



New Classification Results on Surfaces with a Canonical Principal Direction in the Minkowski 3-space

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Abstract. In this paper, we characterize and classify all surfaces endowed with canonical principal direction relative to a space-like and light-like, constant direction in the Minkowski 3-space.

1. Introduction

It is well known that, a helix is a curve whose tangent lines make a constant angle with a fixed vector. After the question ‘Are there any surface making a constant angle with some fixed vector direction?’ was introduced in [5], the concept of constant angle surfaces, called also as helix surfaces, have been studied by geometers. Specially, if one can take the position vector of the surface instead of the fixed vector, in that case, the surface is called as constant slope surface studied in [11, 13, 20]. The applications of constant angle surfaces in the theory of liquid crystals and of layered fluids were firstly considered in [1]. They used for their study of surfaces the Hamilton-Jacobi equation, correlating the surface and the direction field. Further, Munteanu and Nistor gave another approach to classify concerning surfaces in Euclidean spaces for which the unit normal makes a constant angle with a fixed direction in [21]. Moreover, the study of constant angle surfaces was extended in different ambient spaces, e.g. in $\mathbb{S}^2 \times \mathbb{R}$ [5] and $\mathbb{H}^2 \times \mathbb{R}$ [6], in \mathbb{E}_1^3 [9, 15, 17]. In higher dimensional Euclidean space, hypersurfaces whose tangent space makes constant angle with a fixed direction are studied and a local description of how these hypersurfaces are constructed is given. They are called helix hypersurfaces, [3].

One of common geometrical properties of such surfaces is the following. If we denote the projection of the fixed direction k on the tangent plane of the surface by U^T , then U^T is a principal direction of the surface with the corresponding principal curvature 0. Because of this reason, a recent natural problem appearing in the context of constant angle surfaces is to study those surfaces for which U^T remains a principal direction with the corresponding principal curvature being different from zero.

Let N, M and X be a (semi-)Riemannian manifold, a hypersurface of N and a vector field tangent to N , respectively. M is said to have a canonical principal direction (CPD) relative to X if the tangential projection of X to M gives a principal direction, [14]. One of the most common examples of hypersurfaces with CPD is

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rotational hypersurfaces in Euclidean spaces having canonical principal direction relative to a vector field parallel to its rotation axis. We also want to note that a hypersurface in different ambient spaces with CPD relative to its position vector is said to be a generalized constant ratio hypersurface, [8, 10, 12, 26].

The problem of classifying hypersurfaces with CPD relative to a fixed direction k has been studied by some authors recently. For example, in [4], this problem was studied in $\mathbb{S}^2 \times \mathbb{R}$ by Dillen et. al. Further, surfaces with CPD in $\mathbb{H}^2 \times \mathbb{R}$ was studied in [7]. In these two papers, the fixed direction k was chosen to be a unit vector tangent to the second factor. On the other hand, classification of concerning surfaces in semi-Euclidean spaces with CPD relative to a chosen direction was studied in [9, 22, 23]. Before we proceed, we also would like to note that when the codimension of the submanifold is more than one, a generalization of this notion was given by Tojeiro in [25] and a further study appear in [19].

In the present paper, we would like to move the study of CPD hypersurfaces in Euclidean spaces initiated in [22] into semi-Euclidean spaces by obtaining partial classification of CPD surfaces in Minkowski 3-space studied in [9, 23]. This paper is organized as follows. In Sect. 2, we introduce the notation that we will use and give a brief summary of basic definitions in theory of submanifolds of semi-Euclidean spaces. In Sect. 3, we obtain some new characterizations and the complete classification of space-like and Lorentzian CPD surfaces relative to a space-like and light-like, constant direction in the Minkowski 3-space.

2. CPD Hypersurfaces in Minkowski Spaces

In this section, we would like to give some basic equations and facts on hypersurfaces in Minkowski spaces, before we consider on some geometrical properties of hypersurfaces in Minkowski 3-spaces, \mathbb{E}_1^3 endowed with a canonical principal direction.

2.1. Basic Facts and Definitions

First, we would like to give a brief summary of basic definitions, facts and equations in the theory of submanifolds of pseudo-Euclidean space (see for detail, [2, 24]).

Let \mathbb{E}_1^m denotes the Minkowski m -space with the canonical Lorentzian metric tensor given by

$$\tilde{g} = \langle \cdot, \cdot \rangle = \sum_{i=1}^{m-1} dx_i^2 - dx_m^2,$$

where x_1, x_2, \dots, x_m are rectangular coordinates of the points of \mathbb{E}_1^m . We denote the Levi-Civita connection of \mathbb{E}_1^m by $\tilde{\nabla}$.

The causality of a vector in a Minkowski space is defined as following. A non-zero vector v in \mathbb{E}_1^m is said to be space-like, time-like and light-like (null) regarding to $\langle v, v \rangle > 0$, $\langle v, v \rangle < 0$ and $\langle v, v \rangle = 0$, respectively. Note that v is said to be causal if it is not space-like.

Now, let M be an oriented hypersurface in \mathbb{E}_1^{n+1} by considering the case $m = n + 1$. We denote by N and ∇ , the unit normal vector field and Levi-Civita connection of M , respectively. Note that Gauss and Weingarten formulas are given by

$$\begin{aligned}\tilde{\nabla}_X Y &= \nabla_X Y + h(X, Y), \\ \tilde{\nabla}_X N &= -S(X),\end{aligned}$$

respectively, whenever X, Y are tangent to M , where h and S are the second fundamental form and the shape operator (or Weingarten map) of M . The surface M is said to be space-like (resp. time-like) if the induced metric $g = \tilde{g}|_M$ of M is Riemannian (resp. Lorentzian). This is equivalent to being time-like (resp. space-like) of N at each point of M .

The Codazzi equations is given by

$$(\tilde{\nabla}_X h)(Y, Z) = (\tilde{\nabla}_Y h)(X, Z) \tag{1}$$

for any vector fields X, Y, Z tangent to M , where $\bar{\nabla}h$ is defined by

$$(\bar{\nabla}_X h)(Y, Z) = \nabla_X^\perp h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

If M is space-like, then its shape operator S is diagonalizable, i.e., there exists a local orthonormal frame field $\{e_1, e_2\}$ of the tangent bundle of M such that $Se_i = k_i e_i$, $i = 1, 2, \dots, n$. In this case, the vector field e_i and the smooth function k_i are called as a principal direction and a principal curvature of M , respectively.

On the other hand, if M is time-like, then by choosing an appropriated frame field of the tangent bundle of M , the shape operator S can be assumed to have one of the following four matrix representations

$$\begin{aligned} \text{Case I. } S &= \begin{pmatrix} k_1 & & & \\ & \ddots & & \\ & & & k_n \end{pmatrix}, & \text{Case II. } S &= \begin{pmatrix} k_1 & 1 & & \\ 0 & k_1 & & \\ & & k_3 & \\ & & & \ddots \\ & & & & k_n \end{pmatrix}, \\ \text{Case III. } S &= \begin{pmatrix} k_1 & \nu & & \\ -\nu & k_1 & & \\ & & k_3 & \\ & & & \ddots \\ & & & & k_n \end{pmatrix}, & \text{Case IV. } S &= \begin{pmatrix} k_1 & 1 & 1 & \\ 1 & k_1 & 1 & \\ 1 & 1 & k_1 & \\ & & & k_4 \\ & & & & \ddots \\ & & & & & k_n \end{pmatrix} \end{aligned} \tag{2}$$

for some smooth functions $k_1, k_2, \dots, k_n, \nu$ (see for example [18]). We would like to note that in Case I and Case III of (2), the frame field $\{e_1, e_2\}$ is orthonormal, i.e.

$$\langle e_1, e_1 \rangle = -1, \langle e_2, e_2 \rangle = 1, \quad \langle e_i, e_j \rangle = 0 \text{ whenever } i \neq j,$$

and it is pseudo-orthonormal in Case II and Case IV with

$$\langle e_A, e_B \rangle = \delta_{AB} - 1, \quad \langle e_1, e_A \rangle = \langle e_2, e_A \rangle = 0, \quad \langle e_i, e_j \rangle = \delta_{ij} \text{ whenever } A, B = 1, 2, i, j > 2.$$

Now, let M be a surface in the Minkowski 3-space. Then, its mean curvature and Gaussian curvature are defined by $H = \text{trace } S$ and $K = \det S$, respectively. M is said to be flat if K vanishes identically. On the other hand, if $H = 0$ and the surface M is space-like, then it is called maximal while a time-like surface with identically vanishing mean curvature is said to be a minimal surface.

Before we proceed to the next subsection, we would like to notice the notion of angle in the Minkowski 3-space (see for example, [8, 13]):

Definition 2.1. Let v and w be future pointing (past pointing) time-like vectors in \mathbb{E}_1^3 . Then, there is a unique non-negative real number θ such that

$$|\langle v, w \rangle| = \|v\| \|w\| \cosh \theta.$$

The real number θ is called the Lorentzian time-like angle between v and w .

Definition 2.2. Let v and w be a space-like vectors in \mathbb{E}_1^3 that span a space-like vector subspace. Then, we have $|\langle v, w \rangle| \leq \|v\| \|w\|$ and hence, there is a unique real number $\theta \in [0, \pi/2]$ such that

$$|\langle v, w \rangle| = \|v\| \|w\| \cos \theta.$$

The real number θ is called the Lorentzian space-like angle between v and w .

Definition 2.3. Let v and w be a space-like vectors in \mathbb{E}_1^3 that span a time-like vector subspace. Then, we have $|\langle v, w \rangle| > \|v\| \|w\|$ and hence, there is a unique positive real number θ such that

$$|\langle v, w \rangle| = \|v\| \|w\| \cosh \theta.$$

The real number θ is called the Lorentzian time-like angle between v and w .

Definition 2.4. Let v be a space-like vectors and w a future pointing time-like vector in \mathbb{E}_1^3 . Then, there is a unique non-negative real number θ such that

$$|\langle v, w \rangle| = \|v\| \|w\| \sinh \theta.$$

The real number θ is called the Lorentzian time-like angle between v and w .

2.2. A Characterization of CPD Hypersurfaces

First, we would like to recall the following definition (see for example, [9, 14, 23]).

Definition 2.5. Let M be a non-degenerated hypersurface in \mathbb{E}_1^{n+1} and ζ a vector field in \mathbb{E}_1^{n+1} . M is said to be endowed with CPD relative to ζ if its tangential component is a principal direction, i.e., $S(\zeta^T) = k_1 \zeta^T$ for a smooth function k_1 , where ζ^T denotes the tangential component of ζ . In particular if $X = k$ for a fixed direction k in \mathbb{E}_1^{n+1} , we will say that M is a CPD-hypersurface.

As we mentioned before, a surface M in \mathbb{E}^3 was said to be a constant angle surface (CAS), if its unit normal vector field makes a constant angle with a fixed vector, [21] (see also [5, 6, 9]. Later, in [15, 17], this definition is extended to surfaces in Minkowski spaces with obvious restrictions on the causality of the fixed vector and the normal vector because of the definition of ‘angle’ in the Minkowski space (see, Definition 2.1- Definition 2.4).

Remark 2.6. In fact, if the ambient space is pseudo-Euclidean, then a CAS surface is a CPD surface with corresponding principal curvature $k_1 = 0$ (see [15, 17, 21]). Thus, we will exclude this case. Therefore, after this point, we will locally assume that the principal curvature k_1 corresponding to the principal direction of tangential part of k is a non-vanishing function.

Let M be a hypersurface and k be a fixed direction in a Minkowski space \mathbb{E}_1^{n+1} . The fixed vector k can be expressed as

$$k = U + \langle N, N \rangle \langle k, N \rangle N \quad (3)$$

for a tangent vector U . We would like to give the following new characterization of CPD surfaces which is a generalization of [23, Theorem 2.1] and [9, Theorem 3.7 and Theorem 4.5] into hypersurfaces of arbitrary dimensional Minkowsk spaces.

Proposition 2.7. Let M be an oriented hypersurface in the Minkowski space \mathbb{E}_1^{n+1} and k be a fixed vector on the tangent plane to the surface. Consider a unit tangent vector field e_1 along U . Then, M is a CPD hypersurface if and only if a curve α is a geodesic of M whenever it is an integral curve of e_1 .

Proof. We will consider three cases separately subject to causality of U .

Case I. Let e_1 is time-like. Thus, we have

$$k = -\langle k, e_1 \rangle e_1 + \langle k, N \rangle N.$$

Since $\widetilde{\nabla}_{e_1} k = 0$, this equation yields

$$0 = -\langle k, \widetilde{\nabla}_{e_1} e_1 \rangle e_1 - \langle k, e_1 \rangle \widetilde{\nabla}_{e_1} e_1 - \langle k, S e_1 \rangle N - \langle k, N \rangle S e_1.$$

The tangential part of this equation yields $Se_1 = k_1e_1$ if and only if $\nabla_{e_1}e_1 = 0$ which is equivalent to being geodesic of all integral curves of e_1 .

Case II. Let e_1 is space-like. Thus, we have

$$k = \langle k, e_1 \rangle e_1 + \varepsilon \langle k, N \rangle N, \tag{4}$$

where ε is either 1 or -1 regarding to being time-like or space-like of M , respectively.

Similar to Case I, we obtain $Se_1 = k_1e_1$ if and only if $\nabla_{e_1}e_1 = 0$.

Case III. Let e_1 is light-like. In this case, k can be decompose as

$$k = \phi(e_1 - N), \tag{5}$$

for a non-constant function ϕ .

Similar to the other case, we obtain $Se_1 = k_1e_1$ if and only if $\nabla_{e_1}e_1 = 0$. \square

3. New Classifications of CPD Surfaces in \mathbb{E}_1^3

In this section, we want to complete classification of CPD surfaces in \mathbb{E}_1^3 . We would like to note that the complete classification of surfaces endowed with canonical principal direction relative to a time-like constant direction $k = (0, 0, 1)$ was obtained in [9, 23].

3.1. CPD Surfaces Relative to a Space-like, Constant Direction.

In this subsection, we consider surfaces endowed with CPD relative to a space-like, constant direction k . In this case, up to a linear isometry of \mathbb{E}_1^3 , we may assume that $k = (1, 0, 0)$.

First, we will assume that M is a space-like surface endowed with CPD relative to $k = (1, 0, 0)$. In this case, N is time-like and (3) becomes

$$k = \cosh \theta e_1 + \sinh \theta N \tag{6}$$

where θ is a smooth function. Let e_2 be a unit tangent vector field satisfying $\langle e_1, e_2 \rangle = 0$. Considering (6), we obtain the following lemma by a simple computation.

Lemma 3.1. *The Levi-Civita connection ∇ of M is given by*

$$\nabla_{e_1}e_1 = \nabla_{e_1}e_2 = 0, \tag{7a}$$

$$\nabla_{e_2}e_1 = \tanh \theta k_2 e_2, \quad \nabla_{e_2}e_2 = -\tanh \theta k_2 e_1, \tag{7b}$$

and the matrix representation shape operator S of M with respect to $\{e_1, e_2\}$ is

$$S = \begin{pmatrix} e_1(\theta) & 0 \\ 0 & k_2 \end{pmatrix} \tag{8}$$

for a function k_2 satisfying

$$e_1(k_2) = \tanh \theta k_2 (e_1(\theta) - k_2). \tag{9}$$

Furthermore, θ satisfies

$$e_2(\theta) = 0. \tag{10}$$

Proof. By considering (6), one can get

$$0 = X(\cosh \theta)e_1 + \cosh \theta \nabla_X e_1 + \cosh \theta h(e_1, X) - \sinh \theta SX + X(\sinh \theta)N \tag{11}$$

whenever X is tangent to M . (11) for $X = e_1$ gives

$$\begin{aligned} \nabla_{e_1} e_1 &= 0, \\ e_1(\theta) &= k_1, \end{aligned} \tag{12}$$

while (11) for $X = e_2$ is giving

$$\nabla_{e_2} e_1 = \tanh \theta k_2 e_2,$$

where e_2 is the other principal direction of M with the corresponding principal curvature k_2 . Thus, we have (7), (8) and (10) and so the second fundamental form of M becomes

$$h(e_1, e_1) = -k_1 N, \quad h(e_1, e_2) = 0, \quad h(e_2, e_2) = -k_2 N.$$

By considering the Codazzi equation, we obtain (9). \square

Remark 3.2. Because of (12), if $e_1(\theta) \equiv 0$ implies $k_1 = 0$. We will not consider this case because of Remark 2.6.

Now, we consider a point $p \in M$ at which $e_1(\theta)$ does not vanish. First, we would like to prove the following lemma.

Lemma 3.3. There exists a local coordinate system (s, t) defined in a neighborhood \mathcal{N}_p of p such that the induced metric of M is

$$g = ds^2 + m^2 dt^2 \tag{13}$$

for a function m satisfying

$$e_1(m) - \tanh \theta k_2 m = 0. \tag{14}$$

Furthermore, the vector fields e_1, e_2 described as above become $e_1 = \partial_s, e_2 = \frac{1}{m} \partial_t$ in \mathcal{N}_p .

Proof. Because of (7), we have $[e_1, e_2] = -\tanh \theta k_2 e_2$. Thus, if m is a non-vanishing smooth function on M satisfying (14), then we have $[e_1, m e_2] = 0$. Therefore, there exists a local coordinate system (s, t) such that $e_1 = \partial_s$ and $e_2 = \frac{1}{m} \partial_t$. Thus, the induced metric of M is as given in (13). \square

Now, we are ready to obtain the classification theorem.

Theorem 3.4. Let M be an oriented space-like surface in \mathbb{E}_1^3 . Then, M is a surface endowed with a canonical principal direction relative to a space-like constant direction if and only if it is congruent to the surface given by one of the followings:

1. A surface given by

$$x(s, t) = \int^s \cosh \theta(\tau) d\tau (1, 0, 0) + \int^t \sinh \theta(\tau) d\tau (0, \sinh t, \cosh t) + \gamma(t) \tag{15a}$$

where γ is the \mathbb{E}_1^3 -valued function given by

$$\gamma(t) = \left(0, \int^t \Psi(\tau) \cosh \tau d\tau, \int^t \Psi(\tau) \sinh \tau d\tau \right). \tag{15b}$$

for a function $\Psi \in C^\infty(M)$;

2. A flat surface given by

$$x(s, t) = \int^s \cosh \theta(\tau) d\tau (1, 0, 0) + \int^s \sinh \theta(\tau) d\tau (0, \sinh t_0, \cosh t_0) + (0, t \cosh t_0, t \sinh t_0). \quad (16)$$

for a constant t_0 .

Proof. In order to prove the necessary condition, we assume that M is a space-like surface endowed with a CPD relative to $k = (1, 0, 0)$ with the isometric immersion $x : M \rightarrow \mathbb{E}_1^3$. Let $\{e_1, e_2; N\}$ be the local orthonormal frame field described before Lemma 3.1, k_1, k_2 be the principal curvatures of M and (s, t) be a local coordinate system given in Lemma 3.3.

Note that (14) and (9) becomes

$$m_s - m \tanh \theta k_2 = 0, \quad (17)$$

$$(k_2)_s = (\theta' - k_2) \tanh \theta k_2, \quad (18)$$

respectively and $e_2(\theta) = 0$ implies $\theta = \theta(s)$. Thus, we have

$$e_1 = x_s. \quad (19)$$

By combining (17) with (8), the shape operator S of M become

$$S = \begin{pmatrix} \theta' & 0 \\ 0 & \coth \theta \frac{m_s}{m} \end{pmatrix}. \quad (20)$$

Here ' denotes ordinary differentiation with respect to the appropriated variable.

By combining (17) and (18) we obtain

$$m_{ss} - \theta' \coth \theta m_s = 0,$$

whose general solution is

$$m(s, t) = \Psi_1(t) \int^s \sinh \theta(\tau) d\tau + \Psi_2(t),,$$

for some smooth functions Ψ_1, Ψ_2 . Therefore, by re-defining t properly, we may assume either

$$m(s, t) = \int^s \sinh \theta(\tau) d\tau + \Psi(t), \Psi \in C^\infty(M), \quad (21a)$$

or

$$m(s, t) = 1. \quad (21b)$$

Case 1. m satisfies (21a). In this case, by considering the equation (7) with m given in (21a), we get the Levi-Civita connection of M satisfies

$$\nabla_{\partial_s} \partial_s = 0.$$

By combining this equation with (20) and using Gauss formula, we obtain

$$x_{ss} = -\theta' N. \quad (22)$$

On the other hand, from the decomposition (6), we have $\langle x_s, k \rangle = \cosh \theta$ and $\langle x_t, k \rangle = 0$. By considering these equations, we can assume that x has the form of

$$x(s, t) = \left(\int^s \cosh \theta(\tau) d\tau, x_2(s, t), x_3(s, t) \right) + \gamma(t) \quad (23)$$

for a \mathbb{E}_1^3 -valued smooth function $\gamma = (0, \gamma_2, \gamma_3)$. On the other hand, by combining (19) and (22) with (6), we yield

$$(1, 0, 0) = \cosh \theta x_s - \frac{\sinh \theta}{\theta'} x_{ss}. \tag{24}$$

By solving (24) and considering $\langle x_s, x_s \rangle = 1$ in (23), we obtain

$$x(s, t) = \int^s \cosh \theta(\tau) d\tau (1, 0, 0) + \int^s \sinh \theta(\tau) d\tau (0, \sinh \varphi(t), \cosh \varphi(t)) + \gamma(t) \tag{25}$$

for a smooth function $\varphi = \varphi(t)$. Note that (25) implies

$$\begin{aligned} x_s &= \cosh \theta(s) (1, 0, 0) + \int^s \sinh \theta(\tau) d\tau (0, \sinh \varphi(t), \cosh \varphi(t)), \\ x_t &= \varphi'(t) \int^s \sinh \theta(\tau) d\tau (0, \cosh \varphi(t), \sinh \varphi(t)) + (0, \gamma_2'(t), \gamma_3'(t)). \end{aligned} \tag{26}$$

Because of $\langle x_s, x_t \rangle = 0$, we have $(0, \gamma_2', \gamma_3') = h(t)(0, \cosh \varphi, \sinh \varphi)$ for a smooth function $h = h(t)$. Therefore, (26) turns into

$$x_t = (\varphi'(t) \int^s \sinh \theta(\tau) d\tau + h(t)) (0, \cosh \varphi(t), \sinh \varphi(t)).$$

By combining this equation with $\langle x_t, x_t \rangle = m^2$ and using (21a), we obtain $\varphi(t) = t$ and $h(t) = \Psi(t)$ which gives (15b). Thus, we have the Case (1) of the theorem.

Case 2. m is given as (21b). In this case, the induced metric of M becomes $g = ds^2 + dt^2$, the Levi Civita connection of M satisfies

$$\nabla_{\partial_s} \partial_s = 0, \quad \nabla_{\partial_s} \partial_t = 0, \quad \nabla_{\partial_t} \partial_t = 0 \tag{27}$$

and (20) becomes

$$S = \begin{pmatrix} \theta' & 0 \\ 0 & 0 \end{pmatrix}. \tag{28}$$

Therefore, x and N satisfies

$$\begin{aligned} x_{ss} &= -\theta' N, & x_{st} &= 0, & x_{tt} &= 0. \\ N_s &= -\theta' x_s, & N_t &= 0. \end{aligned}$$

A straightforward computation yields that M is congruent to the surface given in Case (2) of the theorem. Hence, the proof for the necessary condition is obtained.

The proof of sufficient condition follows from a direct computation. \square

As a direct result of Theorem 3.4, we obtain the following classification of maximal CPD surfaces.

Proposition 3.5. *A maximal surface in \mathbb{E}_1^3 endowed with CPD relative to a constant, space-like direction is either an open part of a plane or congruents to the surface given by*

$$x(s, t) = \frac{1}{c} (\sin^{-1}(cs), \sqrt{1 - c^2 s^2} \sinh t, \sqrt{1 - c^2 s^2} \cosh t) \tag{29}$$

for a non-zero constant c .

In this case, the angle function θ is

$$\theta(s) = \tanh^{-1}(-cs). \tag{30}$$

Proof. Let M be a space-like CPD surface and assume that it is not an open part of a plane. If M is maximal, then Theorem 3.4 yields that M is congruent to the surface given by (15). Note that the shape operator S of M is (20) for the function m satisfying (21a). Considering the maximality condition $\text{tr } S = 0$ and (20), we have

$$\theta' + \coth \theta \frac{m_s}{m} = 0.$$

Solving this equation, we get

$$\theta(s) = \cosh^{-1} \left(\frac{1}{cm} \right) \tag{31}$$

for a non-zero constant c . Furthermore, one can conclude from (31) that the function m depends only on s . So (21a) implies $\Psi(t) = 0$ which yields $m(s) = \int^s \sinh \theta(\tau) d\tau$ and $\gamma(t) = (0, 0, 0)$. Therefore, (31) becomes

$$\theta' = -\frac{1}{c} \cosh^2 \theta.$$

By solving this equation, we get the expression (30). By a further computation, we obtain (29). Thus, we complete the proof of theorem. \square

In the remaining part of this section, we will assume that M is a Lorentzian surface in the Minkowski 3-space endowed with CPD relative to $k = (1, 0, 0)$.

As we mentioned in the previous subsection, the shape operator S of M can be non-diagonalizable. In this case, we can choose a pseudo-orthonormal frame field $\{e_1, e_2\}$ of the tangent bundle such that S has the matrix representation,

$$S = \begin{pmatrix} k_1 & \mu \\ 0 & k_1 \end{pmatrix}. \tag{32}$$

In this case, (3) becomes

$$k = e_1 + N. \tag{33}$$

By a simple computation we obtain $k_1 = 0$. Thus M is a flat, minimal B-scroll. It is well known that it must be congruent to the surface given by

$$x(s, t) = \left(\frac{s^2}{2} + t, \frac{(2s - 1)^{3/2}}{3}, \frac{s^2}{2} - s + t \right) \tag{34}$$

(see for example, [16]). Hence, we have the following result.

Proposition 3.6. *Let M be an oriented Lorentzian surface in \mathbb{E}_1^3 with non-diagonalizable shape operator. If M is a surface endowed with a canonical principal direction relative to a space-like constant direction, then it is congruent to the surface given by (34).*

Now, assume that M is a Lorentzian surface and its shape operator S can be diagonalizable. Let $\{e_1, e_2\}$ be a local orthonormal frame field of the tangent bundle of M and e_1 be proportional to U . Since the unit normal vector N is a space-like vector, then we have two cases for subject to casuality of e_1 .

Case A. e_1 is a space-like vector. In this case, (3) implies

$$k = \sin \theta e_1 + \cos \theta N. \tag{35}$$

Case B. e_1 is a time-like vector. In this case, (3) implies

$$k = \sinh \theta e_1 + \cosh \theta N. \tag{36}$$

We have the following lemma which is the analogous of Lemma 3.1.

Lemma 3.7. Let M be a Lorentzian surface endowed with CPD relative to $k = (1, 0, 0)$ and $\{e_1, e_2\}$ its principal directions such that $\langle k, e_2 \rangle = 0$. Then, we have the following statements.

1. If e_1 is space-like, then the Levi-Civita connection ∇ of M is given by

$$\nabla_{e_1} e_1 = \nabla_{e_1} e_2 = 0, \tag{37a}$$

$$\nabla_{e_2} e_1 = \cot \theta k_2 e_2, \quad \nabla_{e_2} e_2 = \cot \theta k_2 e_1 \tag{37b}$$

for a function k_2 satisfying

$$e_1(k_2) = \cot \theta k_2 (e_1(\theta) - k_2). \tag{38}$$

2. If e_1 is time-like, then the Levi-Civita connection ∇ of M is given by

$$\nabla_{e_1} e_1 = \nabla_{e_1} e_2 = 0, \tag{39a}$$

$$\nabla_{e_2} e_1 = \coth \theta k_2 e_2, \quad \nabla_{e_2} e_2 = \coth \theta k_2 e_1, \tag{39b}$$

and for a function k_2 satisfying

$$e_1(k_2) = \coth \theta k_2 (e_1(\theta) - k_2). \tag{40}$$

3. In both cases, θ satisfies (10) and the matrix representation shape operator S is

$$S = \begin{pmatrix} e_1(\theta) & 0 \\ 0 & k_2 \end{pmatrix}. \tag{41}$$

Proof. If we consider the decompositions (35) and (36), respectively and follow exactly same way using to prove of Lemma 3.1, then we can get the statement (1) and (2) of the lemma, respectively and in both cases the equations (10) and (41) are satisfied. \square

The proof of the following lemma is similar the proof of Lemma 3.3.

Lemma 3.8. Let M be a Lorentzian surface endowed with CPD relative to $k = (1, 0, 0)$ and $\{e_1, e_2\}$ its principal directions such that $\langle k, e_2 \rangle = 0$. Then there exists a neighborhood \mathcal{N}_p of p on which $e_1 = \partial_s$ and $e_2 = \frac{1}{m} \partial_t$ for a smooth function m . Moreover, if e_1 is space-like then the induced metric of \mathcal{N}_p becomes

$$g = ds^2 - m^2 dt^2 \tag{42}$$

and m satisfies

$$e_1(m) - \cot \theta k_2 m = 0. \tag{43}$$

On the other hand, if e_1 is time-like then the induced metric of \mathcal{N}_m becomes

$$g = -ds^2 + m^2 dt^2 \tag{44}$$

and m satisfies

$$e_1(m) - \coth \theta k_2 m = 0. \tag{45}$$

Theorem 3.9. Let M be an oriented Lorentzian surface in \mathbb{E}_1^3 with diagonalizable shape operator. Then, M is endowed with a canonical principal direction relative to a space-like, constant direction if and only if it is congruent to the surface given by one of the followings

1. A surface given by

$$x(s, t) = \int^s \sin \theta(\tau) d\tau (1, 0, 0) + \int^t \cos \theta(\tau) d\tau (0, \cosh t, \sinh t) + \gamma(t), \tag{46a}$$

where γ is

$$\gamma(t) = (0, \int^t \Psi(\tau) \sinh(\tau) d\tau, \int^t \Psi(\tau) \cosh(\tau) d\tau) \tag{46b}$$

for a function $\Psi \in C^\infty(M)$;

2. A surface given by

$$x(s, t) = \int^s \sin \theta(\tau) d\tau (1, 0, 0) + \int^s \cos \theta(\tau) d\tau (0, \cosh t_0, \sinh t_0) + (0, t \sinh(t_0), t \cosh(t_0)) \tag{47}$$

for a constant t_0 ;

3. A surface given by

$$x(s, t) = \int^s \sinh \theta(\tau) d\tau (-1, 0, 0) + \int^s \cosh \theta(\tau) d\tau (0, \sinh t, \cosh t) + \gamma(t), \tag{48a}$$

where γ is

$$\gamma(t) = (0, \int^t \Psi(\tau) \cosh(\tau) d\tau, \int^t \Psi(\tau) \sinh(\tau) d\tau) \tag{48b}$$

for a function $\Psi \in C^\infty(M)$;

4. A surface given by

$$x(s, t) = \int^s \sinh \theta(\tau) d\tau (-1, 0, 0) + \int^s \cosh \theta(\tau) d\tau (0, \sinh t_0, \cosh t_0) + (0, t \cosh t_0, t \sinh t_0), \tag{49}$$

for a constant t_0 .

Proof. In order to prove the necessary condition, we assume that M is a Lorentzian surface endowed with CPD relative to $k = (1, 0, 0)$. Let $x : M \rightarrow \mathbb{E}_1^3$ be an isometric immersion, $\{e_1, e_2; N\}$ the local orthonormal frame field described before Lemma 3.7, k_1, k_2 principal curvatures of M and (s, t) a local coordinate system given in Lemma 3.8. We will consider two cases described above separately.

Case A. e_1 is a space-like vector. In this case, we have (37),(38), (42) and (43). Note that (43) and (38) turns into, respectively

$$m_s - \cot \theta k_2 m = 0, \tag{50a}$$

$$(k_2)_s = \cot \theta k_2 (\theta' - k_2). \tag{50b}$$

By considering (50a), we obtain $k_2 = \tan \theta \frac{m_s}{m}$. Thus, (41) becomes

$$S = \begin{pmatrix} \theta' & 0 \\ 0 & \tan \theta \frac{m_s}{m} \end{pmatrix}. \tag{51}$$

Furthermore, by differentiating (50a) with respect to s and using (50), we obtain

$$m_{ss} + \theta' \tan \theta m_s = 0.$$

Therefore, m satisfies either

$$m(s, t) = \int^s \cos \theta(\xi) d\xi + \Psi(t) \tag{52a}$$

for a smooth function Ψ or

$$m(s, t) = 1 \tag{52b}$$

Case A1. m satisfies (52a). In this case, similar to the Case (1) in the proof of Theorem 3.4, we consider (37) and (51) to get

$$x_{ss} = \theta' N, \tag{53a}$$

$$x_{st} = \frac{m_s}{m} x_t, \tag{53b}$$

$$x_{tt} = mm_s x_s + \frac{m_t}{m} x_t - mm_s \tan \theta N. \tag{53c}$$

Furthermore, considering (35) we have $\langle e_1, k \rangle = \langle x_s, k \rangle = \sin \theta$ and $\langle x_t, k \rangle = 0$. So we get

$$x(s, t) = \left(\int^s \sin \theta(\tau) d\tau, x_2(s, t), x_3(s, t) \right) + \gamma(t) \tag{54}$$

for a \mathbb{E}_1^3 -valued smooth function $\gamma = (0, \gamma_2, \gamma_3)$. Also (35) and (53a) imply

$$(1, 0, 0) = \sin \theta x_s + \frac{\cos \theta}{\theta'} x_{ss}. \tag{55}$$

By considering (54) and $\langle x_s, x_s \rangle = 1$, we solve (55) and obtain

$$x(s, t) = \int^s \sin \theta(\tau) d\tau (1, 0, 0) + \int^s \cos \theta(\tau) d\tau (0, \cosh \varphi(t), \sinh \varphi(t)) + \gamma(t), \tag{56}$$

for a smooth function φ . By a similar way in the Case (1) in the proof of Theorem 3.4, we could get $\varphi(t) = t$ and (46b) by considering (42) and (56). Furthermore, considering $\varphi(t) = t$ and (46b) in (56) we get (46a). Hence, we get the classification of surface in the case (1) of the Theorem 3.9.

Case A2. m satisfies (52b). In this case, (42) turns into $g = ds^2 - dt^2$. Therefore, by considering (37) and (51), we get

$$\begin{aligned} x_{ss} &= \theta' N, & x_{st} &= 0, & x_{tt} &= 0. \\ N_s &= -\theta' x_s, & N_t &= 0. \end{aligned}$$

A straightforward computation yields that M is congruent to the surface given in Case (2) of the Theorem 3.9. Hence, the proof for the necessary condition is obtained.

Now, we would like to get the case (3) and the case (4) of the Theorem 3.9.

Case B. e_1 is a time-like vector. In this case, we have (39), (40), (44) and (45). By a similar way to Case A, we obtain

$$S = \begin{pmatrix} \theta' & 0 \\ 0 & \tanh \theta \frac{m_s}{m} \end{pmatrix}. \tag{57}$$

Similar to the Case A, we obtain

$$m_{ss} + \theta' \tanh \theta m_s = 0,$$

which yields that m satisfies either

$$m(s, t) = \int^s \cosh \theta(\xi) d\xi + \Psi(t), \tag{58}$$

for a smooth function Ψ or (52b).

If m satisfies (58), we use exactly the same way that we did in the Case A1 and obtain the Case (3) of the theorem. On the other hand, if $m(s, t) = 1$, then we get the Case (4) of the theorem. Hence, the proof of the necessary condition is completed.

The proof of sufficient condition can be obtained by following from a direct computation. \square

Proposition 3.10. *A minimal surface in \mathbb{E}_1^3 endowed with CPD relative to a constant, space-like direction is either an open part of a plane or congruents to one of following two surface given below*

1. *A surface given by*

$$x(s, t) = \frac{1}{c} \left(\sinh^{-1}(cs), \sqrt{c^2s^2 + 1} \cosh t, \sqrt{c^2s^2 + 1} \sinh t \right), \tag{59}$$

for a non-zero constant c. In this case, the angle function θ is

$$\theta(s) = \cot^{-1}(cs). \tag{60}$$

2. *A surface given by*

$$x(s, t) = \frac{1}{c} \left(-\ln \left(\sqrt{c^2s^2 - 1} + cs \right), \sqrt{c^2s^2 - 1} \sinh t, \sqrt{c^2s^2 - 1} \cosh t \right), \tag{61}$$

for a non-zero constant c. In this case, the angle function θ is

$$\theta(s) = \coth^{-1}(cs). \tag{62}$$

Proof. Let M be a Lorentzian CPD surface and assume that it is not an open part of a plane. If M is minimal, then Theorem 3.9 yields that M is congruent to the one of surfaces given by (46) and (48).

Case 1. M is congruent to the surface given by (46). Note that the shape operator S of M satisfies (51) for the function m satisfying (52a). Considering the minimality condition $\text{tr}S = 0$ and (51) yield

$$\theta' + \tan \theta \frac{m_s}{m} = 0$$

which implies

$$\theta(s) = \sin^{-1} \left(\frac{1}{cm} \right), \tag{63}$$

for a non-zero constant c and $m = m(s)$. Therefore, (52a) give $\Psi = 0$. So,

$$m(s, t) = \int^s \cos \theta(\xi) d\xi.$$

By combining this equation with (63) we obtain (60). By a further computation, we obtain (59).

Case 2. M is congruent to the surface given by (48). Note that the shape operator S of M satisfies (57) for the function m satisfying (58). In this case, from the minimality condition $\text{tr}S = 0$ and (51), we obtain

$$\theta' + \tanh \theta \frac{m_s}{m} = 0.$$

By a similar way to Case 1, we obtain (61) and (62). \square

3.2. CPD surfaces relative to a light-like, constant direction.

In this subsection we will consider surfaces endowed with CPD relative to the fixed vector $k = (1, 0, 1)$ which is light-like.

Theorem 3.11. *Let M be an oriented surface in \mathbb{E}_1^3 with diagonalizable shape operator. Then, M is endowed with a canonical principal direction relative to a light-like, constant direction if and only if it is congruent to the surface given by*

$$x(s, t) = \left(\int_{s_0}^s \frac{1}{2\phi(\xi)^2} d\xi \right) (1, 0, 1) + s \left(\gamma_0(t), \sqrt{-2\varepsilon\gamma_0(t) + 1}, \gamma_0(t) - \varepsilon \right) + \int_{t_0}^t b(\xi) \left(\sqrt{-2\varepsilon\gamma_0(\xi) + 1}, -\varepsilon, \sqrt{-2\varepsilon\gamma_0(\xi) + 1} \right) d\xi \tag{64}$$

for some smooth functions b, γ_0 , some constants s_0, t_0 and $\varepsilon \in \{-1, 1\}$ and a non-vanishing function ϕ whose derivative does not vanish. Moreover, the tangential vector field $e_1 = \frac{(1, 0, 1)^T}{\|(1, 0, 1)^T\|}$ is a principal direction of the surfaces given by (64).

Proof. Let N be the unit normal vector field of M associated with its orientation and $x : M \rightarrow \mathbb{E}_1^3$ an isometric immersion. In order to prove necessary condition, assume that M is endowed with a canonical principal direction relative to a light-like, constant direction k . Up to isometries of \mathbb{E}_1^3 , we may assume $k = (1, 0, 1)$. We put $\varepsilon = -\langle N, N \rangle$ and

$$e_1 = \frac{(1, 0, 1)^T}{\|(1, 0, 1)^T\|}.$$

Then, we have

$$(1, 0, 1) = \phi(e_1 - N) \tag{65}$$

for a smooth function ϕ . Note that we have $\langle e_1, e_1 \rangle = \varepsilon$.

Because of the assumption, e_1 is a principal direction of M with corresponding principal curvature k_1 . By a simple computation considering (65) we obtain

$$0 = X(\phi)(e_1 - N) + \phi \nabla_X e_1 + \phi h(e_1, X) + \phi SX, \tag{66}$$

whenever X is tangent to M . Note that (66) for $X = e_1$ gives

$$\nabla_{e_1} e_1 = 0, \tag{67a}$$

$$\nabla_{e_1} e_2 = 0, \tag{67b}$$

$$e_1(\phi) = -\phi k_1, \tag{67c}$$

while (66) for $X = e_2$ is giving

$$\nabla_{e_2} e_1 = -k_2 e_2, \tag{67d}$$

$$\nabla_{e_2} e_2 = -k_2 e_1, \tag{67e}$$

$$e_2(\phi) = 0, \tag{67f}$$

where e_2 is the other principal direction of M with corresponding principal curvature k_2 and $\langle e_2, e_2 \rangle = 1$. In addition, the second fundamental form of M becomes

$$h(e_1, e_1) = -k_1 N, \quad h(e_1, e_2) = 0, \quad h(e_2, e_2) = -\varepsilon k_2 N. \tag{68}$$

Therefore, the Codazzi equation gives

$$e_1(k_2) = k_2^2 - k_1 k_2 \quad \text{and} \quad e_2(k_1) = 0. \tag{69}$$

Note that, because of Remark 2.6, (67c) implies that $e_1(\phi)$ does not vanish on M .

Let $p \in M$. First, we would like to prove the following claim.

Claim 3.11.1. *There exists a neighborhood N_p of p on which the induced metric of M becomes*

$$g = \frac{\varepsilon}{\phi(s)^2} ds^2 + (a(t)s + b(t))^2 dt^2 \tag{70}$$

for some smooth functions a, b such that $e_1 = \phi \partial_s, e_2 = \frac{1}{a(t)s + b(t)} \partial_t$ and

$$k_1(s) = -\phi'(s) \tag{71}$$

Proof of Claim 3.11.1. Note that we have $[e_1, e_2] = k_2e_2$ because of (67a) and (67d). Therefore, (67f) implies $\left[\frac{1}{\phi}e_1, Ge_2 \right] = 0$ for any function G satisfying

$$e_1(G) = -k_2G. \tag{72}$$

Therefore, there exists a local coordinate system (s, t) such that $e_1 = \phi\partial_s$ and $e_2 = \frac{1}{G}\partial_t$. Thus, the induced metric of M is

$$g = \frac{\varepsilon}{\phi^2}ds^2 + G^2dt^2. \tag{73}$$

Note that we have $k_1 = k_1(s)$ and (71) because of (67c), (67f) and (69). In addition, the first equation in (69) and (72) give

$$\phi(k_2)_s = k_2(k_2 - k_1) \tag{74}$$

and

$$\phi(s)G_s = -k_2G \tag{75}$$

respectively. Now, getting derivative of (72) implies

$$\phi'G_s + \phi G_{ss} = -(k_2)_sG - k_2G_s. \tag{76}$$

By combining (75), (71) and (74) with (76), we obtain $\phi G_{ss} = 0$ which yields $G = a(t)s + b(t)$ for some smooth functions a, b . Therefore, (73) becomes (70).

Hence, the proof of the Claim 3.11.1 is completed. ■

Now, let s, t be local coordinates described in the Claim 3.11.1. Note that we have

$$e_1 = \phi x_s. \tag{77}$$

Moreover, (67a) and (68) imply

$$\widetilde{\nabla}_{\phi\partial_s}(\phi\partial_s) = -k_1N$$

from which we get

$$N = \frac{1}{\phi'}(\phi\phi'x_s + \phi^2x_{ss}). \tag{78}$$

By combining (77) and (78) with (65) we get

$$(1, 0, 1) = \phi \left(\phi x_s - \frac{1}{\phi'}(\phi\phi'x_s + \phi^2x_{ss}) \right)$$

which yields

$$x_{ss} = -\frac{\phi'}{\phi^3}(1, 0, 1).$$

By integrating this equation and considering (77), we get

$$x_s = \frac{1}{2\phi^2}(1, 0, 1) + \gamma(t) \tag{79}$$

for an \mathbb{E}_1^3 -valued smooth function γ . As $\langle x_s, x_s \rangle = \frac{\varepsilon}{\phi^2}$, we have

$$\frac{\langle (1, 0, 1), \gamma(t) \rangle - \varepsilon}{\phi(s)^2} + \langle \gamma(t), \gamma(t) \rangle = 0.$$

Since ϕ is not constant, the above equation implies $\langle (1, 0, 1), \gamma(t) \rangle = \varepsilon$ and $\langle \gamma(t), \gamma(t) \rangle = 0$. By considering these equations, we obtain

$$\gamma(t) = (\gamma_0(t), \sqrt{-2\varepsilon\gamma_0(t) + 1}, \gamma_0(t) - \varepsilon)$$

for a smooth function γ_0 . Therefore (79) becomes

$$x_s = \frac{1}{2\phi^2}(1, 0, 1) + (\gamma_0(t), \sqrt{-2\varepsilon\gamma_0(t) + 1}, \gamma_0(t) - \varepsilon). \tag{80}$$

(77) and (78) imply

$$e_1 = \phi x_s = \frac{1}{2\phi(s)}(1, 0, 1) + \phi(s)(\gamma_0(t), \sqrt{-2\varepsilon\gamma_0(t) + 1}, \gamma_0(t) - \varepsilon), \tag{81a}$$

$$N = -\frac{1}{2\phi(s)}(1, 0, 1) + \phi(s)(\gamma_0(t), \sqrt{-2\varepsilon\gamma_0(t) + 1}, \gamma_0(t) - \varepsilon). \tag{81b}$$

Since e_2 is a unit vector satisfying $\langle e_1, e_2 \rangle = \langle N, e_2 \rangle = 0$, we may assume

$$e_2 = \frac{1}{a(t)s + b(t)}x_t = (\sqrt{-2\varepsilon\gamma_0(t) + 1}, -\varepsilon, \sqrt{-2\varepsilon\gamma_0(t) + 1}). \tag{81c}$$

By integrating (80), we obtain

$$x(s, t) = \left(\int_{s_0}^s \frac{1}{2\phi(\xi)^2} d\xi \right) (1, 0, 1) + s(\gamma_0(t), \sqrt{-2\varepsilon\gamma_0(t) + 1}, \gamma_0(t) - \varepsilon) + \Gamma(t) \tag{82}$$

for a smooth \mathbb{E}_1^3 -valued function Γ . By combining (82) and (81c), we get

$$\Gamma'(t) = \left(\left(a(t) - \frac{\gamma_0'(t)}{\sqrt{-2\varepsilon\gamma_0(t) + 1}} \right) s + b(t) \right) (\sqrt{-2\varepsilon\gamma_0(t) + 1}, -\varepsilon, \sqrt{-2\varepsilon\gamma_0(t) + 1})$$

from which we conclude

$$a(t) = \frac{\gamma_0'(t)}{\sqrt{-2\varepsilon\gamma_0(t) + 1}}$$

and

$$\Gamma'(t) = b(t) (\sqrt{-2\varepsilon\gamma_0(t) + 1}, -\varepsilon, \sqrt{-2\varepsilon\gamma_0(t) + 1}).$$

By combining the last equation with (82), we obtain (64). Hence, the proof of the necessary condition is completed.

Conversely, consider the surface M given by (64) whose derivative does not vanish. A direct computation yields that unit normal of M is

$$N = \left(\frac{\varepsilon}{2\phi(s)} - \varepsilon\phi(s)\gamma_0(t), -\varepsilon\phi(s)\sqrt{1 - 2\varepsilon\gamma_0(t)}, \varepsilon(-\phi(s))\gamma_0(t) + \frac{\varepsilon}{2\phi(s)} + \phi(s) \right)$$

and the principal curvatures of M are

$$e_1 = \phi(s) \frac{\partial}{\partial s} \quad \text{and} \quad e_2 = \frac{\sqrt{-2\varepsilon\gamma_0(t) + 1}}{s\gamma_0'(t) + b(t)\sqrt{-2\varepsilon\gamma_0(t) + 1}} \frac{\partial}{\partial t}. \tag{83}$$

Moreover, we have $\langle e_2, (1, 0, 1) \rangle = 0$ which yields that $(1, 0, 1)^T$ is a principal direction. Hence the proof of sufficient condition is completed. \square

Remark 3.12. As stated in Theorem 3.11, the surface M given by (64) is endowed with a canonical principal direction relative to the light-like vector $(1, 0, 1)$. Now, consider an arbitrary light-like vector \tilde{k} in \mathbb{E}_1^3 . Let l_1, l_2 vectors in \mathbb{E}_1^3 satisfying

$$\langle l_1, l_1 \rangle = \langle l_1, l_2 \rangle = \langle \tilde{k}, l_2 \rangle = 0, \quad \langle \tilde{k}, l_1 \rangle = -1, \quad \langle l_2, l_2 \rangle = 1.$$

Consider the surface \tilde{M} in \mathbb{E}_1^3 given by $\tilde{x}(s, t) = Lx(s, t)$, where $x(s, t)$ is the position vector of M and L is the linear isometry given by

$$L(1, 0, 1) = \tilde{k}, \quad L\left(-\frac{1}{2}, 0, \frac{1}{2}\right) = l_1, \quad L(0, 1, 0) = l_2.$$

It is straightforward to show that \tilde{M} is endowed with a canonical principal direction relative to the light-like vector \tilde{k} .

By considering the proof of Theorem 3.11, we obtain the following proposition.

Proposition 3.13. Let M be the surface given by (64). Then, the matrix representation of the shape operator S of M with respect to $\{e_1, e_2\}$ is

$$S = \begin{pmatrix} \varepsilon\phi'(s) & 0 \\ 0 & \frac{\varepsilon\phi(s)\gamma_0'(t)}{\sqrt{1-2\varepsilon\gamma_0(t)b(t)+s}\gamma_0'(t)} \end{pmatrix}, \quad (84)$$

where e_1, e_2 are vector fields given by (83)

From Proposition 3.13 we conclude the following characterization results.

Corollary 3.14. A flat surface with diagonalizable shape operator in \mathbb{E}_1^3 endowed with CPD relative to a light-like direction is congruent to the surface given by

$$x(s, t) = \left(cs + \sqrt{1-2c\varepsilon t} + \int_{s_0}^s \frac{1}{2\phi(\xi)^2} d\xi, -\varepsilon t + s\sqrt{1-2c\varepsilon}, \right. \\ \left. s(c-\varepsilon) + \sqrt{1-2c\varepsilon t} + \int_{s_0}^s \frac{1}{2\phi(\xi)^2} d\xi \right) \quad (85)$$

for some constants $s_0, \varepsilon \in \{-1, 1\}$ and a non-vanishing function ϕ whose derivative does not vanish.

Corollary 3.15. A minimal (resp. maximal) surface with diagonalizable shape operator in \mathbb{E}_1^3 endowed with CPD relative to a light-like direction is congruent to the surface given by

$$x(s, t) = \left(\frac{(c_1+s)^3}{c_2} + c_1 t, (s-c_1)\sqrt{1-2t\varepsilon}, \frac{(c_1+s)^3}{c_2} + c_1 t + s(t-\varepsilon) \right) \quad (86)$$

for some constants $c_2 > 0, c_1$ with $\varepsilon = -1$ (resp. $\varepsilon = 1$).

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