Filomat 32:2 (2018), 693–704
https://doi.org/10.2298/FIL1802693A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Bounds for Generalized Normalized δ-Casorati Curvatures for Submanifolds in Bochner Kaehler Manifold

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Abstract. In this paper, we prove sharp inequalities between the normalized scalar curvature and the generalized normalized δ-Casorati curvatures for different submanifolds in Bochner Kaehler manifold. Moreover, We also characterize submanifolds on which the equalities hold.

1. Introduction

The theory of Chen invariants, one of the most interesting research area of differential geometry was started in 1993 by Chen [8]. In the initial paper Chen established inequalities between the scalar curvature and the sectional curvature (intrinsic invariants) and the squared norm of the mean curvature (the main extrinsic invariant) of a submanifold in a real space form. The same author obtained the inequalities for submanifolds between the k-Ricci curvature, the squared mean curvature and the shape operator in the real space form with arbitrary codimension [7]. Since then different geometers proved the similar inequalities for different submanifolds and ambient spaces.

The Casorati curvature (an extrinsic invariant) of a submanifold of a Riemannian manifold introduced by Casorati defined as the normalized square length of the second fundamental form [5]. The concept of Casorati curvature extends the concept of the principal direction of a hypersurface of a Riemannian manifold [11]. The geometrical meaning and the importance of the Casorati curvature discussed by some distinguished geometers [9, 13, 16]. Therefore it attracts the attention of geometers to obtain the optimal inequalities for the Casorati curvatures of the submanifolds of different ambient spaces [1, 10, 14].

In this paper, we will study the inequalities for the generalized normalized δ-Casorati curvatures for different submanifolds of Bochner Kaehler manifold.

2. Preliminaries

Let $\mathcal{N}$ be a n-dimensional submanifold of a Bochner Kaehler manifold $\overline{\mathcal{N}}$ of dimension $2m$. Let ∇ and $\overline{\nabla}$ be the Levi-Civita connection on $\mathcal{N}$ and $\overline{\mathcal{N}}$ respectively. Let J be the complex structure on $\overline{\mathcal{N}}$. Then the Gauss and Weingarten formulas are given respectively by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1}$$

2010 *Mathematics Subject Classification*. Primary 53B05; Secondary 53B20, 53C40
Keywords. Casorati curvature, Inequalities, Bochner Kaehler manifold
Received: 03 April 2017; Revised: 27 November 2017; Accepted: 03 December 2017
Communicated by Ljubica Velimirović
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$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X Y, \tag{2}$$

for all X, Y tangent to $\mathcal{N}$ and vector field N normal to $\mathcal{N}$. Where h, $\nabla^{\perp}_X$, A_N denotes the second fundamental form, normal connection and the shape operator respectively. The second fundamental form and the shape operator are related by

$$g(h(X, Y), N) = g(A_N X, Y). \tag{3}$$

Let R be the curvature tensor of $\mathcal{N}$ and $\overline{R}$ be the curvature tensor of $\overline{\mathcal{N}}$. Then the Gauss equation is given by [8]

$$\overline{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, Z), h(Y, W)) - g(h(X, W), h(Y, Z)),$$

for any vector fields X, Y, Z, W tangent to $\mathcal{N}$.

Let $p \in \mathcal{N}$ and $\{e_1, ..., e_n\}$ be an orthonormal basis of the tangent space $T_p\mathcal{N}$ and $\{e_{n+1}, ..., e_{2m}\}$ be an orthonormal basis of $T^{\perp}_p\mathcal{N}$. We denote by $\mathcal{H}$, the mean curvature vector at p, that is

$$\mathcal{H}(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i). \tag{4}$$

Also, we set

$$h^r_{ij} = g(h(e_i, e_j), e_r), \qquad i, j \in \{1, ..., n\}, \quad r \in \{n+1, ..., 2m\}$$

and

$$||h||^2 = \sum_{i,j=1}^{n} (h(e_i, e_j), h(e_i, e_j)). \tag{5}$$

For any $p \in \mathcal{N}$ and $X \in T_p\mathcal{N}$, we put $JX = PX + FX$, where PX and FX are the tangential and normal components of JX, respectively.

We denote by

$$||P||^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j). \tag{6}$$

Definition 2.1 ([6]). *A submanifold $\mathcal{N}$ of an almost Hermitian manifold $\overline{\mathcal{N}}$ is said to be a slant submanifold if for any $p \in \mathcal{N}$ and a non zero vector $X \in T_p\mathcal{N}$, the angle between JX and X is constant, i.e., the angle does not depend on the choice of $p \in \mathcal{N}$ and $X \in T_p\mathcal{N}$. The angle $\theta \in [0, \frac{\pi}{2}]$ is called the slant angle of $\mathcal{N}$ in $\overline{\mathcal{N}}$.*

Definition 2.2 ([3]). *A differential distribution D on $\mathcal{N}$ is called a slant distribution, if for each $p \in \mathcal{N}$ and each non-zero vector $X \in D_p$, the angle $\theta_D(X)$ between JX ans X is constant and is independent of the choice of $p \in \mathcal{N}$ and $X \in D_p$.*

Definition 2.3 ([4]). *A submanifold $\mathcal{N}$ of an almost Hermitian manifold $\overline{\mathcal{N}}$ is said to be a bi-slant submanifold, if there exist two orthogonal distributions D_1 and D_2, such that (i) $T\mathcal{N}$ admits the orthogonal direct decomposition i.e $T\mathcal{N} = D_1 + D_2$. (ii) For i=1,2, D_i is the slant distribution with slant angle θ_i.*

In fact, semi-slant submanifolds, hemi-slant submanifolds, CR-submanifolds, slant submanifolds can be obtained from bi-slant submanifolds in particular. We can see the case in the following table [1]:

Table 1: Defination

S.N.	$\overline{\mathcal{N}}$	$\mathcal{N}$	D_1	D_2	θ_1	θ_2
(1)	$\overline{\mathcal{N}}$	bi-slant	slant	slant	slant angle	slant angle
(2)	$\overline{\mathcal{N}}$	semi-slant	invariant	slant	0	slant angle
(3)	$\overline{\mathcal{N}}$	hemi-slant	slant	anti-invariant	slant angle	$\frac{\pi}{2}$
(4)	$\overline{\mathcal{N}}$	CR	invariant	anti-invariant	0	$\frac{\pi}{2}$
(5)	$\overline{\mathcal{N}}$	slant	either $D_1 = 0$ or $D_2 = 0$		either $\theta_1 = \theta_2 = \theta$ or $\theta_1 = \theta_2 \neq \theta$	

Invariant and anti-invariant submanifolds are the slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively and when $0 < \theta < \frac{\pi}{2}$, then the slant submanifold is called proper slant submanifold and said to be proper bi-slant if θ_i lies between 0 and $\frac{\pi}{2}$.

If $\mathcal{N}$ is a bi-slant submanifold in a Bochner Kaehler manifold $\overline{\mathcal{N}}$, then

$$||P||^2 = \sum_{i,j}^{n} g^2(Pe_i, e_j) = 2(d_1 cos^2\theta_1 + d_2 cos^2\theta_2), \tag{7}$$

where $dimD_1 = 2d_1$ and $dimD_2 = 2d_2$.

Let $\mathcal{N}$ be a Riemannian manifold. Denote by $\mathcal{K}(\pi)$ the sectional curvature of $\mathcal{N}$ of the plane section $\pi \subset T_p\mathcal{N}, \quad p \in \mathcal{N}$. The scalar curvature τ for an orthonormal basis $\{e_1, e_2, ..., e_n\}$ of the tangent space $T_p\mathcal{N}$ at p is defined by

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j)$$

and the normalized scalar curvature ρ of $\mathcal{N}$ is defined by

$$\rho = \frac{2\tau}{n(n-1)}.$$

The curvature tensor $\overline{R}$ of a Bochner Kaehler manifold $\overline{\mathcal{N}}$ is given by [15]

$$\begin{aligned} \overline{R}(X, Y, Z, W) &= L(Y, Z)g(X, W) - L(X, Z)g(Y, W) + L(X, W)g(Y, Z) - L(Y, W)g(X, Z) \\ &\quad + M(Y, Z)g(JX, W) - M(X, Z)g(JY, W) + M(X, W)g(JY, Z) \\ &\quad - M(Y, W)g(JX, Z) - 2M(X, Y)g(JZ, W) - 2M(Z, W)g(JX, Y), \end{aligned} \tag{8}$$

where

$$L(Y, Z) = \frac{1}{2m+4}\overline{Ric}(Y, Z) - \frac{\overline{\rho}}{2(2m+2)(2m+4)}g(Y, Z), \tag{9}$$

$$M(Y, Z) = -L(Y, JZ), \tag{10}$$

$$L(Y, Z) = L(Z, Y), \quad L(Y, Z) = L(JY, JZ), \quad L(Y, JZ) = -L(JY, Z), \tag{11}$$

where $\overline{Ric}$ and $\overline{\rho}$ are the Ricci tensor and scalar curvature of $\overline{\mathcal{N}}$.

The norm of the squared mean curvature of the submanifold is defined by

$$\|\mathcal{H}\|^2 = \frac{1}{n^2}\sum_{\gamma=n+1}^{2m}\left(\sum_{i=1}^{n} h_{ii}^{\gamma}\right)^2 \tag{12}$$

and the squared norm of second fundamental form h is denoted by C defined as

$$C = \frac{1}{n}\sum_{\gamma=n+1}^{2m}\sum_{i,j=1}^{n}\left(h_{ij}^{\gamma}\right)^2 \tag{13}$$

known as Casorati curvature of the submanifold.

If we suppose that Π is an r-dimensional subspace of $T\mathcal{N}$, $r \geq 2$, and $\{e_1, e_2, \ldots, e_r\}$ is an orthonormal basis of Π. then the scalar curvature of the r-plane section Π is given as

$$\tau(\Pi) = \sum_{1\leq\gamma<\beta\leq r} K(e_\gamma \wedge e_\beta)$$

and the Casorati curvature C of the subspace Π is as follows

$$C(\Pi) = \frac{1}{r}\sum_{\gamma=n+1}^{2m}\sum_{i,j=1}^{n}\left(h_{ij}^{\gamma}\right)^2$$

A point $p \in \mathcal{N}$ is said to be an *invariantly quasi-umbilical point* if there exist $2m - n$ mutually orthogonal unit normal vectors $\xi_{n+1}, \ldots, \xi_{2m}$ such that the shape operators with respect to all the directions ξ_γ have an eigenvalue of multiplicity $n - 1$ and that for each ξ_γ the distinguished eigen direction is the same. The submanifold is said to be an *invariantly quasi-umbilical submanifold* if each of its points is an invariantly quasi-umbilical point.

The normalized δ-Casorati curvature $\delta_c(n-1)$ and $\widehat{\delta_c}(n-1)$ are defined as

$$[\delta_c(n-1)]_p = \frac{1}{2}C_p + \frac{n+1}{2n} inf\{C(\Pi)|\Pi : \text{a hyperplane of } T_p\mathcal{N}\} \tag{14}$$

and

$$[\widehat{\delta_c}(n-1)]_p = 2C_p + \frac{2n-1}{2n} sup\{C(\Pi)|\Pi : \text{a hyperplane of } T_p\mathcal{N}\}. \tag{15}$$

For a positive real number $t \neq n(n-1)$, put

$$b(t) = \frac{1}{nt}(n-1)(n+t)(n^2-n-t) \tag{16}$$

then the generalized normalized δ-Casorati curvatures $\delta_c(t; n-1)$ and $\widehat{\delta_c}(t; n-1)$ are given as

$$[\delta_c(t; n-1)]_p = tC_p + b(t) inf\{C(\Pi)|\Pi : \text{a hyperplane of } T_p\mathcal{N}\}$$

if $0 < t < n^2 - n$, and

$$[\widehat{\delta_c}(t; n-1)]_p = rC_p + b(t) sup\{C(\Pi)|\Pi : \text{a hyperplane of } T_p\mathcal{N}\}.$$

if $t > n^2 - n$.

3. Main Theorem

Theorem 3.1. *Let $\mathcal{N}$ be a submanifold of a Bochner Kaehler manifold $\widetilde{\mathcal{N}}$. Then*

(i) The generalized normalized δ-Casorati curvature $\delta_c(t; n-1)$ satisfies

$$\begin{aligned}\rho \quad \leq \quad & \frac{\delta_c(t;n-1)}{n(n-1)} + \frac{4mn+5n-n^2-3\|P\|^2}{n(n-1)(2m+2)(2m+4)}\overline{\rho} \\ & -\frac{1}{n(n-1)(m+2)}\sum_{i,j=1}^{n}\overline{Ric}(e_i,e_j)g(e_i,e_j) + \frac{3}{n(n-1)(m+2)}\sum_{i,j=1}^{n}\overline{Ric}(e_i,Je_j)g(e_i,Je_j) \qquad (17)\end{aligned}$$

for any real number t such that $0 < t < n(n-1)$.

(ii) The generalized normalized δ-Casorati curvature $\widehat{\delta_c}(t; n-1)$ satisfies

$$\begin{aligned}\rho \quad \leq \quad & \frac{\widehat{\delta_c}(t;n-1)}{n(n-1)} + \frac{4mn+5n-n^2-3\|P\|^2}{n(n-1)(2m+2)(2m+4)}\overline{\rho} \\ & -\frac{1}{n(n-1)(m+2)}\sum_{i,j=1}^{n}\overline{Ric}(e_i,e_j)g(e_i,e_j) + \frac{3}{n(n-1)(m+2)}\sum_{i,j=1}^{n}\overline{Ric}(e_i,Je_j)g(e_i,Je_j) \qquad (18)\end{aligned}$$

for any real number $t > n(n-1)$. Moreover , the equality holds in (17) and (18) if and only if $\mathcal{N}$ is an invariantly quasi-umbilical submanifold with trivial normal connection in $\widetilde{\mathcal{N}}$, such that with respect to suitable tangent orthonormal frame $\{e_1, \ldots, e_n\}$ and normal orthonormal frame $\{e_{n+1}, \ldots, e_{2m}\}$, the shape operator $A_r \equiv A_{e_r}$, $r \in \{n+1, \ldots, 2m\}$, take the following form

$$A_{n+1} = \begin{pmatrix} b & 0 & 0 & \ldots & 0 & 0 \\ 0 & b & 0 & \ldots & 0 & 0 \\ 0 & 0 & b & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & b & 0 \\ 0 & 0 & 0 & \ldots & 0 & \frac{n(n-1)}{t}b \end{pmatrix}, A_{n+2} = \cdots = A_{2m} = 0. \qquad (19)$$

Proof. Let $\{e_1, \ldots, e_n\}$ and $\{e_{n+1}, \ldots, e_{2m}\}$ be an orthonormal basis of $T_p\mathcal{N}$ and $T_p^{\perp}\mathcal{N}$ respectively at any point $p \in \mathcal{N}$. Putting $X = W = e_i$, $Y = Z = e_j$ in (8) and take $i \neq j$, we have

$$\begin{aligned}\overline{R}(e_i,e_j,e_j,e_i) \quad = \quad & L(e_j,e_j)g(e_i,e_i) - L(e_i,e_j)g(e_j,e_i) + L(e_i,e_i)g(e_j,e_j) - L(e_j,e_i)g(e_i,e_j) \\ & +M(e_j,e_j)g(Je_i,e_i) - M(e_i,e_j)g(Je_j,e_i) + M(e_i,e_i)g(Je_j,e_j) \\ & -M(e_j,e_i)g(Je_i,e_j) - 2M(e_i,e_j)(Je_j,e_i) - 2M(e_j,e_i)g(Je_i,e_j) \qquad (20)\end{aligned}$$

From Gauss equation and (20), we have

$$\begin{aligned}R(e_i,e_j,e_j,e_i) \quad = \quad & L(e_j,e_j)g(e_i,e_i) - L(e_i,e_j)g(e_j,e_i) + L(e_i,e_i)g(e_j,e_j) - L(e_j,e_i)g(e_i,e_j) \\ & +M(e_j,e_j)g(Je_i,e_i) - M(e_i,e_j)g(Je_j,e_i) + M(e_i,e_i)g(Je_j,e_j) \\ & -M(e_j,e_i)g(Je_i,e_j) - 2M(e_i,e_j)(Je_j,e_i) - 2M(e_j,e_i)g(Je_i,e_j) \\ & +g(h(e_j,e_j),h(e_i,e_i)) - g(h(e_i,e_j),h(e_j,e_i)). \qquad (21)\end{aligned}$$

By taking summation $1 \leq i, j \leq n$ and using (10) and (11) in (21), we obtain

$$\begin{aligned} 2\tau &= 2n\sum_{i=1}^{n} L(e_i,e_i) - 2\sum_{i,j=1}^{n} L(e_i,e_j)g(e_i,e_j) + 6\sum_{i,j=1}^{n} L(e_i,Je_j)g(e_i,Je_j) \\ &\quad + \sum_{i,j=1}^{n} g(h(e_i,e_i),h(e_j,e_j)) - \sum_{i,j=1}^{n} g(h(e_i,e_j),h(e_j,e_i)). \end{aligned} \tag{22}$$

Now, combining (22) and (9), we get

$$\begin{aligned} 2\tau &= n^2\|\mathcal{H}\|^2 - nC + \frac{4mn+5n-n^2-3\|P\|^2}{(2m+2)(2m+4)}\overline{\rho} \\ &\quad - \frac{1}{m+2}\sum_{i,j=1}^{n} \overline{Ric}(e_i,e_j)g(e_i,e_j) + \frac{3}{m+2}\sum_{i,j=1}^{n} \overline{Ric}(e_i,Je_j)g(e_i,Je_j). \end{aligned} \tag{23}$$

Define the following function, denoted by Q, a quadratic polynomial in the components of the second fundamental form

$$\begin{aligned} Q &= tC + b(t)C(\Pi) - 2\tau + \frac{4mn+5n-n^2-3\|P\|^2}{(2m+2)(2m+4)}\overline{\rho} \\ &\quad - \frac{1}{m+2}\sum_{i,j=1}^{n} \overline{Ric}(e_i,e_j)g(e_i,e_j) + \frac{3}{m+2}\sum_{i,j=1}^{n} \overline{Ric}(e_i,Je_j)g(e_i,Je_j), \end{aligned} \tag{24}$$

where Π is the hyperplane of $T_p\mathcal{N}$. Without loss of generality, we suppose that Π is spanned by $e_1,\ldots,e_{n-1}$, it follows from (24) that

$$Q = \frac{n+t}{n}\sum_{\gamma=n+1}^{2m}\sum_{i,j=1}^{n}(h_{ij}^{\gamma})^2 + \frac{b(t)}{n-1}\sum_{\gamma=n+1}^{2m}\sum_{i,j=1}^{n-1}(h_{ij}^{\gamma})^2 - \sum_{\gamma=n+1}^{2m}\left(\sum_{i=1}^{n} h_{ii}^{\gamma}\right)^2$$

which can be easily written as

$$\begin{aligned} Q &= \sum_{\gamma=n+1}^{2m}\sum_{i=1}^{n-1}\left[\left(\frac{n+t}{n} + \frac{b(t)}{n-1}\right)(h_{ii}^{\gamma})^2 + \frac{2(n+t)}{n}(h_{in}^{\gamma})^2\right] \\ &\quad + \sum_{\gamma=n+1}^{2m}\left[2\left(\frac{n+t}{n} + \frac{b(t)}{n-1}\right)\sum_{(i<j)=1}^{n}(h_{ij}^{\gamma})^2 - 2\sum_{(i<j)=1}^{n} h_{ii}^{\gamma}h_{jj}^{\gamma} + \frac{t}{n}(h_{nn}^{\gamma})^2\right]. \end{aligned} \tag{25}$$

From(25), we can see that the critical points

$$h^c = (h_{11}^{n+1}, h_{12}^{n+1}, \ldots, h_{nn}^{n+1}, \ldots, h_{11}^{2m}, \ldots, h_{nn}^{2m})$$

of Q are the solutions of the following system of homogenous equations:

$$\begin{cases} \frac{\partial Q}{\partial h_{ii}^{\gamma}} = 2\left(\frac{n+t}{n} + \frac{b(t)}{n-1}\right)(h_{ii}^{\gamma}) - 2\sum_{k=1}^{n} h_{kk}^{\gamma} = 0 \\ \frac{\partial Q}{\partial h_{nn}^{\gamma}} = \frac{2t}{n}h_{nn}^{\gamma} - 2\sum_{k=1}^{n-1} h_{kk}^{\gamma} = 0 \\ \frac{\partial Q}{\partial h_{ij}^{\gamma}} = 4\left(\frac{n+t}{n} + \frac{b(t)}{n-1}\right)(h_{ij}^{\gamma}) = 0 \\ \frac{\partial Q}{\partial h_{in}^{\gamma}} = 4(\frac{n+t}{n})(h_{in}^{\gamma}) = 0, \end{cases} \tag{26}$$

where $i, j = \{1, 2, \ldots, n-1\}, i \neq j$, and $\gamma \in \{n+1, n+2, \ldots, 2m\}$.

Hence, from (26) every solution h^c has $h^{\gamma}_{ij} = 0$ for $i \neq j$ and the corresponding determinant to the first two equations of the above system is zero. Moreover, the Hessian matrix of $\mathcal{Q}$ is of the following form

$$\mathbb{H}(\mathcal{Q}) = \begin{pmatrix} H_1 & O & O \\ O & H_2 & O \\ O & O & H_3 \end{pmatrix}$$

where

$$H_1 = \begin{pmatrix} 2\left(\frac{n+t}{n} + \frac{b(t)}{n-1}\right) - 2 & -2 & \ldots & -2 & -2 \\ -2 & 2\left(\frac{n+t}{n} + \frac{b(t)}{n-1}\right) - 2 & \ldots & -2 & -2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & \ldots & 2\left(\frac{n+t}{n} + \frac{b(t)}{n-1}\right) - 2 & -2 \\ -2 & -2 & \ldots & -2 & \frac{2t}{n} \end{pmatrix}$$

and H_2 and H_3 are the next diagonal matrices and O is the null matrix of the respective dimensions. H_2 and H_3 are respectively given as

$$H_2 = diag\left(4\left(\frac{n+t}{n} + \frac{b(t)}{n-1}\right), 4\left(\frac{n+t}{n} + \frac{b(t)}{n-1}\right), \ldots, 4\left(\frac{n+t}{n} + \frac{b(t)}{n-1}\right)\right)$$

and

$$H_3 = diag\left(\frac{4(n+t)}{n}, \frac{4(n+t)}{n}, \ldots, \frac{4(n+t)}{n}\right).$$

Therefore we find that $\mathbb{H}(\mathcal{Q})$ has the following eigenvalues

$$\lambda_{11} = 0,\ \lambda_{22} = 2\left(\frac{2t}{n} + \frac{b(t)}{n-1}\right),\ \lambda_{33} = \cdots = \lambda_{nn} = 2\left(\frac{n+t}{n} + \frac{b(t)}{n-1}\right),$$

$$\lambda_{ij} = 4\left(\frac{n+t}{n} + \frac{b(t)}{n-1}\right), \lambda_{in} = \lambda_{ni} = \frac{4(n+t)}{n},\ \forall\, i, j \in \{1, 2, \ldots, n-1\},\ i \neq j.$$

Thus, $\mathcal{Q}$ is parabolic and reaches a minimum $\mathcal{Q}(h^c) = 0$ for the solution h^c of the system (26). Hence $\mathcal{Q} \geq 0$ and hence

$$\begin{aligned} 2\tau \quad \leq \quad & tC + b(t)C(\Pi) + \frac{4mn + 5n - n^2 - 3||P||^2)}{(2m+2)(2m+4)}\overline{\rho} \\ & -\frac{1}{m+2}\sum_{i,j=1}^{n} \overline{Ric}(e_i, e_j)g(e_i, e_j) + \frac{3}{m+2}\sum_{i,j=1}^{n} \overline{Ric}(e_i, Je_j)g(e_i, Je_j) \end{aligned}$$

whereby, we obtain

$$\begin{aligned} \rho \quad \leq \quad & \frac{t}{n(n-1)}C + \frac{b(t)}{n(n-1)}C(\Pi) + \frac{4mn + 5n - n^2 - 3||P||^2)}{n(n-1)(2m+2)(2m+4)}\overline{\rho} \\ & -\frac{1}{n(n-1)(m+2)}\sum_{i,j=1}^{n} \overline{Ric}(e_i, e_j)g(e_i, e_j) + \frac{3}{n(n-1)(m+2)}\sum_{i,j=1}^{n} \overline{Ric}(e_i, Je_j)g(e_i, Je_j), \end{aligned}$$

for every tangent hyperplane Π of $\mathcal{N}$. If we take the infimum over all tangent hyperplanes Π, the result trivially follows. Moreover the equality sign holds if and only if

$$h^{\gamma}_{ij} = 0,\ \forall\, i, j \in \{1, \ldots, n\},\ i \neq j \text{ and } \gamma \in \{n+1, \ldots, 2m\} \tag{27}$$

and

$$h^{\gamma}_{nn} = \frac{n(n-1)}{t}h^{\gamma}_{11} = \cdots = \frac{n(n-1)}{t}h^{\gamma}_{n-1n-1}, \forall \gamma \in \{n+1,\ldots,2m\} \tag{28}$$

From (27) and (28), we obtain that the equality holds if and only if the submanifold is invariantly quasi-umbilical with normal connections in $\widetilde{\mathcal{N}}$, such that the shape operator takes the form (19) with respect to the orthonormal tangent and orthonormal normal frames.

In the same way, we can prove (ii).

□

Corollary 3.2. *Let $\mathcal{N}$ be a submanifold of a Bochner Kaehler manifold $\widetilde{\mathcal{N}}$. Then*

(i) The normalized δ-Casorati curvature $\delta_c(n-1)$ satisfies

$$\begin{aligned}\rho \quad \leq \quad & \delta_c(n-1) + \frac{4mn+5n-n^2-3\|P\|^2)}{n(n-1)(2m+2)(2m+4)}\overline{\rho} \\ & -\frac{1}{n(n-1)(m+2)}\sum_{i,j=1}^{n}\overline{Ric}(e_i,e_j)g(e_i,e_j) + \frac{3}{n(n-1)(m+2)}\sum_{i,j=1}^{n}\overline{Ric}(e_i,Je_j)g(e_i,Je_j)\end{aligned}$$

Moreover, the equality sign holds if and only if $\mathcal{N}$ is an invariantly quasi-umbilical submanifold with trivial normal connection in $\widetilde{\mathcal{N}}$, such that with respect to suitable tangent orthonormal frame $\{e_1,\ldots,e_n\}$ and normal orthonormal frame $\{e_{n+1},\ldots,e_{2m}\}$, the shape operator $A_r \equiv A_{e_r}$, $r \in \{n+1,\ldots,2m\}$, take the following form

$$A_{n+1} = \begin{pmatrix} b & 0 & 0 & \ldots & 0 & 0 \\ 0 & b & 0 & \ldots & 0 & 0 \\ 0 & 0 & b & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & b & 0 \\ 0 & 0 & 0 & \ldots & 0 & 2b \end{pmatrix}, A_{n+2} = \cdots = A_{2m} = 0.$$

(ii) The normalized δ-Casorati curvature $\widehat{\delta_c}(n-1)$ satisfies

$$\begin{aligned}\rho \quad \leq \quad & \widehat{\delta_c}(n-1) + \frac{4mn+5n-n^2-3\|P\|^2)}{n(n-1)(2m+2)(2m+4)}\overline{\rho} \\ & -\frac{1}{n(n-1)(m+2)}\sum_{i,j=1}^{n}\overline{Ric}(e_i,e_j)g(e_i,e_j) + \frac{3}{n(n-1)(m+2)}\sum_{i,j=1}^{n}\overline{Ric}(e_i,Je_j)g(e_i,Je_j)\end{aligned}$$

Moreover, the equality sign holds if and only if $\mathcal{N}$ is an invariantly quasi-umbilical submanifold with trivial normal connection in $\widetilde{\mathcal{N}}$, such that with respect to suitable tangent orthonormal frame $\{e_1,\ldots,e_n\}$ a and normal orthonormal frame $\{e_{n+1},\ldots,e_{2m}\}$, the shape operator $A_r \equiv A_{e_r}$, $r \in \{n+1,\ldots,2m\}$, take the following form

$$A_{n+1} = \begin{pmatrix} 2b & 0 & 0 & \ldots & 0 & 0 \\ 0 & 2b & 0 & \ldots & 0 & 0 \\ 0 & 0 & 2b & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2b & 0 \\ 0 & 0 & 0 & \ldots & 0 & b \end{pmatrix}, A_{n+2} = \cdots = A_{2m} = 0.$$

Proof. (i) One can easily see that

$$\Big[\delta_c(\frac{n(n-1)}{2} : n-1)\Big]_p = n(n-1)\Big[\delta_c(n-1)\Big]_p, \tag{29}$$

at any point $p \in M$. Therefore, putting $t = \frac{n(n-1)}{2}$ in (17) and taking into account (29) we have our assertion.

Similarly, we obtain (ii). □

Next, we prove the following.

Theorem 3.3. *Let $\mathcal{N}$ be a bi-slant submanifold of a Bochner Kaehler manifold $\widetilde{\mathcal{N}}$. Then*

(i) The generalized normalized δ-Casorati curvature $\delta_c(t; n-1)$ satisfies

$$\begin{aligned}\rho \leq\ & \frac{\delta_c(t;n-1)}{n(n-1)} + \frac{4mn+5n-n^2-6(d_1cos^2\theta_1+d_2cos^2\theta_2)}{n(n-1)(2m+2)(2m+4)}\overline{\rho} \\ & -\frac{1}{n(n-1)(m+2)}\sum_{i,j=1}^{n}\overline{Ric}(e_i,e_j)g(e_i,e_j) + \frac{3}{n(n-1)(m+2)}\sum_{i,j=1}^{n}\overline{Ric}(e_i,Je_j)g(e_i,Je_j)\end{aligned} \tag{30}$$

for any real number t such that $0 < t < n(n-1)$.

(ii) The generalized normalized δ-Casorati curvature $\widehat{\delta_c}(t; n-1)$ satisfies

$$\begin{aligned}\rho \leq\ & \frac{\widehat{\delta_c}(t;n-1)}{n(n-1)} + \frac{4mn+5n-n^2-6(d_1cos^2\theta_1+d_2cos^2\theta_2)}{n(n-1)(2m+2)(2m+4)}\overline{\rho} \\ & -\frac{1}{n(n-1)(m+2)}\sum_{i,j=1}^{n}\overline{Ric}(e_i,e_j)g(e_i,e_j) + \frac{3}{n(n-1)(m+2)}\sum_{i,j=1}^{n}\overline{Ric}(e_i,Je_j)g(e_i,Je_j)\end{aligned} \tag{31}$$

for any real number t > $n(n-1)$. Moreover , the equality holds in (30) and (31) if and only if $\mathcal{N}$ is an invariantly quasi-umbilical submanifold with trivial normal connection in $\widetilde{\mathcal{N}}$, such that with respect to suitable tangent orthonormal frame $\{e_1, \dots, e_n\}$ and normal orthonormal frame $\{e_{n+1}, \dots, e_{2m}\}$, the shape operator $A_r \equiv A_{e_r}$, $r \in \{n+1, \dots, 2m\}$, take the following form

$$A_{n+1} = \begin{pmatrix} b & 0 & 0 & \dots & 0 & 0 \\ 0 & b & 0 & \dots & 0 & 0 \\ 0 & 0 & b & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{n(n-1)}{t}b \end{pmatrix}, A_{n+2} = \cdots = A_{2m} = 0. \tag{32}$$

Proof. Proof of the above theorem is similar to the proof of Theorem 3.1 and is obtained just using (7) in Theorem 3.1. □

Further, we state and prove the following Theorems.

Theorem 3.4. *Let $\mathcal{N}$ be a semi-slant submanifold of a Bochner Kaehler manifold $\widetilde{\mathcal{N}}$. Then* *(i) The generalized normalized δ-Casorati curvature* $\delta_c(t; n-1)$ *satisfies*

$$\rho \leq \frac{\delta_c(t;n-1)}{n(n-1)} + \frac{4mn+5n-n^2-6(d_1+d_2cos^2\theta_2)}{n(n-1)(2m+2)(2m+4)}\overline{\rho} - \frac{1}{n(n-1)(m+2)}\sum_{i,j=1}^{n}\overline{Ric}(e_i,e_j)g(e_i,e_j) + \frac{3}{n(n-1)(m+2)}\sum_{i,j=1}^{n}\overline{Ric}(e_i,Je_j)g(e_i,Je_j) \quad (33)$$

for any real number t such that $0 < t < n(n-1)$.

(ii) The generalized normalized δ-Casorati curvature $\widehat{\delta_c}(t; n-1)$ *satisfies*

$$\rho \leq \frac{\widehat{\delta_c}(t;n-1)}{n(n-1)} + \frac{4mn+5n-n^2-6(d_1+d_2cos^2\theta_2)}{n(n-1)(2m+2)(2m+4)}\overline{\rho} - \frac{1}{n(n-1)(m+2)}\sum_{i,j=1}^{n}\overline{Ric}(e_i,e_j)g(e_i,e_j) + \frac{3}{n(n-1)(m+2)}\sum_{i,j=1}^{n}\overline{Ric}(e_i,Je_j)g(e_i,Je_j) \quad (34)$$

for any real number $t > n(n-1)$. *Moreover , the equality holds in (33) and (34) if and only if* $\mathcal{N}$ *is an invariantly quasi-umbilical submanifold with trivial normal connection in* $\widetilde{\mathcal{N}}$, *such that with respect to suitable tangent orthonormal frame* $\{e_1,\ldots,e_n\}$ *and normal orthonormal frame* $\{e_{n+1},\ldots,e_{2m}\}$, *the shape operator* $A_r \equiv A_{e_r}$, $r \in \{n+1,\ldots,2m\}$, *take the following form*

$$A_{n+1} = \begin{pmatrix} b & 0 & 0 & \ldots & 0 & 0 \\ 0 & b & 0 & \ldots & 0 & 0 \\ 0 & 0 & b & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & b & 0 \\ 0 & 0 & 0 & \ldots & 0 & \frac{n(n-1)}{t}b \end{pmatrix}, A_{n+2} = \cdots = A_{2m} = 0. \quad (35)$$

Theorem 3.5. *Let* $\mathcal{N}$ *be a hemi-slant submanifold of a Bochner Kaehler manifold* $\widetilde{\mathcal{N}}$. *Then*

(i) The generalized normalized δ-Casorati curvature $\delta_c(t; n-1)$ *satisfies*

$$\rho \leq \frac{\delta_c(t;n-1)}{n(n-1)} + \frac{4mn+5n-n^2-6(d_1cos^2\theta_1)}{n(n-1)(2m+2)(2m+4)}\overline{\rho} - \frac{1}{n(n-1)(m+2)}\sum_{i,j=1}^{n}\overline{Ric}(e_i,e_j)g(e_i,e_j) + \frac{3}{n(n-1)(m+2)}\sum_{i,j=1}^{n}\overline{Ric}(e_i,Je_j)g(e_i,Je_j) \quad (36)$$

for any real number t such that $0 < t < n(n-1)$.

(ii) The generalized normalized δ-Casorati curvature $\widehat{\delta_c}(t; n-1)$ *satisfies*

$$\rho \leq \frac{\widehat{\delta_c}(t;n-1)}{n(n-1)} + \frac{4mn+5n-n^2-6(d_1cos^2\theta_1)}{n(n-1)(2m+2)(2m+4)}\overline{\rho} - \frac{1}{n(n-1)(m+2)}\sum_{i,j=1}^{n}\overline{Ric}(e_i,e_j)g(e_i,e_j) + \frac{3}{n(n-1)(m+2)}\sum_{i,j=1}^{n}\overline{Ric}(e_i,Je_j)g(e_i,Je_j) \quad (37)$$

for any real number $t > n(n-1)$. *Moreover, the equality holds in (36) and (37) if and only if* $\mathcal{N}$ *is an invariantly quasi-umbilical submanifold with trivial normal connection in* $\widetilde{\mathcal{N}}$, *such that with respect to suitable tangent orthonormal frame* $\{e_1,\ldots,e_n\}$ *and normal orthonormal frame* $\{e_{n+1},\ldots,e_{2m}\}$, *the shape operator* $A_r \equiv A_{e_r}$, $r \in \{n+1,\ldots,2m\}$, *take the following form*

$$A_{n+1} = \begin{pmatrix} b & 0 & 0 & \dots & 0 & 0 \\ 0 & b & 0 & \dots & 0 & 0 \\ 0 & 0 & b & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{n(n-1)}{t}b \end{pmatrix}, A_{n+2} = \dots = A_{2m} = 0. \tag{38}$$

Theorem 3.6. *Let N be a CR-submanifold of a Bochner Kaehler manifold $\widetilde{N}$. Then*

(i) The generalized normalized δ-Casorati curvature $\delta_c(t; n-1)$ satisfies

$$\begin{aligned} \rho \leq \; & \frac{\delta_c(t;n-1)}{n(n-1)} + \frac{4mn + 5n - n^2 - 6d_1}{n(n-1)(2m+2)(2m+4)}\overline{\rho} \\ & - \frac{1}{n(n-1)(m+2)} \sum_{i,j=1}^{n} \overline{Ric}(e_i, e_j) g(e_i, e_j + \frac{3}{n(n-1)(m+2)} \sum_{i,j=1}^{n} \overline{Ric}(e_i, Je_j) g(e_i, Je_j) \end{aligned} \tag{39}$$

for any real number t such that $0 < t < n(n-1)$.

(ii) The generalized normalized δ-Casorati curvature $\widehat{\delta_c}(t; n-1)$ satisfies

$$\begin{aligned} \rho \leq \; & \frac{\widehat{\delta_c}(t;n-1)}{n(n-1)} + \frac{4mn + 5n - n^2 - 6d_1}{n(n-1)(2m+2)(2m+4)}\overline{\rho} \\ & - \frac{1}{n(n-1)(m+2)} \sum_{i,j=1}^{n} \overline{Ric}(e_i, e_j) g(e_i, e_j) + \frac{3}{n(n-1)(m+2)} \sum_{i,j=1}^{n} \overline{Ric}(e_i, Je_j) g(e_i, Je_j) \end{aligned} \tag{40}$$

for any real number $t > n(n-1)$. Moreover , the equality holds in (39) and (40) if and only if N is an invariantly quasi-umbilical submanifold with trivial normal connection in $\widetilde{N}$, such that with respect to suitable tangent orthonormal frame $\{e_1, \dots, e_n\}$ and normal orthonormal frame $\{e_{n+1}, \dots, e_{2m}\}$, the shape operator $A_r \equiv A_{e_r}$, $r \in \{n+1, \dots, 2m\}$, take the following form

$$A_{n+1} = \begin{pmatrix} b & 0 & 0 & \dots & 0 & 0 \\ 0 & b & 0 & \dots & 0 & 0 \\ 0 & 0 & b & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{n(n-1)}{t}b \end{pmatrix}, A_{n+2} = \dots = A_{2m} = 0. \tag{41}$$

Theorem 3.7. *Let N be a slant submanifold of a Bochner Kaehler manifold $\widetilde{N}$. Then*

(i) The generalized normalized δ-Casorati curvature $\delta_c(t; n-1)$ satisfies

$$\begin{aligned} \rho \leq \; & \frac{\delta_c(t;n-1)}{n(n-1)} + \frac{4mn + 5n - n^2 - 3cos^2\theta}{n(n-1)(2m+2)(2m+4)}\overline{\rho} \\ & - \frac{1}{n(n-1)(m+2)} \sum_{i,j=1}^{n} \overline{Ric}(e_i, e_j) g(e_i, e_j) + \frac{3}{n(n-1)(m+2)} \sum_{i,j=1}^{n} \overline{Ric}(e_i, Je_j) g(e_i, Je_j) \end{aligned} \tag{42}$$

for any real number t such that $0 < t < n(n-1)$.

(ii) The generalized normalized δ-Casorati curvature $\widehat{\delta_c}(t; n-1)$ satisfies

$$\begin{aligned}\rho \quad \leq \quad & \frac{\widehat{\delta_c}(t;n-1)}{n(n-1)} + \frac{4mn+5n-n^2-3cos^2\theta}{n(n-1)(2m+2)(2m+4)}\overline{\rho} \\ & -\frac{1}{n(n-1)(m+2)}\sum_{i,j=1}^{n}\overline{Ric}(e_i,e_j)g(e_i,e_j) + \frac{3}{n(n-1)(m+2)}\sum_{i,j=1}^{n}\overline{Ric}(e_i,Je_j)g(e_i,Je_j) \end{aligned} \quad (43)$$

for any real number $t > n(n-1)$. Moreover , the equality holds in (42) and (43) if and only if N is an invariantly quasi-umbilical submanifold with trivial normal connection in $\widetilde{N}$, such that with respect to suitable tangent orthonormal frame $\{e_1, \dots, e_n\}$ and normal orthonormal frame $\{e_{n+1}, \dots, e_{2m}\}$, the shape operator $A_r \equiv A_{e_r}$, $r \in \{n+1, \dots, 2m\}$, take the following form

$$A_{n+1} = \begin{pmatrix} b & 0 & 0 & \dots & 0 & 0 \\ 0 & b & 0 & \dots & 0 & 0 \\ 0 & 0 & b & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{n(n-1)}{t}b \end{pmatrix}, A_{n+2} = \dots = A_{2m} = 0. \quad (44)$$

Remark 3.8. *We obtain proof of Theorem 3.4 - Theorem 3.7 just using Table 1 and result of Theorem 3.3.*

Acknowledgment

The authors are grateful to the referees for their constructive input.

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