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On Decompositions via Generalized Closedness in Ideal Spaces

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Abstract. The aim of the present paper is to introduce and study the notions of $\mathcal{RP}_I$-sets, $\mathcal{RPC}_I$-sets and $\mathcal{RC}_I$-sets. Properties of $\mathcal{RP}_I$-sets, $\mathcal{RPC}_I$-sets and $\mathcal{RC}_I$-sets are investigated. Also, various decompositions in ideal spaces are established via generalized closedness with $\mathcal{RP}_I$-sets, $\mathcal{RPC}_I$-sets and $\mathcal{RC}_I$-sets.

1. Introduction and Preliminaries

Recently, the notions of weakly I_{rg}-closed sets [4], strongly-$\mathcal{I}$-LC sets [6], pre$^*_I$-open sets [3] and I-R closed sets [1] and properties of them have been introduced and studied in the literature. In the present paper, the notions of $\mathcal{RP}_I$-sets, $\mathcal{RPC}_I$-sets and $\mathcal{RC}_I$-sets and properties of $\mathcal{RP}_I$-sets, $\mathcal{RPC}_I$-sets and $\mathcal{RC}_I$-sets are introduced and studied. Meanwhile, various decompositions in ideal spaces are established via generalized closedness with the notions of $\mathcal{RP}_I$-sets, $\mathcal{RPC}_I$-sets and $\mathcal{RC}_I$-sets.

Throughout the present paper, (X, τ) or (Y, σ) represent topological space on which no separation axioms are assumed unless explicitly stated. The closure and the interior of a subset T of a topological space X will be denoted by $Cl(T)$ and $Int(T)$, respectively.

An ideal I on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (1) If $S \in I$ and $N \subset S$, then $N \in I$, (2) If $S \in I$ and $N \in I$, then $S \cup N \in I$ [8]. Let (X, τ) be a topological space with an ideal I on X. A set operator $(.)^* : P(X) \to P(X)$ where $P(X)$ is the set of all subsets of X, said to be a local function [8] of S with respect to τ and I is defined as follows:

$$S^*(I, \tau) = \{x \in X : N \cap S \notin I \text{ for each } N \in \tau(x)\}$$

where $\tau(x) = \{N \in \tau : x \in N\}$ for $S \subset X$.

A Kuratowski closure operator $Cl^*(.)$ for a topology $\tau^*(I, \tau)$, said to be the $\star$-topology and finer than τ, is defined by $Cl^*(S) = S \cup S^*(I, \tau)$ [7]. They are denoted by S^* for $S^*(I, \tau)$ and τ^* for $\tau^*(I, \tau)$. Meanwhile, (X, τ, I) is called an ideal topological space or simply an ideal space for an ideal I on X [8].

A subset T of a topological space (X, τ) is called regular open [11] (resp. regular closed [11]) if $T = Int(Cl(T))$ (resp. $T = Cl(Int(T))$).

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Definition 1.1. *A subset T of an ideal topological space (X, τ, I) is called*
(1) a strongly-$\mathcal{I}$-LC set [6] if there exist a regular open subset S and a $\star$-closed subset N of X such that $T = S \cap N$.
(2) I_g-closed [2] if $T^* \subset N$ whenever $T \subset N$ and N is an open subset of X.
(3) semi-I-open [5] if $T \subset Cl^*(Int(T))$.
(4) I_{rg}-closed [10] if $T^* \subset N$ whenever $T \subset N$ and N is a regular open subset of X.
(5) I_g-open [2] (resp. I_{rg}-open [10]) if $X \backslash T$ is an I_g-closed subset (resp. an I_{rg}-closed subset) of X.

Definition 1.2. *Let (X, τ, I) be an ideal topological space. A subset T of (X, τ, I) is called*
(1) a weakly I_{rg}-closed set [4] if $(Int(T))^* \subset N$ whenever $T \subset N$ and N is a regular open subset of X.
(2) a weakly I_{rg}-open set [4] if $X \backslash T$ is a weakly I_{rg}-closed subset of X.
(3) pre_I^*-open [3] if $T \subset Int^*(Cl(T))$.
(4) pre_I^*-closed [3] if $X \backslash T$ is a pre_I^*-open subset of X.
(5) I-R closed [1] if $T = Cl^*(Int(T))$.

Remark 1.3. *([4]) The following diagram holds for a subset T of an ideal topological space (X, τ, I):*

$$\begin{tikzcd}
\text{an } I_g\text{-closed set} \arrow[r] & \text{an } I_{rg}\text{-closed set} \arrow[r] & \text{a weakly } I_{rg}\text{-closed set} \\
\text{a } \star\text{-closed set} \arrow[u] \arrow[rr] & & \text{a } pre_I^*\text{-closed set} \arrow[u] \\
\text{an I-R closed set} \arrow[u] & &
\end{tikzcd}$$

Theorem 1.4. *([4]) The following properties are equivalent for a subset T of an ideal topological space (X, τ, I):*
(1) T is a weakly I_{rg}-closed subset of X,
(2) $Cl^*(Int(T)) \subset S$ whenever $T \subset S$ and S is a regular open subset of X.

2. Decompositions in Ideal Spaces

Definition 2.1. *A subset T of an ideal topological space (X, τ, I) is said to be an $\mathcal{RP}_I$-set if there exist a regular open subset S and a pre_I^*-closed subset N of X such that $T = S \cap N$.*

Remark 2.2. *Let (X, τ, I) be an ideal topological space and $T \subset X$. The following properties hold:*
(1) If T is a pre_I^*-closed subset of X, then T is an $\mathcal{RP}_I$-set
(2) If T is a regular open subset of X, then T is an $\mathcal{RP}_I$-set.
(3) These implications are not reversible as shown in the following example.

Example 2.3. *Suppose that $X = \{x, y, z, w\}$, $\tau = \{X, \emptyset, \{x\}, \{y, z\}, \{x, y, z\}\}$ and $I = \{\emptyset, \{x\}, \{w\}, \{x, w\}\}$. Then $T = \{y, z, w\}$ is an $\mathcal{RP}_I$-set in X but it is not a regular open subset of X. Also, $N = \{y, z\}$ is an $\mathcal{RP}_I$-set in X but it is not a pre_I^*-closed subset of X.*

Theorem 2.4. *The following properties are equivalent for a subset T of an ideal topological space (X, τ, I):*
(1) T is a pre_I^*-closed set,
(2) T is an $\mathcal{RP}_I$-set and a weakly I_{rg}-closed set.

Proof. (1) $\Rightarrow$ (2) : Let T be a pre_I^*-closed subset of X. Since T is a pre_I^*-closed set, by Remark 1.3 and 2.2, T is an $\mathcal{RP}_I$-set and a weakly I_{rg}-closed subset of X.

(2) $\Rightarrow$ (1) : Let T be an $\mathcal{RP}_I$-set and a weakly I_{rg}-closed subset of X. Since T is an $\mathcal{RP}_I$-set, it follows that there exist a regular open subset S and a pre_I^*-closed subset N of X such that $T = S \cap N$. We have $T \subset S$. Since T is a weakly I_{rg}-closed subset of X, then $(Int(T))^* \subset S$. Meanwhile, we have $T \subset N$. Since N is a pre_I^*-closed set, then $Cl^*(Int(T)) \subset N$. This implies that $Cl^*(Int(T)) \subset S \cap N = T$. Thus, T is a pre_I^*-closed subset of X. □

Definition 2.5. *A subset T of an ideal topological space (X, τ, I) is said to be*
(1) pre_I^*-clopen if T is a pre_I^*-open subset and a pre_I^*-closed subset of X.
(2) an $\mathcal{RPC}_I$-set if there exist a regular open subset S and a pre_I^*-clopen subset N of X such that $T = S \cap N$.

Theorem 2.6. *Let (X, τ, I) be an ideal topological space and $T \subset X$. If T is an $\mathcal{RPC}_I$-set in X, then T is a pre_I^*-open subset of X.*

Proof. Let T be an $\mathcal{RPC}_I$-set in X. This implies that there exist a regular open subset S and a pre_I^*-clopen subset N of X such that $T = S \cap N$. We have

$$\begin{aligned} T &= S \cap N \\ &\subset S \cap Int^*(Cl(N)) \\ &= Int^*(S \cap Cl(N)) \\ &\subset Int^*(Cl(S \cap N)) \\ &= Int^*(Cl(T)). \end{aligned}$$

It follows that $T \subset Int^*(Cl(T))$. Thus, $T = S \cap N$ is a pre_I^*-open subset of X. □

Remark 2.7. *Theorem 2.6 is not reversible as shown in the following example.*

Example 2.8. *Let $X = \{x, y, z, w\}$, $\tau = \{X, \emptyset, \{x\}, \{y, z\}, \{x, y, z\}\}$ and $I = \{\emptyset, \{x\}, \{w\}, \{x, w\}\}$. Then $T = \{x, y, z\}$ is a pre_I^*-open subset of X but it is not an $\mathcal{RPC}_I$-set in X.*

Remark 2.9. *Let (X, τ, I) be an ideal topological space and $T \subset X$. The following properties hold:*
(1) If T is a regular open subset of X, then T is an $\mathcal{RPC}_I$-set.
(2) If T is a pre_I^*-clopen subset of X, then T is an $\mathcal{RPC}_I$-set.
(3) These implications are not reversible as shown in the following example.

Example 2.10. *Suppose that $X = \{x, y, z, w\}$, $\tau = \{X, \emptyset, \{x\}, \{y, z\}, \{x, y, z\}\}$ and $I = \{\emptyset, \{x\}, \{w\}, \{x, w\}\}$. Then $T = \{x, y, w\}$ is an $\mathcal{RPC}_I$-set in X but it is not a regular open subset of X. Meanwhile, $N = \{y, z\}$ is an $\mathcal{RPC}_I$-set in X but it is not a pre_I^*-clopen subset of X.*

Remark 2.11. *Let (X, τ, I) be an ideal topological space and $T \subset X$. The following diagram holds for T by Remark 2.9 and Theorem 2.6:*

$$\begin{array}{ccc} & & \text{a } pre_I^*\text{-open set} \\ & & \uparrow \\ \text{a } pre_I^*\text{-clopen set} & \longrightarrow & \text{an } \mathcal{RPC}_I\text{-set} \\ & & \uparrow \\ & & \text{a regular open set} \end{array}$$

Theorem 2.12. *The following properties are equivalent for a subset T of an ideal topological space (X, τ, I):*
(1) T is a pre_I^*-clopen subset of X,
(2) T is an $\mathcal{RPC}_I$-set and a pre_I^*-closed subset of X,
(3) T is an $\mathcal{RPC}_I$-set and a weakly I_{rg}-closed subset of X.

Proof. (1) ⇒ (2) : Let T be a pre_I^*-clopen subset of X. By Remark 2.9, T is an $\mathcal{RPC}_I$-set and also a pre_I^*-closed subset of X.

(2) ⇒ (3) : Let T be an $\mathcal{RPC}_I$-set and a pre_I^*-closed subset of X. By Remark 1.3, T is a weakly I_{rg}-closed subset of X.

(3) ⇒ (1) : Let T be an $\mathcal{RPC}_I$-set and a weakly I_{rg}-closed subset of X. Since T is an $\mathcal{RPC}_I$-set, then there exist a regular open subset S of X and a pre$_I^*$-clopen subset N of X such that $T = S \cap N$. This implies that T is an $\mathcal{RP}_I$-set in X. Since T is an $\mathcal{RP}_I$-set and a weakly I_{rg}-closed subset of X, by Theorem 2.4, T is a pre$_I^*$-closed subset of X. On the other hand, since T is an $\mathcal{RPC}_I$-set in X, it follows from Theorem 2.6 that T is a pre$_I^*$-open subset of X. Thus, T is a pre$_I^*$-clopen subset of X. □

Definition 2.13. *A subset T of an ideal topological space (X, τ, I) is said to be an $\mathcal{RC}_I$-set if there exist a regular open subset S and an I-R closed subset N of X such that $T = S \cap N$.*

Remark 2.14. *Let (X, τ, I) be an ideal topological space and $T \subset X$. The following properties hold:*
(1) If T is an I-R closed subset of X, then T is an $\mathcal{RC}_I$-set
(2) If T is a regular open subset of X, then T is an $\mathcal{RC}_I$-set.
(3) These implications are not reversible as shown in the following example.

Example 2.15. *Let $X = \{x, y, z, w\}$, $\tau = \{X, \varnothing, \{x\}, \{y, z\}, \{x, y, z\}\}$ and $I = \{\varnothing, \{x\}, \{w\}, \{x, w\}\}$. Then $T = \{y, z\}$ is an $\mathcal{RC}_I$-set in X but it is not an I-R closed subset of X. Also, $N = \{y, z, w\}$ is an $\mathcal{RC}_I$-set in X but it is not a regular open subset of X.*

Remark 2.16. *(1) The following diagram holds for a subset T of an ideal topological space (X, τ, I):*

$$\begin{array}{ccc} \text{an } \mathcal{RC}_I\text{-set} & \longrightarrow & \text{a strongly-}\mathcal{I}\text{-LC set} \\ \downarrow & \swarrow & \\ \text{an } \mathcal{RP}_I\text{-set} & & \\ \uparrow & & \\ \text{an } \mathcal{RPC}_I\text{-set} & & \end{array}$$

(2) These implications are not reversible as shown in the following example.

Example 2.17. *Suppose that $X = \{x, y, z, w\}$, $\tau = \{X, \varnothing, \{x\}, \{y, z\}, \{x, y, z\}\}$ and $I = \{\varnothing, \{x\}, \{w\}, \{x, w\}\}$. Then $T = \{x, w\}$ is both a strongly-$\mathcal{I}$-LC set and an $\mathcal{RP}_I$-set in X but it is neither an $\mathcal{RPC}_I$-set nor an $\mathcal{RC}_I$-set in X. Meanwhile, $N = \{x, y, w\}$ is an $\mathcal{RP}_I$-set in X but it is not a strongly-$\mathcal{I}$-LC subset of X. The set $S = \{x, z, w\}$ is an $\mathcal{RPC}_I$-set in X but it is not a strongly-$\mathcal{I}$-LC subset of X.*

Theorem 2.18. *The following properties are equivalent for a subset T of an ideal topological space (X, τ, I):*
(1) T is an $\mathcal{RC}_I$-set in X,
(2) T is an $\mathcal{RP}_I$-set and a semi-I-open subset of X,
(3) For a regular open subset S of X, $T = S \cap Cl^*(Int(T))$.

Proof. (1) ⇒ (2) : Let T be an $\mathcal{RC}_I$-set in X. By Remark 2.16, T is an $\mathcal{RP}_I$-set in X.

Since T is an $\mathcal{RC}_I$-set in X, then there exist a regular open subset S and a subset N of X such that $N = Cl^*(Int(N))$ and $T = S \cap N$. It follows that

$$\begin{aligned} T &= S \cap N \\ &= S \cap Cl^*(Int(N)) \\ &\subset Cl^*(S \cap Int(N)) \\ &= Cl^*(Int(S \cap N)) \\ &= Cl^*(Int(T)). \end{aligned}$$

We have $T \subset Cl^*(Int(T))$. Thus, T is a semi-I-open subset of X.

(2) ⇒ (3) : Let T be an $\mathcal{RP}_I$-set and a semi-I-open subset of X. Since T is an $\mathcal{RP}_I$-set, it follows that there exist a regular open subset S and a pre$_I^*$-closed subset N of X such that $T = S \cap N$. This implies $T \subset N$. Then

we have $Cl^*(Int(T)) \subset Cl^*(Int(N))$. Since N is a pre_I^*-closed subset of X, then we have $Cl^*(Int(N)) \subset N$. Since T is a semi-I-open subset of X, then $T \subset Cl^*(Int(T))$. We have

$$\begin{aligned} T &= T \cap Cl^*(Int(T)) \\ &= S \cap N \cap Cl^*(Int(T)) \\ &= S \cap Cl^*(Int(T)). \end{aligned}$$

Thus, for a regular open subset S of X, we have $T = S \cap Cl^*(Int(T))$.

(3) $\Rightarrow$ (1) : Let $T = S \cap Cl^*(Int(T))$ for a regular open subset S of X. We have $Cl^*(Int(T)) = Cl^*(Int(Cl^*(Int(T))))$. Consequently, T is an $\mathcal{RC}_I$-set in X. □

Definition 2.19. *Let (X, τ, I) be an ideal topological space and $T \subset X$. The pre_I^*-closure of T is defined by the intersection of all pre_I^*-closed sets of X containing T and is denoted by $p_I^*Cl(T)$.*

Theorem 2.20. *The following properties are equivalent for a subset T of an ideal topological space (X, τ, I):*
(1) *T is an $\mathcal{RP}_I$-set in X,*
(2) *For a regular open subset S of X, $T = S \cap p_I^*Cl(T)$*

Proof. (1) $\Rightarrow$ (2) : Let T be an $\mathcal{RP}_I$-set in X. This implies that there exist a regular open subset S and a pre_I^*-closed subset N of X such that $T = S \cap N$. We have $T \subset N$. This implies $T \subset p_I^*Cl(T) \subset N$. Consequently, we have

$$T = T \cap p_I^*Cl(T) = S \cap N \cap p_I^*Cl(T) = S \cap p_I^*Cl(T).$$

Thus, $T = S \cap p_I^*Cl(T)$ for a regular open subset S of X.

(2) $\Rightarrow$ (1) : Let $T = S \cap p_I^*Cl(T)$ for a regular open subset S of X. We have $p_I^*Cl(T) \subset N$, for any pre_I^*-closed set N containing T. This implies

$$Cl^*(Int(p_I^*Cl(T))) \subset Cl^*(Int(N)) \subset N.$$

It follows that $Cl^*(Int(p_I^*Cl(T))) \subset \cap\{N : T \subset N, N \text{ is } pre_I^*\text{-closed}\}$. Consequently, $Cl^*(Int(p_I^*Cl(T))) \subset p_I^*Cl(T)$. Thus, $p_I^*Cl(T)$ is a pre_I^*-closed subset of X and hence T is an $\mathcal{RP}_I$-set in X. □

Theorem 2.21. *Suppose that (X, τ, I) is an ideal topological space and $T \subset X$. If T is an $\mathcal{RP}_I$-set in X, then $p_I^*Cl(T) \backslash T$ is a pre_I^*-closed subset of X.*

Proof. Let T be an $\mathcal{RP}_I$-set in X. This implies $T = S \cap p_I^*Cl(T)$ for a regular open subset S of X by Theorem 2.20. It follows that

$$\begin{aligned} p_I^*Cl(T) \backslash T &= p_I^*Cl(T) \backslash (S \cap p_I^*Cl(T)) = p_I^*Cl(T) \cap (X \backslash (S \cap p_I^*Cl(T))) \\ &= p_I^*Cl(T) \cap ((X \backslash S) \cup (X \backslash p_I^*Cl(T))) \\ &= (p_I^*Cl(T) \cap (X \backslash S)) \cup (p_I^*Cl(T) \cap (X \backslash p_I^*Cl(T))) \\ &= p_I^*Cl(T) \cap (X \backslash S). \end{aligned}$$

We have $p_I^*Cl(T) \backslash T = p_I^*Cl(T) \cap (X \backslash S)$. Thus, $p_I^*Cl(T) \backslash T$ is a pre_I^*-closed subset of X. □

Theorem 2.22. *The following properties are equivalent for a subset T of an ideal topological space (X, τ, I):*
(1) T is an $\mathcal{RC}_I$-set in X,
(2) T is a strongly-$\mathcal{I}$-LC set and a semi-I-open subset of X.

Proof. (1) $\Rightarrow$ (2) : Suppose that T is an $\mathcal{RC}_I$-set in X. It follows from Remark 2.16 and Theorem 2.18 that T is a strongly-$\mathcal{I}$-LC set and a semi-I-open subset of X.

(2) $\Rightarrow$ (1) : Let T be a strongly-$\mathcal{I}$-LC set and a semi-I-open subset of X. It follows from Remark 2.16 that T is an $\mathcal{RP}_I$-set in X. Since T is an $\mathcal{RP}_I$-set and a semi-I-open subset of X, by Theorem 2.18, T is an $\mathcal{RC}_I$-set in X. □

In the next two theorems, we have obtained some characterizations of the notion of *I-R* closed sets.

Theorem 2.23. *The following properties are equivalent for a subset T of an ideal topological space* (X, τ, I)*:*
(1) T is an I-R closed subset of X,
(2) T is a strongly-$\mathcal{I}$-LC set, an I_g-closed subset and a semi-I-open subset of X,
(3) T is a strongly-$\mathcal{I}$-LC set, an I_{rg}-closed subset and a semi-I-open subset of X,
(4) T is a strongly-$\mathcal{I}$-LC set, a weakly I_{rg}-closed subset and a semi-I-open subset of X,
(5) T is an $\mathcal{RP}_I$-set, a weakly I_{rg}-closed subset and a semi-I-open subset of X.

Proof. (1) ⇒ (2) : Let T be an *I-R* closed subset of X. Since T is an *I-R* closed set, then T is a ⋆-closed subset and a semi-*I*-open subset of X. This implies that T is a strongly-$\mathcal{I}$-*LC* set in X. Also, since T is *I-R* closed set, by Remark 1.3, T is an I_g-closed subset of X.

(2) ⇒ (3) ⇒ (4) : It follows from Remark 1.3.

(4) ⇒ (5) : Since T is a strongly-$\mathcal{I}$-*LC* subset of X, by Remark 2.16, T is an $\mathcal{RP}_I$-set in X.

(5) ⇒ (1) : Suppose that T is an $\mathcal{RP}_I$-set, a weakly I_{rg}-closed subset and a semi-*I*-open subset of X. Since T is a semi-*I*-open subset of X, we have $T \subset Cl^*(Int(T))$. Since T is an $\mathcal{RP}_I$-set and a weakly I_{rg}-closed subset of X, by Theorem 2.4, T is a pre$^*_I$-closed subset of X. It follows that $Cl^*(Int(T)) \subset T$. Consequently, we have $T = Cl^*(Int(T))$. Thus, T is an *I-R* closed subset of X. □

Theorem 2.24. *The following properties are equivalent for a subset T of an ideal topological space* (X, τ, I)*:*
(1) T is an I-R closed subset of X,
(2) T is an $\mathcal{RC}_I$-set and an I_g-closed subset of X.
(3) T is an $\mathcal{RC}_I$-set and an I_{rg}-closed subset of X.
(4) T is an $\mathcal{RC}_I$-set and a pre$^*_I$-closed subset of X.
(5) T is an $\mathcal{RC}_I$-set and a weakly I_{rg}-closed subset of X.

Proof. (1) ⇒ (2) : Since T is an *I-R* closed subset of X, by Remark 1.3 and 2.14, T is an $\mathcal{RC}_I$-set and an I_g-closed subset of X.

(2) ⇒ (3) ⇒ (5) : It follows from the fact that any I_g-closed subset and any I_{rg}-closed subset of X is a weakly I_{rg}-closed subset of X by Remark 1.3.

(1) ⇒ (4) : Since T is an *I-R* closed subset of X, by Remark 2.14, T is an $\mathcal{RC}_I$-set and a pre$^*_I$-closed subset of X.

(4) ⇒ (5) : By Remark 1.3, T is a weakly I_{rg}-closed subset of X.

(5) ⇒ (1) : Let T be an $\mathcal{RC}_I$-set and a weakly I_{rg}-closed subset of X. It follows from Theorem 2.18 that T is an $\mathcal{RP}_I$-set and a semi-*I*-open subset of X. This implies by Theorem 2.23 that T is an *I-R* closed subset of X. □

3. Decompositions and Continuities in Ideal Spaces

Definition 3.1. *Suppose that* (X, τ, I) *is an ideal topological space. A function* $f : (X, \tau, I) \to (Y, \sigma)$ *is called pre$^*_I$-continuous (resp. $P^*_I C$-continuous, $\mathcal{RPC}_I$-continuous, WI_{rg}-continuous, $\mathcal{RP}_I$-continuous) if* $f^{-1}(T)$ *is a pre$^*_I$-closed subset (resp. a pre$^*_I$-clopen subset, an $\mathcal{RPC}_I$-set, a weakly I_{rg}-closed subset, an $\mathcal{RP}_I$-set) of X for each closed subset T of Y.*

Theorem 3.2. *Suppose that* (X, τ, I) *is an ideal topological space and* $f : (X, \tau, I) \to (Y, \sigma)$ *is a function. Then the following properties are equivalent for f:*
(1) f is pre$^*_I$-continuous,
(3) f is $\mathcal{RP}_I$-continuous and WI_{rg}-continuous.

Proof. It follows from Theorem 2.4. □

Theorem 3.3. *Suppose that (X, τ, I) is an ideal topological space and $f : (X, \tau, I) \to (Y, \sigma)$ is a function. Then the following properties are equivalent for f:*
(1) f is P_I^*C-continuous,
(3) f is $\mathcal{RPC}_I$-continuous and pre_I^*-continuous,
(3) f is $\mathcal{RPC}_I$-continuous and WI_{rg}-continuous.

Proof. It follows from Theorem 2.12. □

Definition 3.4. *Suppose that (X, τ, I) is an ideal topological space. A function $f : (X, \tau, I) \to (Y, \sigma)$ is called*
(1) $\mathcal{RC}_I$-continuous if $f^{-1}(T)$ is an $\mathcal{RC}_I$-set in X for each closed subset T of Y.
(2) strongly-$\mathcal{I}$-LC-continuous [6] if $f^{-1}(T)$ is a strongly-$\mathcal{I}$-LC set in X for each closed subset T of Y.

Remark 3.5. *(1) Suppose that (X, τ, I) is an ideal topological space and $f : (X, \tau, I) \to (Y, \sigma)$ is a function. Then we have the following diagram for f by using the diagram in Remark 2.16 (1) and Definitions 3.1 and 3.4.*

$$\begin{tikzcd}
\mathcal{RC}_I\text{-continuous} \arrow[r] \arrow[d] & \text{strongly-}\mathcal{I}\text{-LC-continuous} \arrow[dl] \\
\mathcal{RP}_I\text{-continuous} & \\
\mathcal{RPC}_I\text{-continuous} \arrow[u] &
\end{tikzcd}$$

(2) None of these implications is reversible as shown by the following example.

Example 3.6. *Suppose that $X = \{x, y, z, w\}$, $\tau = \{X, \emptyset, \{x\}, \{y, z\}, \{x, y, z\}\}$ and $I = \{\emptyset, \{x\}, \{w\}, \{x, w\}\}$. Then the function $f : (X, \tau, I) \to (X, \tau)$, defined by $f(x) = w$, $f(y) = z$, $f(z) = y$, $f(w) = w$ is both strongly-$\mathcal{I}$-LC-continuous and $\mathcal{RP}_I$-continuous but f is neither $\mathcal{RPC}_I$-continuous nor $\mathcal{RC}_I$-continuous. The function $g : (X, \tau, I) \to (X, \tau)$, defined by $g(x) = y$, $g(y) = z$, $g(z) = x$, $g(w) = y$ is $\mathcal{RP}_I$-continuous but g is not strongly-$\mathcal{I}$-LC-continuous. The function $h : (X, \tau, I) \to (X, \tau)$, defined by $h(x) = y$, $h(y) = x$, $h(z) = z$, $h(w) = z$ is $\mathcal{RPC}_I$-continuous but h is not strongly-$\mathcal{I}$-LC-continuous.*

Definition 3.7. *Suppose that (X, τ, I) is an ideal topological space. A function $f : (X, \tau, I) \to (Y, \sigma)$ is called contra semi-I-continuous [9] (resp. IR-continuous, I_{rg}-continuous [6], I_g-continuous [6]) if $f^{-1}(T)$ is a semi-I-open subset (resp. an I-R-closed subset, an I_{rg}-closed subset, an I_g-closed subset) of X for each closed subset T of Y.*

Theorem 3.8. *Suppose that (X, τ, I) is an ideal topological space and $f : (X, \tau, I) \to (Y, \sigma)$ is a function. Then the following properties are equivalent for f:*
(1) f is $\mathcal{RC}_I$-continuous,
(2) f is strongly-$\mathcal{I}$-LC-continous and contra semi-I-continuous,
(3) f is $\mathcal{RP}_I$-continuous and contra semi-I-continuous.

Proof. It follows from Theorems 2.18 and 2.22. □

Theorem 3.9. *Let (X, τ, I) be an ideal topological space. For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following properties are equivalent:*
(1) f is IR-continuous,
(2) f is $\mathcal{RC}_I$-continuous and I_g-continuous,
(3) f is $\mathcal{RC}_I$-continuous and I_{rg}-continuous,
(4) f is $\mathcal{RC}_I$-continuous and pre_I^*-continuous,
(5) f is $\mathcal{RC}_I$-continuous and WI_{rg}-continuous.

Proof. It follows from Theorem 2.24. □

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