *The rows of* T *belong to* $\ell^1(I)$. *Moreover* $\sum_{j\in I} |Te_j(i_0)| \le \|Te_{j_0}\|_\infty$, *for any fixed* $i_0, j_0 \in I$.

Proof. If $T \equiv 0$, then the assertions follow, otherwise let $j, i_0, j_0, j_0' \in I$, with $j_0 \neq j_0'$. Set

$$\delta_j = \begin{cases} 1 & \text{if } Te_j(i_0) \ge 0, \\ -1 & \text{if } Te_j(i_0) < 0. \end{cases}$$

and $F = \{j_1, \dots, j_n\} \subseteq I$. Then $\sum_{j\in F} \delta_j e_j$ is convex equivalent to either $\pm e_{j_0}$ or $e_{j_0} - e_{j_0'}$. Since $T \in \mathcal{P}_e$, it follows that $\sum_{j\in I} |Te_j(i_0)| < \infty$. Thus for each $\epsilon > 0$, there exists $j^* \in I \setminus F$ with $|Te_{j^*}(i_0)| < \epsilon$. Now define

$$\delta^* = \begin{cases} -1 & \text{if } \delta_{j_1} = \dots = \delta_{j_n} = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Since $\sum_{k=1}^n \delta_{j_k} e_{j_k} + \delta^* e_{j^*} \sim_c e_{j_0} - e_{j_0'}$, we have

$$\sum_{k=1}^n \delta_{j_k} Te_{j_k} + \delta^* Te_{j^*} \sim_c Te_{j_0} - Te_{j_0'}.$$

Therefore

$$\sum_{k=1}^n |Te_{j_k}(i_0)| + \delta^* Te_{j^*}(i_0) \in \mathrm{co}\left(\sum_{k=1}^n \delta_{j_k} Te_{j_k} + \delta^* Te_{j^*}\right) = \mathrm{co}\Big(Te_{j_0} - Te_{j_0'}\Big),$$

which implies

$$\mathrm{dist}\left(\sum_{k=1}^n |Te_{j_k}(i_0)|, \mathrm{co}(Te_{j_0} - Te_{j_0'})\right) \le |\delta^* Te_{j^*}(i_0)| < \epsilon.$$

As ϵ is arbitrary, we have $\sum_{k=1}^n |Te_{j_k}(i_0)| \in \overline{\mathrm{co}(Te_{j_0} - Te_{j_0'})}$, which follows that

$$\sum_{k=1}^n |Te_{j_k}(i_0)| \le \|Te_{j_0} - Te_{j_0'}\|_\infty.$$

Since $F \subseteq I$ is an arbitrary finite set, we conclude that

$$\sum_{j\in I} |Te_j(i_0)| \le \|Te_{j_0} - Te_{j_0'}\|_\infty. \tag{1}$$

That is, the rows of T belong to $\ell^1(I)$. The inequality (1) concludes that for any $j, i \in I$, we have $|Te_j(i)| \le \|Te_{j_0} - Te_{j_0'}\|_\infty$, which follows

$$\|Te_j\|_\infty \le \|Te_{j_0} - Te_{j_0'}\|_\infty. \tag{2}$$

The proof is completed by showing that

$$\|Te_{j_0} - Te_{j_0'}\|_\infty \le \|Te_{j_0}\|_\infty. \tag{3}$$

Let $\epsilon > 0$. Since $Te_j \in \ell^p(I)$, there are $i_1, \ldots, i_M \in I$ such that for each $i \in I \setminus \{i_1, \ldots, i_M\}$ we have $|Te_j(i)| < \frac{\epsilon}{2}$. On the other hand (1) shows that all the series

$$\sum_{j\in I} |Te_j(i_1)|, \ldots, \sum_{j\in I} |Te_j(i_M)|$$

converge. So there exist $j_1, \ldots, j_N \in I$ such that for all $j \in I \setminus \{j_0, \ldots, j_N\}$,

$$|Te_j(i_1)|, \ldots, |Te_j(i_M)| < \frac{\epsilon}{2}.$$

Now if $j^* \neq j_0, j_1, \ldots, j_N$, then for all $i \in I$, we have

$$|Te_{j_0}(i) - Te_{j^*}(i)| \leq \begin{cases} |Te_{j_0}(i)| + \epsilon \\ \epsilon + |Te_{j^*}(i)| \end{cases} \leq \|Te_{j_0}\|_\infty + \epsilon,$$

which implies

$$\|Te_{j_0} - Te_{j_0'}\|_\infty = \|Te_{j_0} - Te_{j^*}\|_\infty \leq \|Te_{j_0}\|_\infty + \epsilon.$$

Since ϵ is arbitrary, it follows (3). This completes the proof. □

In the following, we obtain some properties of $\mathcal{P}_e$.

Lemma 2.2. *Let $T \in \mathcal{P}_e$ and $i \in I$. Then we have*

$$a \leq \sum_{j\in I^-} Te_j(i) \leq 0 \leq \sum_{j\in I^+} Te_j(i) \leq b,$$

where $I^+ = \{j \in I;\ Te_j(i) > 0\}$, $I^- = \{j \in I;\ Te_j(i) < 0\}$.

Proof. Let $F \subseteq I^+$ be a nonempty finite set. Since for $j_0 \in I$, we have $\mathrm{co}\left(\sum_{j\in F} Te_j\right) = \mathrm{co}(Te_{j_0})$. It implies that

$$0 \leq \sum_{j\in F} Te_j(i) \in \mathrm{Im}\left(\sum_{j\in F} Te_j\right) \subseteq \mathrm{co}\left(\sum_{j\in F} Te_j\right) = \mathrm{co}(Te_{j_0}).$$

Thus $0 \leq \sum_{j\in F} Te_j(i) \leq \sup_{i\in I} Te_{j_0}(i) = b$. Since the last inequality holds for all finite subsets $F \subseteq I^+$, we conclude that

$$0 \leq \sum_{j\in I^+} Te_j(i) \leq b.$$

Similar arguments apply to the other inequality. □

Lemma 2.3. *If $T \in \mathcal{P}_e$, and $j_0 \in I$, then $0 \in \mathrm{Im}(Te_{j_0})$ and $\mathrm{co}(Te_{j_0}) = [a, b]$.*

Proof. Let $j_0, j_1 \in I$ with $j_0 \neq j_1$. If $a = b = 0$, then $Te_{j_0} = 0$ and we are done. Otherwise, $a < 0$ or $b > 0$. Since $a = \inf_{i\in I} Te_j(i)$ and $b = \sup_{i\in I} Te_j(i)$, we have $\|Te_{j_1}\|_\infty = \max\{b, -a\} > 0$.
Now if $\|Te_{j_1}\|_\infty = b > 0$, then there is $i_0 \in I$ such that $Te_{j_1}(i_0) = b$. Applying Theorem 2.1, it implies that $b = |Te_{j_1}(i_0)| \leq \sum_{j\in I} |Te_j(i_0)| \leq \|Te_{j_1}\|_\infty = b$. These inequalities imply $|Te_j(i_0)| = 0$ for all $j \neq j_1$. Therefore $Te_{j_0}(i_0) = 0$, which implies $0 \in \mathrm{Im}(Te_{j_0})$.
For $\|Te_{j_1}\|_\infty = -a > 0$, the result follows by a similar argument. □

Theorem 2.4. *Let $T \in \mathcal{P}_e$. Then for $a < 0 < b$, we have*

$$\frac{1}{a}\sum_{j\in I^-} Te_j(i) + \frac{1}{b}\sum_{j\in I^+} Te_j(i) \leq 1,$$

for $a < 0 = b$, we have $\sum_{j\in I} Te_j(i) \geq a$, and for $a = 0 < b$, we have $\sum_{j\in I} Te_j(i) \leq b$, where $(Te_j(i))_{j\in I}$ is an arbitrary row of T.

Proof. The proof is similar to the proof of Theorem 18 in [3]. □

Theorem 2.5. *Every $T : \ell^p(I) \to \ell^p(I)$ which preserves convex equivalent, preserves convex majorization, i.e. $\mathcal{P}_c = \mathcal{P}_e$.*

Proof. Suppose that $T \in \mathcal{P}_e$. According to the first of this section, the values of $a := \inf Te_j$ and $b := \sup Te_j$ are constants. On the other hand Lemma 2.3 implies that $a = \min Te_j$ and $b = \max Te_j$. The proof now follows by using the previous theorem and Theorem 1.3. □

3. The Structure of the Operators in $\mathcal{P}_e$ when I is a Finite set

In this section, we wish to investigate the structure of the operators on $\ell^p(I)$ that preserve convex equivalent when I is a finite set. Let $\mathrm{card}(I) = n \in \mathbb{N}$. Using Remark 22 in [3], one can replace $\mathbb{R}^n$ with $\ell^p(I)$ and assume that $I = \{1, \dots, n\}$. In this section, we assume that $(t_{ij})_{i,j\in I}$ is the matrix representation of the linear map $T : \mathbb{R}^n \to \mathbb{R}^n$. We recall that for each $T \in \mathcal{P}_e$ the values $a := \inf Te_j$ and $b := \sup Te_j$ are constants, independent of the choice of $j \in I$.

Now for $n = 1$, it is easy to see that every linear map $T : \mathbb{R} \to \mathbb{R}$ lies in $\mathcal{P}_e$, and for $n = 2$ an easy computation shows that a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ belongs to $\mathcal{P}_e$, if and only if the matrix representation of T is either of the form

$$T = \begin{bmatrix} \alpha & \alpha \\ \beta & \beta \end{bmatrix} \qquad \text{or} \qquad T = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix},$$

for some $\alpha, \beta \in \mathbb{R}$. For the case $n \geq 3$, we will show that $T \in \mathcal{P}_e$, if and only if T is a coefficient of a permutation on $\mathbb{R}^n$ [3].

Lemma 3.1. *Let $n \geq 3$, $T \in \mathcal{P}_e$ and $b > 0$. If $I_j := \{i \in I; Te_j(i) = b\}$ for $j = 1, \dots, n$, then each I_j is a singleton and $\cup_{j\in I} I_j = I$.*

Proof. For each $j \in I$, the relation $\mathrm{co}(Te_j) = [a, b]$ implies that there exists $i \in I$ such that $Te_j(i) = b$. Thus each I_j is a nonempty set. Suppose that $i_0 \in I_{j_1} \cap I_{j_2}$, for some distinct elements $j_1, j_2 \in I$. Then

$$2b = Te_{j_1}(i_0) + Te_{j_2}(i_0) \in \mathrm{co}(Te_{j_1} + Te_{j_2}) = [a, b],$$

that implies $2b \leq b$. This contradiction shows $I_{j_1} \cap I_{j_2} = \emptyset$, for $j_1 \neq j_2$. Therefore by using the relations

$$n \leq \mathrm{card}(I_1) + \cdots + \mathrm{card}(I_n) = \mathrm{card}(\cup_{j\in I} I_j) \leq \mathrm{card}(I) = n,$$

each I_j must be a singleton. □

Lemma 3.2. *Let $n \geq 3$, $T \in \mathcal{P}_e$, and $b > 0$. If $t_{i_0 j_0} = b$ for some $i_0, j_0 \in I$, then $t_{i_0 j} = 0$, for all $j \neq j_0$ in I.*

Proof. Let $j \in I$ and $j \neq j_0$. For each $\lambda \in [0, 1]$, we have $Te_{j_0} + \lambda Te_j \sim_c Te_{j_0}$, since $e_{j_0} + \lambda e_j \sim_c e_{j_0}$. Therefore

$$\max(Te_{j_0} + \lambda Te_j) = \max\left\{t_{1j_0} + \lambda t_{1j}, \dots, t_{nj_0} + \lambda t_{nj}\right\} = \max(Te_{j_0}) = b.$$

This shows that for infinite values of $\lambda \in [0, 1]$ and for a constant $i \in I$, we have $t_{ij_0} + \lambda t_{ij} = b$. Thus $t_{ij} = 0$ and $t_{ij_0} = b$. Now, by the assumption $t_{i_0 j_0} = b$ and Lemma 3.1, it implies $i = i_0$. Therefore $t_{i_0 j} = 0$, for all $j \neq j_0$. □

Theorem 3.3. *For $n \geq 3$, $T \in \mathcal{P}_e$ if and only if T is a coefficient of a permutation.*

Proof. Suppose that $T \in \mathcal{P}_e$. If $T = 0$, then the assertion is clear. Let $0 \not\equiv T \in \mathcal{P}_e$. By replacing $-T$ by T if necessary, we can assume that $b > 0$. Since $I = \cup_{j \in I} I_j$, where I_j is as in Lemma 3.1, then for each $i \in I$ there is $j \in I$ with $i \in I_j$. So we have $t_{ij} = b$. Moreover, the previous lemma implies $t_{ij_1} = 0$, for each $j_1 \neq j$. This means that in any row of T, we have exactly one time b and other entries of this row are equal to zero. Now, if b appears more than one time in some columns of T, then there is at least one column that is completely zero, which is not possible. Thus in each row and column of T, b appears exactly one time and other entries are all zero. Thus T is a coefficient of a permutation.
The converse is obvious. $\square$

4. Characterization of Strong Preservers of Convex Majorization

We first recall that $\mathcal{P}_{sc}$ is denoted for the set of all bounded linear operators $T : \ell^p(I) \to \ell^p(I)$ which strongly preserve convex majorization, i.e. $f \prec_c g$, if and only if $Tf \prec_c Tg$, for $f, g \in \ell^p(I)$. We also use the notation $\mathcal{P}_{se}$ for the set of all operators $T : \ell^p(I) \to \ell^p(I)$ which strongly preserve convex equivalent, that is $f \sim_c g$ if and only if $Tf \sim_c Tg$.

Let us mention some direct consequences of $\mathcal{P}_{sc}$, $\mathcal{P}_{se}$, and $\mathcal{E}$.

- $\mathcal{E} \subseteq \mathcal{P}_{sc} \subseteq \mathcal{P}_{se}$.
- $\mathcal{P}_{sc}$ and $\mathcal{P}_{se}$ are both closed under the combination and (nonzero) scaler multiplication.
- If $T \in \mathcal{P}_{se}$, then $\ker(T) = \{0\}$.

Example 4.1. *Suppose (T_n) is a sequence of operators on $\ell^p := \ell^p(\mathbb{N})$, which is defined by $T_n(f) = \left(\frac{1}{n} f_1, f_1, f_2, \ldots\right)$, for each $f = (f_1, f_2, f_3, \ldots) \in \ell^p$. The sequence (T_n) converges to $T : \ell^p \to \ell^p$, where $Tf = \left(0, f_1, f_2, \ldots\right)$, the right shift operator. By using Theorem 1.3, we have $T_n, T \in \mathcal{P}_c$, and so $T_n, T \in \mathcal{P}_e$.*

Example 4.2. *Let $T : \ell^p \to \ell^p$ be the bounded linear operator represented by the matrix form*

$$T = \begin{bmatrix} 1 & 0 & \cdots \\ -1 & 0 & \cdots \\ 0 & 1 & \cdots \\ 0 & -1 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

Then $Tf = (f_1, -f_1, f_2, -f_2, \ldots)$, for each $f = (f_1, f_2, \ldots) \in \ell^p$. Theorem 1.3 implies that $T \in \mathcal{P}_c$. However, $T \notin \mathcal{P}_{se}$, since $Te_1 \sim_c T(-e_1)$, but e_1 is not convex equivalent to $-e_1$.

Lemma 4.3. *If $T \in \mathcal{P}_{se}$, then $a \neq -b$.*

Proof. On the contrary, suppose that $a = -b$. Since $\mathrm{co}(Te_j) = \mathrm{co}(T(-e_j)) = [a, b]$, it follows that $Te_j \sim_c T(-e_j)$, but we have $e_j \not\sim_c -e_j$. $\square$

Example 4.4. *Let $T : \ell^p \to \ell^p$ be presented by the matrix form*

$$T = \begin{bmatrix} 1 & 0 & \cdots \\ -2 & 0 & \cdots \\ 0 & 1 & \cdots \\ 0 & -2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

For $f = (-2, \frac{1}{2}, 0, 0, \dots)$ and $g = (-2, 1, 0, 0, \dots)$ in ℓ^p, we have

$$Tf = (-2, 4, \frac{1}{2}, -1, 0, 0, \dots) \sim_c (-2, 4, 1, -2, 0, 0, \dots) = Tg,$$

however, $f \nsim_c g$. Therefore $T \notin \mathcal{P}_{se}$.

Lemma 4.5. *Let $T \in \mathcal{P}_c$, $a < 0 < b$, $\alpha \leq \min\{\frac{a}{b}, \frac{b}{a}\}$, and $j_1, j_2 \in I$ be such that $j_1 \neq j_2$, then for $g = \alpha e_{j_1} + e_{j_2}$, we have*

$$\alpha b \leq \inf Tg \leq \sup Tg \leq \alpha a.$$

Proof. Let $0 < \epsilon \leq \min\{-a, b\}$. Then there exists a finite set $F \subseteq I$ with $|Te_{j_1}(i)| < \frac{\epsilon}{-\alpha} \leq \epsilon$, for each $i \in I \setminus F$. By using Theorem 8 in [3], each rows of T lies in $\ell^1(I)$. On the other hand, F is a finite set. Thus there exists $j_0 \in I$ (with $j_0 \neq j_1$) such that $|Te_{j_0}(i)| < \epsilon$, for each $i \in F$. Now we consider the following two cases for $i \in I$:
Case 1. Let $i \in F$. Then $a \leq Te_{j_1}(i) \leq b$ and $|Te_{j_0}(i)| < \epsilon$. So, we have

$$\alpha b - \epsilon \leq \alpha Te_{j_1}(i) + Te_{j_0}(i) \leq \alpha a + \epsilon. \tag{4}$$

Case 2. Let $i \in I \setminus F$. Then $|Te_{j_1}(i)| < \frac{\epsilon}{-\alpha}$ and $a \leq Te_{j_0}(i) \leq b$. Therefore

$$a - \epsilon \leq \alpha Te_{j_1}(i) + Te_{j_0}(i) \leq b + \epsilon. \tag{5}$$

Thus (4) and (5) imply that

$$\begin{aligned} \alpha b - \epsilon = \min\{\alpha b - \epsilon, a - \epsilon\} &\leq \alpha Te_{j_1}(i) + Te_{j_0}(i) \\ \leq \max\{\alpha a + \epsilon, b + \epsilon\} &= \alpha a + \epsilon. \end{aligned}$$

Since ϵ is arbitrary, we have

$$\mathrm{co}(Tg) = \mathrm{co}(\alpha Te_{j_1} + Te_{j_2}) = \mathrm{co}(\alpha Te_{j_1} + Te_{j_0}) \subseteq [\alpha b, \alpha a],$$

which proves the assertion. □

Lemma 4.6. *Let $T \in \mathcal{P}_c$ and $\min Te_j = a < 0 < b = \max Te_j$. Then $T \notin \mathcal{P}_{se}$.*

Proof. For $\alpha := \min\{\frac{a}{b}, \frac{b}{a}\}$ and for any distinct elements $j_1, j_2 \in I$, define $f = \alpha e_{j_1}$ and $g = \alpha e_{j_1} + e_{j_2}$. Then $Tf = \alpha Te_{j_1}$, which implies

$$\mathrm{co}(Tf) = \mathrm{co}(\alpha Te_{j_1}) = \alpha[a, b] = [\alpha b, \alpha a].$$

On the other hand, there exist $i_1, i_1^* \in I$ with $Te_{j_1}(i_1) = a$ and $Te_{j_1}(i_1^*) = b$. Hence Theorem 1.3 implies that $Te_{j_2}(i_1) = Te_{j_2}(i_1^*) = 0$, and so

$$\alpha a = \alpha Te_{j_1}(i_1) + Te_{j_2}(i_1) \in \mathrm{co}(Tg), \tag{6}$$

$$\alpha b = \alpha Te_{j_1}(i_1^*) + Te_{j_2}(i_1^*) \in \mathrm{co}(Tg). \tag{7}$$

Lemma 4.5 implies that

$$\alpha b \leq \inf Tg \leq \sup Tg \leq \alpha a.$$

From (6) and (7) we conclude that

$$\mathrm{co}(Tg) = [\alpha b, \alpha a] = \mathrm{co}(Tf).$$

It follows that $Tf \sim_c Tg$, although $f \nsim_c g$. That is $T \notin \mathcal{P}_{se}$. □

Lemma 4.7. *Let $T \in \mathcal{P}_{se}$. Then $a = 0 < b$ or $a < 0 = b$.*

Proof. It is clear that $a \leq 0 \leq b$. Now Lemma 4.6 implies that $a = 0 \leq b$, or $a \leq b = 0$. This gives the claim, because if $a = b = 0$, then T must be zero, and so is not a strong preserver of $\sim_c$. □

Lemma 4.8. *Let I be an infinitely countable set and $T \in \mathcal{P}_{se}$. Then (the matrix representation of) T does not contain any zero row.*

Proof. Without loss of generality we may assume that $I = \mathbb{N}$. On the contrary, suppose that there exists $i_0 \in I$ such that the i_0th row of T is equal to zero. Define $f = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$. Then for each $j \in I$ we have

$$\mathrm{co}(Te_j) = [a, 0] = \mathrm{co}(Tf), \text{ or } \mathrm{co}(Te_j) = [0, b] = \mathrm{co}(Tf),$$

that is $Tf \sim_c Te_j$, but $f \nsim_c e_j$. This implies that $T \notin \mathcal{P}_{se}$. □

Notice that whenever I is an uncountable set, then T may contain a zero row.
In the next theorem, we characterize the elements of $\mathcal{P}_{se}$.

Theorem 4.9. *Let I be an infinite set. Then $\mathcal{P}_{se} = \{\lambda T;\ \ \lambda \neq 0,\ T \in \mathcal{E}\}$.*

Proof. It is easily verified that $\{\lambda T;\ \ \lambda \neq 0,\ T \in \mathcal{E}\} \subseteq \mathcal{P}_{se}$. To prove $\mathcal{P}_{se} \subseteq \{\lambda T; \lambda \neq 0, T \in \mathcal{E}\}$, we consider the following two cases. Let I be a countable set, then by using Theorem 1.4 and Lemma 4.8, we have $\frac{1}{b}T \in \mathcal{E}$, when $a = 0 < b$, and $\frac{1}{a}T \in \mathcal{E}$, whenever $a < 0 = b$.
Now, let I be an uncountable set, then the assertion follows by using Theorem 29[3], Lemmas 4.7 and 4.8. □

Theorem 4.10. *Let I be an infinite set. Then $\mathcal{P}_{sc} = \mathcal{P}_{se}$.*

Proof. It is easily seen that $\mathcal{P}_{sc} \subseteq \mathcal{P}_{se}$. Now let $T \in \mathcal{P}_{se}$. Hence Theorem 4.9 yields $T = \lambda T_1$, for some $\lambda \neq 0$, and $T_1 \in \mathcal{E}$. So

$$\mathrm{co}(Tf) = \mathrm{co}\left(\lambda T_1(f)\right) = \lambda\, \mathrm{co}(T_1(f)) = \lambda\, \mathrm{co}(f).$$

Therefore $f \prec_c g$ if and only if $\mathrm{co}(Tf) = \lambda\, \mathrm{co}(f) \subseteq \lambda\, \mathrm{co}(g) = \mathrm{co}(Tg)$. That is $T \in \mathcal{P}_{sc}$. □

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